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INTEGRAL CURVES OF A LINEAR VECTOR FIELD IN SEMI-EUCLIDEAN SPACES

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ABSTRACT. In this paper, we study integral curves or flow lines of a linear vector field in $(2n+1)$-dimensional semi-Euclidean space $\mathbb{E}_\nu^{2n+1}$. The skew symmetric matrix has been found depending on the number of timelike vectors are odd or even. Taking into consideration of the structure, we obtained the linear first order system of differential equations. This system gives rise to integral curves of linear vector fields. Meanwhile solution of the system has also been presented and discussed.

Keywords. Integral curve, linear vector field, semi-Euclidean space, skew-symmetric matrix.

AMS (MOS) Subject Classification. 53A04, 53A35, 53A40, 53B30

1. Introduction

A vector field is an assignment of a vector to each point in a subset of Euclidean space. As vector fields exist at all points of space, they can be specified along curves and surfaces as well. This is especially important because all laws of electricity and magnetism can be formulated through the behavior of vector fields along curves and surfaces. Vector fields are often used to model, for example, the speed and direction of a moving fluid throughout space, or the strength and direction of some force, such as the magnetic or gravitational force, as it changes from point to point, [4].

Several authors studied the integral curves by using matrix of a linear vector field in Euclidean and 3-dimensional Lorentz-Minkowski spaces. Karger and Novak [5] classified the integral curves of a linear vector field in 3-dimensional Euclidean space. They showed that the integral curves of the linear vector field in $\mathbb{E}^3$ are helixes, circles or parallel straight lines. Acratalishian [1] has shown that these results of Karger and Novak are extended to $(2n+1)$-dimensional Euclidean space. In [10], Yaylacı has given a classification of the integral curves of a linear vector field in 3-dimensional Lorentz-Minkowski space.

In this paper, we investigate integral curves of a linear vector field in $(2n+1)$-dimensional semi-Euclidean space $\mathbb{E}_\nu^{2n+1}$. The results can be easily transferred to the Euclidean and Lorentz-Minkowski spaces.

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Section 2 is concerned with some basic geometric notations.

Section 3 deals with the classification of the integral curves of a linear vector field on $\mathbb{E}_\nu^{2n+1}$. This classification depends on whether the number of timelike vectors (equivalently the index of the $(2n+1)$-dimensional semi-Euclidean space $\mathbb{E}_\nu^{2n+1}$) is odd or even as well as the rank of the matrix of the linear vector in $\mathbb{E}_\nu^{2n+1}$. Also in this section, it is given some examples with respect to special cases of n and ν.

2. **Preliminaries**

We review briefly the basic concepts of the semi-Euclidean space that will be required in this paper.

Definition 2.1. The semi-Euclidean space $\mathbb{E}_\nu^{2n+1}$ is the $(2n+1)$-dimensional vector space $\mathbb{E}^{2n+1}$ endowed with the pseudo scalar product

$$\langle v, w\rangle = -\sum_{i=1}^{\nu} v_i w_i + \sum_{j=\nu+1}^{2n+1} v_j w_j, \quad \forall v, w \in \mathbb{E}_\nu^{2n+1}.$$

We say that the vector $v \in \mathbb{E}_\nu^{2n+1}$ is spacelike, lightlike or timelike if $\langle v, v\rangle > 0$ or $v = 0$, $\langle v, v\rangle = 0$ and $v \neq 0$, and $\langle v, v\rangle < 0$, respectively, [8]. We define the signature of a vector v as

$$\varepsilon = \begin{cases} 1, & v \text{ is spacelike} \\ 0, & v \text{ is lightlike} \\ -1, & v \text{ is timelike.} \end{cases}$$

The norm of a vector $v \in \mathbb{E}_\nu^{2n+1}$ is defined by $\|v\| = \sqrt{|\langle v, v\rangle|}$.

Definition 2.2. A frame field $\{u_1, \ldots, u_n, \ldots, u_{2n}, u_{2n+1}\}$ in $\mathbb{E}_\nu^{2n+1}$ is called a pseudo orthonormal frame field, [9], if

$$\begin{gathered} \langle u_{2n}, u_{2n}\rangle = -\langle u_{2n+1}, u_{2n+1}\rangle = -1,\ \langle u_{2n}, u_{2n+1}\rangle = 0, \\ \langle u_{2n}, u_i\rangle = \langle u_{2n+1}, u_i\rangle = 0,\ \langle u_i, u_j\rangle = \delta_{ij},\ i, j = 1, \ldots, 2n-1. \end{gathered}$$

Definition 2.3. Let $\alpha(s)$, s being the arclength parameter, be a non-null regular curve in semi-Euclidean space $\mathbb{E}_\nu^{2n+1}$. The changing of a pseudo orthonormal frame field $\{u_1(s), \ldots, u_n(s), \ldots, u_{2n}(s), u_{2n+1}(s)\}$ of $\mathbb{E}_\nu^{2n+1}$ along α is given by

$$\begin{aligned} u_1'(s) &= \kappa_1(s) u_2(s) \\ u_i'(s) &= -\varepsilon_{i-1}\varepsilon_i \kappa_{i-1}(s) u_{i-1}(s) + \kappa_i(s) u_{i+1}(s), \quad 2 \leq i \leq 2n, \\ u_{2n+1}'(s) &= -\varepsilon_{2n}\varepsilon_{2n+1}\kappa_{2n}(s) u_{2n}(s). \end{aligned}$$

These equations are called the Frenet-Serret type formulae for $\alpha(s)$, where $\kappa_i(s)$, $1 \leq i \leq 2n$, is the curvature function of α, $\kappa_i(s) = \varepsilon_{i+1} < u_i'(s), u_{i+1}(s) >$, and ε_i is the signature of the vector u_i, $1 \leq i \leq 2n$, [6].

