Dynamic Systems and Applications 24 (2015) 375-388

MONOTONE ITERATIVE TECHNIQUE AND EXISTENCE RESULTS FOR FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper is concerned with the existence of positive solutions for boundary value problems of fractional functional differential equations involving the Caputo fractional derivative. The proof is based on the monotone iterative technique. As an application, an example is worked out to demonstrate the main result.

AMS (MOS) Subject Classification. 34B10; 34B18; 34B15, 34K37.

1. INTRODUCTION

The purpose of this paper is to investigate the following boundary value problem of fractional functional differential equation with p-Laplacian operator

$$(1.1)\qquad \begin{cases} {}^cD^{\beta}[\phi_p({}^cD^{\alpha}u(t))] + f(t, u(t-\tau), u(t+\theta)) = 0, \quad t \in (0,1), \\ {}^cD^{\alpha}u(0) = 0, \\ au(t) - bu'(t) = \eta(t), \quad t \in [-\tau, 0], \\ cu(t) + du'(t) = \xi(t), \quad t \in [1, 1+\theta], \end{cases}$$

where $1 < \alpha \leq 2$, $0 < \beta \leq 1$, ${}^cD^{\alpha}$ and ${}^cD^{\beta}$ are the Caputo fractional derivatives, $0 < \tau, \theta < 1$, $a, d \geq 0$, $b, c > 0$ are real constants satisfying $b > \frac{2-\alpha}{\alpha-1}a$ and $\phi_p(x)$ is a p-Laplacian operator defined by $\phi_p(x) = |x|^{p-2}x$, $p > 1$, $\phi_q = {\phi_p}^{-1}$, $\frac{1}{p} + \frac{1}{q} = 1$.

We will suppose that the following assumptions are satisfied:

$(A1)$ $f \in \mathcal{C}([0,1] \times [0,+\infty) \times [0,+\infty), \mathbb{R}^+)$, $f(t,u,v) > 0$ for all $(t,u,v) \in [0,1] \times [0,+\infty) \times [0,+\infty)$.

$(A2)$ $\eta \in \mathcal{C}([-\tau, 0], [0,\infty))$, $\xi \in \mathcal{C}([1, 1+\theta], [0,\infty))$ and $\eta(0) = \xi(1) = 0$.

Recently, the study of nonlinear fractional boundary value problems has gained much attention because of their applications in various research areas of applied sciences and engineering. In particular, many authors have investigated the existence results of positive solutions of nonlinear boundary value problems for fractional differential equations. (See [1, 2, 8, 9, 10, 11, 12] and the references therein.) But there are relatively few works available for the existence of positive solutions for fractional

Received April 2, 2015

functional differential equations. For instance, in [5], Li *et al.* considered the following boundary value problem of fractional functional Sturm-Liouville differential equation

$$
\begin{cases} D^{\alpha}u(t)+a(t)f(t,u_t)=0, \quad t\in(0,1), \\ -au(t)+bu'(t)=\eta(t), \quad t\in[-\tau,0], \\ cu(t)+du'(t)=\xi(t), \quad t\in[1,1+\theta], \end{cases} \tag{1.2}
$$

where $1<\alpha\leq 2$ and D^{α} is the Caputo fractional derivative. By means of the Guo Krasnoselskii fixed point theorem, they obtained the existence of positive solutions for the fractional functional BVP (1.2).

In [4], by means of fixed point theorems on cones, Zhao *et al.* investigated the following fractional functional boundary value problem

$$
\begin{cases} D^{\alpha}u(t)+r(t)f(u_t)=0, \quad t\in(0,1), \quad q\in(n-1,n], \\ u^{i}(0)=0, \quad 0\leq i\leq n-3, \\ \alpha u^{(n-2)}(t)-\beta u^{(n-1)}(t)=