Definition 2.4. The signature matrix S in the $(2n+1)$-dimensional semi-Euclidean space $\mathbb{E}_\nu^{2n+1}$ is the diagonal matrix whose diagonal entries are $s_1 = s_2 = \cdots = s_\nu = -1$ and $s_{\nu+1} = s_{\nu+2} = \cdots = s_{2n+1} = +1$. We call that A is a skew-symmetric matrix in $(2n+1)$-dimensional semi-Euclidean space if its transpose satisfies the equation $A^t = -SAS$, [8].

Let X be a vector field in the $\mathbb{E}_\nu^{2n+1}$. By an integral curve of the vector field X we understand a curve $\alpha : (a, b) \to \mathbb{E}_\nu^{2n+1}$ such that its every tangent vector belongs to the vector field X. If $\dfrac{d\alpha}{dt} = X(\alpha(t))$, $\forall t \in I$, is satisfied, then the curve α is called an integral curve of the vector field X. A vector field X in $\mathbb{E}_\nu^{2n+1}$ is called linear if $X_v = SAS(v)$ for all $v \in \mathbb{E}_\nu^{2n+1}$, where A is a linear mapping from $\mathbb{E}_\nu^{2n+1}$ into $\mathbb{E}_\nu^{2n+1}$ and S is the signature matrix.

Definition 2.5. A curve is called a general helix or cylindrical helix if its tangents makes a constant angle with a fixed line in space. A curve is a general helix if and only if the ratio of curvature to torsion is constant, [7].

For n-dimensional case, we know from Hayden in 1831 that if

$$\frac{\kappa_{n-1}}{\kappa_{n-2}} = cons., \ \frac{\kappa_{n-3}}{\kappa_{n-4}} = cons., \ \cdots \ \frac{\kappa_2}{\kappa_1} = cons.,$$

the curve is called as generalized helix where $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$ are curvatures of the curve, [3].

3. **Classification of Integral Curves of A Linear Vector Field**

In this section, we will classify the integral curves of a linear vector field X in $\mathbb{E}_\nu^{2n+1}$.

Let $\mathbb{E}_\nu^{2n+1}$ be a $(2n+1)$-dimensional semi-Euclidean vector space over $\mathbb{R}$ and X be a linear vector field in $\mathbb{E}_\nu^{2n+1}$. The classification will be done in either case in terms of whether the number of timelike vectors is odd and even.

★ **Case** 1. **Let the number of timelike vectors be odd:** Let X be a linear mapping in $\mathbb{E}_\nu^{2n+1}$ given by a skew-symmetric matrix A with respect to a pseudo-orthonormal basis ϕ. So, the normal form of this matrix can be written as

$$
\begin{bmatrix}
0 & -\lambda_1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\lambda_1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & -\lambda_{\frac{\nu-1}{2}} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & \lambda_{\frac{\nu-1}{2}} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \lambda_{\frac{\nu+1}{2}} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \lambda_{\frac{\nu+1}{2}} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \lambda_{\frac{\nu+3}{2}} & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & -\lambda_{\frac{\nu+3}{2}} & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_n & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\lambda_n & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix}
$$

where $\lambda_i \in \mathbb{R} - \{0\}$, $1 \leq i \leq n$. So, we have the following theorem.

Theorem 3.1. *Let X be a linear vector field in $\mathbb{E}_\nu^{2n+1}$ determined by the matrix*

$$
\begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix}
$$

with respect to a pseudo-orthonormal frame $\{O;\ u_1, u_2, \ldots, u_{2n+1}\}$, whose A is the normal formed skew-symmetric matrix and C is a $(2n+1) \times 1$ column matrix such that

$$
C = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2n-1} \\ a_{2n} \\ a_{2n+1} \end{bmatrix}.
$$

Then the integral curves of X have the following properties:

i) If the rank of the matrix $[AC]$ is equal to $2n+1$, then the integral curves are the generalized helixes,

ii) If the rank of the matrix $[AC]$ is equal to $2k$, $1 \leq k \leq n$, then the integral curves are Lorentzian circles in parallel planes whose centres lie on a same straight line perpendicular to those planes,

iii) If the rank of the matrix $[AC]$ is equal to $2k+1$, $1 \leq k \leq n$, then the integral curves are the generalized helixes, *iv) If the rank of the matrix $[AC]$ is equal to 1, then the integral curves are the parallel straight lines.*

Proof. i) Let X be a linear vector field in $\mathbb{E}_\nu^{2n+1}$. Then the value of the linear vector field X for all points $P = (x_1, x_2, \dots, x_{2n+1}) \in \mathbb{E}_\nu^{2n+1}$ can be written as

$$\begin{bmatrix} X(P) \\ 1 \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P \\ 1 \end{bmatrix}$$

or

$$\begin{aligned} X(P) \;=\; & (-\lambda_1 x_2 + a_1,\ \lambda_1 x_1 + a_2, \dots, -\lambda_{\frac{\nu-1}{2}} x_{\nu-1} + a_{\nu-2},\ \lambda_{\frac{\nu-1}{2}} x_{\nu-2} + a_{\nu-1}, \\ & \lambda_{\frac{\nu+1}{2}} x_{\nu+1} + a_\nu,\ \lambda_{\frac{\nu+1}{2}} x_\nu + a_{\nu+1},\ \lambda_{\frac{\nu+3}{2}} x_{\nu+3} + a_{\nu+2},\ -\lambda_{\frac{\nu+3}{2}} x_{\nu+2} + a_{\nu+3}, \\ & \dots, \lambda_n x_{2n} + a_{2n-1},\ -\lambda_n x_{2n-1} + a_{2n},\ a_{2n+1}). \end{aligned}$$

Here, for the sake of simplicity, we choose $\lambda_i = 1$, $1 \le i \le n$. If $\alpha : I \subset \mathbb{R} \to \mathbb{E}_\nu^{2n+1}$ is an integral curve of the linear vector field X, then by the definition of the integral curve we can write

$$\frac{d\alpha(t)}{dt} = X(\alpha(t)),\ \forall t \in I.$$

So, the integral curve with the initial condition $\alpha(t) = P = (x_1, x_2, \dots, x_{2n+1})$ is a solution of the differential equation

$$\frac{d\alpha(t)}{dt} = X(P).$$

Hence, we get the system of differential equations

$$\frac{d\alpha_i(t)}{dt} = \begin{cases} -x_{i+1} + a_i & i = 2k-1,\ 1 \le k < \frac{\nu+1}{2} \\ x_{i-1} + a_i & i = 2k,\ 1 \le k < \frac{\nu+1}{2} \\ x_{i+1} + a_i & i = 2k-1,\ k = \frac{\nu+1}{2} \\ x_{i-1} + a_i & i = 2k,\ k = \frac{\nu+1}{2} \\ x_{i+1} + a_i & i = 2k-1,\ \frac{\nu+3}{2} \le k \le n \\ -x_{i-1} + a_i & i = 2k,\ \frac{\nu+3}{2} \le k \le n \\ a_i & i = 2n+1. \end{cases}$$

If we solve this system of differential equations, we get the integral curve of the linear vector field X as

$$\begin{aligned} \alpha(t) \;=\; & (-A_i \sin t + B_i \cos t - a_{2i},\ A_i \cos t + B_i \sin t + a_{2i-1},\ A_{\frac{\nu+1}{2}} \sinh t + \\ & B_{\frac{\nu+1}{2}} \cosh t - a_{\nu+1},\ A_{\frac{\nu+1}{2}} \cosh t + B_{\frac{\nu+1}{2}} \sinh t + a_\nu,\ A_j \sin t - B_j \cos t + a_{2j}, \\ & A_j \cos t + B_j \sin t - a_{2j-1},\ ct + d), \end{aligned}$$

where $1 \le i \le \frac{\nu-1}{2}$ and $\frac{\nu+3}{2} \le j \le n$. Now, we can examine the character of the integral curve . If we take into consideration the derivations of $\alpha(t)$, we get linearly independent vectors α', α'', α''', $\alpha^{(4)}$ and $\alpha^{(5)}$. The other higher order derivations are linear dependent. So, it can be constructed only Frenet quintette on the curve $\alpha(t)$.

Therefore, there exists four curvature functions k_1, k_2, k_3 and k_4. Now we show that $\alpha(t)$ is a generalized helixes in $\mathbb{E}_\nu^{2n+1}$. For this aim, firstly, let us calculate the velocity of $\alpha(t)$. The velocity of the curve is obtained as

$$\langle \alpha'(t), \alpha'(t) \rangle = \sum_{i=1}^{\frac{\nu-1}{2}} -(A_i^2 + B_i^2) - A_{\frac{\nu+1}{2}}^2 + B_{\frac{\nu+1}{2}}^2 + \sum_{j=\frac{\nu+3}{2}}^{n} (A_j^2 + B_j^2) + c^2,$$

Assume that $\alpha(t)$ is a timelike curve, that is $\langle \alpha'(t), \alpha'(t) \rangle = -1$. Thus, we have a pseudo-orthogonal system by the Gramm-Schmidt method:

$$\begin{aligned}
u_1 &= (-A_i \cos t - B_i \sin t,\ -A_i \sin t + B_i \cos t, \ldots, A_{\frac{\nu+1}{2}} \cosh t + B_{\frac{\nu+1}{2}} \sinh t, \\
&\qquad A_{\frac{\nu+1}{2}} \sinh t + B_{\frac{\nu+1}{2}} \cosh t,\ A_j \cos t + B_j \sin t,\ -A_j \sin t + B_j \cos t, \ldots, c) \\
u_2 &= (A_i \sin t - B_i \cos t,\ -A_i \cos t - B_i \sin t, \ldots, A_{\frac{\nu+1}{2}} \sinh t + B_{\frac{\nu+1}{2}} \cosh t, \\
&\qquad A_{\frac{\nu+1}{2}} \cosh t + B_{\frac{\nu+1}{2}} \sinh t,\ -A_j \sin t + B_j \cos t,\ -A_j \cos t - B_j \sin t, \ldots, 0) \\
u_3 &= (1-\gamma) \Big[A_i \cos t + B_i \sin t,\ A_i \sin t - B_i \cos t, \ldots, \frac{1+\gamma}{1-\gamma}(A_{\frac{\nu+1}{2}} \cosh t + B_{\frac{\nu+1}{2}} \sinh t, \\
&\qquad A_{\frac{\nu+1}{2}} \sinh t + B_{\frac{\nu+1}{2}} \cosh t),\ -A_j \cos t - B_j \sin t,\ A_j \sin t - B_j \cos t, \ldots, \frac{\gamma}{1-\gamma} c \Big] \\
u_4 &= \frac{\gamma - \theta}{\gamma} \Big[-A_i \sin t + B_i \cos t,\ A_i \cos t + B_i \sin t, \ldots, \frac{\gamma+\theta}{\gamma-\theta}(A_{\frac{\nu+1}{2}} \sinh t + B_{\frac{\nu+1}{2}} \cosh t, \\
&\qquad A_{\frac{\nu+1}{2}} \cosh t + B_{\frac{\nu+1}{2}} \sinh t),\ A_j \sin t - B_j \cos t,\ A_j \cos t + B_j \sin t, \ldots, 0 \Big] \\
u_5 &= \Big(\mu_1(-A_i \cos t - B_i \sin t),\ \mu_1(-A_i \sin t + B_i \cos t), \ldots, \mu_2(A_{\frac{\nu+1}{2}} \cosh t + B_{\frac{\nu+1}{2}} \sinh t), \\
&\qquad \mu_2(A_{\frac{\nu+1}{2}} \sinh t + B_{\frac{\nu+1}{2}} \cosh t),\ \mu_1(A_j \cos t + B_j \sin t),\ \mu_1(-A_j \sin t + B_j \cos t), \\
&\qquad \ldots, \Big(\frac{\gamma^2(1-\theta)}{(\theta-\gamma^2)(\gamma-1)} - \theta\Big) c \Big)
\end{aligned}$$

where

$$\begin{aligned}
\gamma &= \sum_{i=1}^{\frac{\nu-1}{2}} (A_i^2 + B_i^2) - A_{\frac{\nu+1}{2}}^2 + B_{\frac{\nu+1}{2}}^2 - \sum_{j=\frac{\nu+3}{2}}^{n} (A_j^2 + B_j^2), \\
\theta &= \sum_{i=1}^{\frac{\nu-1}{2}} (A_i^2 + B_i^2) + A_{\frac{\nu+1}{2}}^2 - B_{\frac{\nu+1}{2}}^2 - \sum_{j=\frac{\nu+3}{2}}^{n} (A_j^2 + B_j^2),
\end{aligned}$$

$$\mu_1 = \frac{(\theta-\gamma)(1-\theta)}{\theta-\gamma^2},\ \mu_2 = \frac{(\theta+\gamma)(1-\theta)}{\theta-\gamma^2}$$

and $1 \leq i \leq \frac{\nu-1}{2}$ and $\frac{\nu+3}{2} \leq j \leq n$. If we use

$$\kappa_i(s) = \varepsilon_{i+1} \langle u_i'(s), u_{i+1}(s) \rangle$$

for the curvature functions $k_i(s)$, $1 \leq i \leq 4$, we get

$$k_1 = -\varepsilon_2 \gamma,\ k_2 = \varepsilon_3(\gamma^2 - \theta),\ k_3 = -\varepsilon_4 \frac{\gamma^2 - \theta^2}{\gamma} \text{ and } k_4 = \varepsilon_5 \frac{(\gamma^2-\theta^2)(1-\theta)}{\theta-\gamma^2}.$$

So, we obtain

$$\frac{k_1}{k_2} = const \text{ and } \frac{k_3}{k_4} = const.$$

This means that the curve $\alpha(t)$ is the generalized helix.

ii) Let $rank\ [AC] = 2k$, $1 \leq k \leq n$, then:

a) If $rank\ [AC] = 2n$, $k = n$, then the linear first order system of differential equations became

$$\frac{d\alpha_i(t)}{dt} = \begin{cases} -x_{i+1} + a_i & i = 2k-1,\ 1 \leq k < \frac{\nu+1}{2} \\ x_{i-1} + a_i & i = 2k,\ 1 \leq k < \frac{\nu+1}{2} \\ x_{i+1} + a_i & i = 2k-1,\ k = \frac{\nu+1}{2} \\ x_{i-1} + a_i & i = 2k,\ k = \frac{\nu+1}{2} \\ x_{i+1} + a_i & i = 2k-1,\ \frac{\nu+3}{2} \leq k \leq n \\ -x_{i-1} + a_i & i = 2k,\ \frac{\nu+3}{2} \leq k \leq n \\ 0 & i = 2n+1. \end{cases}$$

Hence, the solution of this system is

$$\begin{aligned} \alpha(t) \ = \ & (-A_i \sin t + B_i \cos t - a_{2i},\ A_i \cos t + B_i \sin t + a_{2i-1},\ A_{\frac{\nu+1}{2}} \sinh t + \\ & B_{\frac{\nu+1}{2}} \cosh t - a_{\nu+1},\ A_{\frac{\nu+1}{2}} \cosh t + B_{\frac{\nu+1}{2}} \sinh t + a_\nu,\ A_j \sin t - B_j \cos t + a_{2j}, \\ & A_j \cos t + B_j \sin t - a_{2j-1},\ d), \end{aligned}$$

where $1 \leq i \leq \frac{\nu-1}{2}$ and $\frac{\nu+3}{2} \leq j \leq n$. It is easy to show that the curve $\alpha(t)$ is a Lorentzian circle.

b) Let $rank\ [AC] = r$, $r = 2, 4, \ldots, 2n-2$. Then, the linear first order system of differential equations became

$$\frac{d\alpha_i(t)}{dt} = \begin{cases} -x_{i+1} + a_i & i = 2k-1,\ 1 \leq k < \frac{\nu+1}{2} \\ x_{i-1} + a_i & i = 2k,\ 1 \leq k < \frac{\nu+1}{2} \\ x_{i+1} + a_i & i = 2k-1,\ k = \frac{\nu+1}{2} \\ x_{i-1} + a_i & i = 2k,\ k = \frac{\nu+1}{2} \\ x_{i+1} + a_i & i = 2k-1,\ \frac{\nu+3}{2} \leq k \leq \frac{r}{2} \\ -x_{i-1} + a_i & i = 2k,\ \frac{\nu+3}{2} \leq k \leq \frac{r}{2} \\ 0 & r+1 \leq i \leq 2n+1. \end{cases}$$

If we solve this system, we get

$$\begin{aligned} \alpha(t) \ = \ & (-A_i \sin t + B_i \cos t - a_{2i},\ A_i \cos t + B_i \sin t + a_{2i-1},\ A_{\frac{\nu+1}{2}} \sinh t + \\ & B_{\frac{\nu+1}{2}} \cosh t - a_{\nu+1},\ A_{\frac{\nu+1}{2}} \cosh t + B_{\frac{\nu+1}{2}} \sinh t + a_\nu,\ A_j \sin t - B_j \cos t + a_{2j}, \\ & A_j \cos t + B_j \sin t - a_{2j-1},\ d_{r+1},\ d_{r+2}, \ldots,\ d_{2n+1}), \end{aligned}$$

where $1 \leq i \leq \frac{\nu-1}{2}$ and $\frac{\nu+3}{2} \leq j \leq \frac{r}{2}$. So, the curve $\alpha(t)$ is a Lorentzian circle.

iii) Let $rank\ [AC] = 2k+1$, $1 \leq k \leq n$. In this case,

a) If $rank\ [AC] = 2n+1$, for $k = n$, then $\alpha(t)$ is the same as the first part of the theorem.

b) If *rank* $[AC] = 2k+1 = r+1$, $r = 2,4,\ldots,2n-2$, then the linear first order system of differential equations became

$$\frac{d\alpha_i(t)}{dt} = \begin{cases} -x_{i+1} + a_i & i = 2k-1,\ 1 \le k < \frac{\nu+1}{2} \\ x_{i-1} + a_i & i = 2k,\ 1 \le k < \frac{\nu+1}{2} \\ x_{i+1} + a_i & i = 2k-1,\ k = \frac{\nu+1}{2} \\ x_{i-1} + a_i & i = 2k,\ k = \frac{\nu+1}{2} \\ x_{i+1} + a_i & i = 2k-1,\ \frac{\nu+3}{2} \le k \le \frac{r}{2} \\ -x_{i-1} + a_i & i = 2k,\ \frac{\nu+3}{2} \le k \le \frac{r}{2} \\ a_i & i = r+1, \\ 0 & r+2 \le i \le 2n+1. \end{cases}$$

This system of differential equations has the solution

$$\begin{aligned} \alpha(t) \;=\; & (-A_i \sin t + B_i \cos t - a_{2i},\ A_i \cos t + B_i \sin t + a_{2i-1},\ A_{\frac{\nu+1}{2}} \sinh t + \\ & B_{\frac{\nu+1}{2}} \cosh t - a_{\nu+1},\ A_{\frac{\nu+1}{2}} \cosh t + B_{\frac{\nu+1}{2}} \sinh t + a_\nu,\ A_j \sin t - B_j \cos t + a_{2j}, \\ & A_j \cos t + B_j \sin t - a_{2j-1},\ a_{r+1} t + d,\ d_{r+2},\ d_{r+3}, \ldots,\ d_{2n+1}), \end{aligned}$$

where $1 \le i \le \frac{\nu-1}{2}$ and $\frac{\nu+3}{2} \le j \le \frac{r}{2}$. It is easy to show that the curve $\alpha(t)$ is the generalized helix.

iv) If *rank* $[AC] = 1$, then $\lambda_i = 0$ which gives us a linear first order system of the differential equations. This system has the solution $\alpha(t)$ which is the parallel straight lines in $\mathbb{E}_\nu^{2n+1}$. □

★ **Case** 2. **Let the number of timelike vectors be even:** In this case, a skew-symmetric matrix A with respect to a pseudo-orthonormal basis ϕ can be written as

$$A = \begin{bmatrix} 0 & \lambda_1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & \ldots & 0 & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_n & 0 \\ 0 & 0 & 0 & 0 & \ldots & -\lambda_n & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_{(2n+1)\times(2n+1)}$$

where $\lambda_i \in \mathbb{R} - \{0\}$. So, we have the following theorem.

Theorem 3.2. *Let X be a linear vector field in $\mathbb{E}_\nu^{2n+1}$. Then the integral curves of X have the following properties:*

i) If the rank of the matrix $[AC]$ is equal to $2n+1$, then the integral curves are same parametrized circular helixes,

ii) If the rank of the matrix $[AC]$ is equal to $2k$, $1 \leq k \leq n$, then the integral curves are Lorentzian circles in parallel planes,

iii) If the rank of the matrix $[AC]$ is equal to $2k+1$, $1 \leq k \leq n$, then the integral curves are circular helixes,

iv) If the rank of the matrix $[AC]$ is equal to 1, then the integral curves are the parallel straight lines.

Proof. i) Let X be a linear vector field in $\mathbb{E}_\nu^{2n+1}$. Then the value of the linear vector field X for all points $P = (x_1, x_2, \ldots, x_{2n+1}) \in \mathbb{E}_\nu^{2n+1}$ can be written as

$$\begin{bmatrix} X(P) \\ 1 \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P \\ 1 \end{bmatrix}$$

or

$$X(P) = (\lambda_1 x_2 + a_1, \ -\lambda_1 x_1 + a_2, \ldots, \lambda_n x_{2n} + a_{2n-1}, \ -\lambda_n x_{2n-1} + a_{2n}, \ a_{2n+1}).$$

If we choose $\lambda_i = 1$, $1 \leq i \leq n$, and use the definition of the integral curve, we get the system of differential equations

$$\begin{aligned} \frac{d\alpha_1}{dt} &= x_2 + a_1, \ \frac{d\alpha_2}{dt} = -x_1 + a_2, \ldots, \frac{d\alpha_{2n-1}}{dt} = x_{2n} + a_{2n-1}, \\ \frac{d\alpha_{2n}}{dt} &= -x_{2n-1} + a_{2n}, \ \frac{d\alpha_{2n+1}}{dt} = a_{2n+1} = c. \end{aligned}$$

If we solve the differential equation $\frac{d\alpha_{2n+1}}{dt} = c$, we get $\alpha_{2n+1} = ct + d$. The other $2n$ equations can be solved in pairs. So, the general solution of these differential equations are

$$\begin{aligned} \alpha(t) = (&A_1 \sin t - B_1 \cos t + a_2, \ A_1 \cos t + B_1 \sin t - a_1, \ldots, \\ &A_{\frac{\nu}{2}} \sin t - B_{\frac{\nu}{2}} \cos t + a_\nu A_{\frac{\nu}{2}} \cos t + B_{\frac{\nu}{2}} \sin t - a_{\nu-1}, \\ &A_{\frac{\nu+2}{2}} \sin t - B_{\frac{\nu+2}{2}} \cos t + a_{\nu+2}, \\ &A_{\frac{\nu+2}{2}} \cos t + B_{\frac{\nu+2}{2}} \sin t - a_{\nu+1}, \ldots, \ A_n \sin t - B_n \cos t + a_{2n}, \\ &A_n \cos t + B_n \sin t - a_{2n-1}, \ ct + d). \end{aligned}$$

Now, we can examine the character of the integral curve . If we take into consideration the derivations of $\alpha(t)$, we get linearly independent vectors α', α'' and α'''. The other higher order derivations are linear dependent. Therefore, there are two curvature functions k_1 and k_2. Now we show that $\alpha(t)$ is a circular helix in $\mathbb{E}_\nu^{2n+1}$. Let us calculate the velocity of $\alpha(t)$. The velocity of the curve is obtained as

$$\langle \alpha'(t), \alpha'(t)\rangle = -\sum_{i=1}^{\frac{\nu}{2}}(A_i^2 + B_i^2) + \sum_{j=\frac{\nu+2}{2}}^{n}(A_j^2 + B_j^2) + c^2.$$

Assume that $\alpha(t)$ is a timelike curve, that is $\langle \alpha'(t), \alpha'(t)\rangle = -1$. Thus, we have an orthogonal system by the Gramm-Schmidt method:

$$\begin{aligned}
u_1 &= (A_1\cos t + B_1\sin t,\ -A_1\sin t + B_1\cos t, \ldots,\ A_{\frac{\nu}{2}}\cos t + B_{\frac{\nu}{2}}\sin t,\\
&\quad -A_{\frac{\nu}{2}}\sin t + B_{\frac{\nu}{2}}\cos t, \ldots, A_n\cos t + B_n\sin t,\ -A_n\sin t + B_n\cos t,\ c),\\
u_2 &= (-A_1\sin t + B_1\cos t,\ -A_1\cos t - B_1\sin t, \ldots,\ -A_{\frac{\nu}{2}}\sin t + B_{\frac{\nu}{2}}\cos t,\\
&\quad -A_{\frac{\nu}{2}}\cos t - B_{\frac{\nu}{2}}\sin t, \ldots, -A_n\sin t + B_n\cos t,\ -A_n\cos t - B_n\sin t,\ 0),\\
u_3 &= (\gamma-1)(A_1\cos t + B_1\sin t,\ -A_1\sin t + B_1\cos t, \ldots,\ A_{\frac{\nu}{2}}\cos t + B_{\frac{\nu}{2}}\sin t,\\
&\quad -A_{\frac{\nu}{2}}\sin t + B_{\frac{\nu}{2}}\cos t, \ldots, A_n\cos t + B_n\sin t,\ -A_n\sin t + B_n\cos t,\ \frac{\gamma}{\gamma-1}c),
\end{aligned}$$

where

$$\gamma = \sum_{i=1}^{\frac{\nu}{2}}(A_i^2 + B_i^2) - \sum_{j=\frac{\nu+2}{2}}^{n}(A_j^2 + B_j^2).$$

So, we have

$$k_1 = -\epsilon_2\gamma, \quad k_2 = \epsilon_3\gamma(\gamma-1) \quad \text{and} \quad \frac{k_1}{k_2} = \frac{\epsilon_2}{\epsilon_3}\frac{1}{(1-\gamma)}.$$

This means that the curve $\alpha(t)$ is the helix. If the curve $\alpha(t)$ is translated by $T = (-a_1,\ a_2,\ -a_3,\ a_4, \ldots, -a_{2n-1},\ a_{2n},\ 0)$ we get

$$\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_{2n}^2 = -\gamma = const.$$

So, we can say that $\alpha(t)$ is a circular helix.
$ii)$ Let $rank\ [AC] = 2k$, $1 \le k \le n$, then:
$a)$ If $rank\ [AC] = 2n$, $k = n$, then the differential equations system became

$$\begin{aligned}
\frac{d\alpha_1}{dt} &= x_2 + a_1,\ \frac{d\alpha_2}{dt} = -x_1 + a_2, \ldots, \frac{d\alpha_{2n-1}}{dt} = x_{2n} + a_{2n-1},\\
\frac{d\alpha_{2n}}{dt} &= -x_{2n-1} + a_{2n},\ \frac{d\alpha_{2n+1}}{dt} = 0.
\end{aligned}$$

This system has the solution

$$\begin{aligned}
\alpha(t) &= (A_1\sin t - B_1\cos t + a_2,\ A_1\cos t + B_1\sin t - a_1, \ldots, A_{\frac{\nu}{2}}\sin t - B_{\frac{\nu}{2}}\cos t + a_\nu\\
&\quad A_{\frac{\nu}{2}}\cos t + B_{\frac{\nu}{2}}\sin t - a_{\nu-1},\ A_{\frac{\nu+2}{2}}\sin t - B_{\frac{\nu+2}{2}}\cos t + a_{\nu+2},\ A_{\frac{\nu+2}{2}}\cos t +\\
&\quad B_{\frac{\nu+2}{2}}\sin t - a_{\nu+1}, \ldots,\ A_n\sin t - B_n\cos t + a_{2n},\ A_n\cos t + B_n\sin t - a_{2n-1},\ d).
\end{aligned}$$

It is easy to show that the curve $\alpha(t)$ is a Lorentzian circle.
b) Let $rank\ [AC] = r$, $r = 2, 4, \ldots, 2n-2$. Then, the differential equations system became

$$\begin{aligned} \frac{d\alpha_1}{dt} &= x_2 + a_1,\ \frac{d\alpha_2}{dt} = -x_1 + a_2, \ldots, \frac{d\alpha_{r-1}}{dt} = x_r + a_{r-1}, \\ \frac{d\alpha_r}{dt} &= -x_{r-1} + a_r,\ \frac{d\alpha_j}{dt} = 0,\ r+1 \leq j \leq 2n+1. \end{aligned}$$

If we solve this system, we get

$$\begin{aligned} \alpha(t) &= (A_1 \sin t - B_1 \cos t + a_2,\ A_1 \cos t + B_1 \sin t - a_1, \ldots, A_{\frac{r}{2}} \sin t - \\ &\quad B_{\frac{r}{2}} \cos t + a_r,\ A_{\frac{r}{2}} \cos t + B_{\frac{r}{2}} \sin t - a_{r-1},\ d_{r+1},\ d_{r+2}, \ldots, d_{2n}). \end{aligned}$$

The curve $\alpha(t)$ is a Lorentzian circle.
iii) Let $rank\ [AC] = 2k+1$, $1 \leq k \leq n$. In this case,
a) If $rank\ [AC] = 2n+1$, for $k = n$, then $\alpha(t)$ is the same as the first part of the theorem.

b) If $rank\ [AC] = 2k+1 = r+1$, $r = 2, 4, \ldots, 2n-2$, then the differential equations system became

$$\begin{aligned} \frac{d\alpha_1}{dt} &= x_2 + a_1,\ \frac{d\alpha_2}{dt} = -x_1 + a_2, \ldots, \frac{d\alpha_{r-1}}{dt} = x_r + a_{r-1}, \\ \frac{d\alpha_r}{dt} &= -x_{r-1} + a_r,\ \frac{d\alpha_{r+1}}{dt} = a_{r+1},\ \frac{d\alpha_j}{dt} = 0,\ r+2 \leq j \leq 2n+1. \end{aligned}$$

This system has the solution

$$\begin{aligned} \alpha(t) &= (A_1 \sin t - B_1 \cos t + a_2,\ A_1 \cos t + B_1 \sin t - a_1, \ldots, A_{\frac{r}{2}} \sin t - \\ &\quad B_{\frac{r}{2}} \cos t + a_r,\ A_{\frac{r}{2}} \cos t + B_{\frac{r}{2}} \sin t - a_{r-1},\ a_{r+1} t + d,\ d_{r+2},\ d_{r+3}, \ldots, d_{2n}). \end{aligned}$$

It is easy to show that the curve $\alpha(t)$ is the circular helix.
iv) If $rank\ [AC] = 1$, then $\lambda_i = 0$ which gives us a system of the differential equations. This system of differential equations has the solution $\alpha(t)$ which are parallel straight lines in $\mathbb{E}^{2n+1}_\nu$. □

Now, let us give two examples for some special cases.

Example 3.3. The case $\nu = 0$ and $n = 1$. Consider the vector field $X : \mathbb{E}^3 \to \mathbb{E}^3$ defined by $X(x, y, z) = (y, -x, 0)$. We can take skew-symmetric matrix A and 3×1 column matrix C as below

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}.$$

Then the value of the vector field X for all points $P = (x, y, z) \in \mathbb{E}^3$ can be written as

$$\begin{bmatrix} X(P) \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & a \\ -1 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}.$$

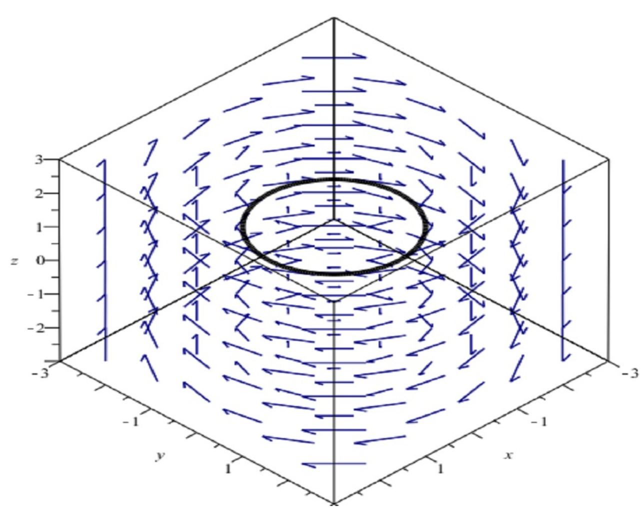
If $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ is an integral curve of the vector field X, then by the definition of the integral curve, we get the system of differential equations as

$$\begin{aligned} \frac{dx}{dt} &= y + a, \\ \frac{dy}{dt} &= -x + b, \\ \frac{dz}{dt} &= 0. \end{aligned}$$

So, the solution of this system is

$$\alpha(t) = (B \sin t - D \cos t + b,\ B \cos t + D \sin t - a,\ d),$$

where B and D are any constants. The curve $\alpha(t)$ is a circle in $\mathbb{E}^3$. In the following figure, the vector field X and its integral curve is demonstrated.



Example 3.4. The case $\nu = 1$ and $n = 1$. Consider the vector field $X : \mathbb{E}^3_1 \to \mathbb{E}^3_1$ defined by $X(x, y, z) = (y, x, 0)$. We can take skew-symmetric matrix A and 3×1 column matrix C as below

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}.$$

Then the value of the vector field X for all points $P = (x, y, z) \in \mathbb{E}_1^3$ can be written as

$$\begin{bmatrix} X(P) \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & a \\ 1 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}.$$

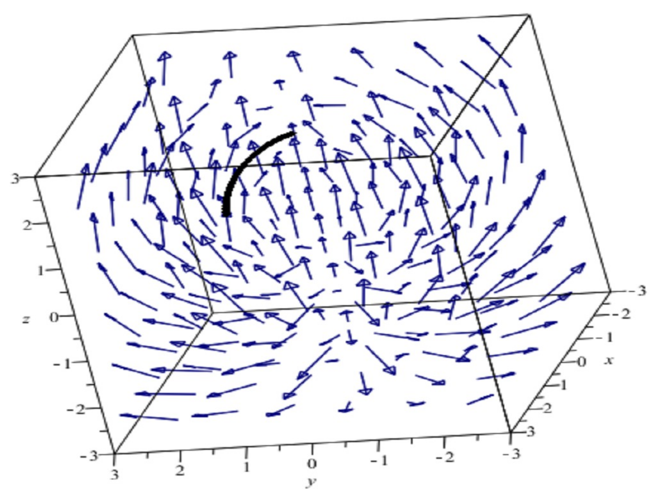
If $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^3$ is an integral curve of the vector field X, then by the definition of the integral curve, we get the system of differential equations as

$$\begin{aligned} \frac{dx}{dt} &= y + a, \\ \frac{dy}{dt} &= x + b, \\ \frac{dz}{dt} &= 0. \end{aligned}$$

So, the solution of this system is

$$\alpha(t) = (B \sinh t + D \cosh t - b,\ B \cosh t + D \sinh t + a,\ d),$$

where B and D are any constants. The curve $\alpha(t)$ is a Lorentzian circle in $\mathbb{E}_1^3$. Figure of vector field X and its integral curve is as follows:



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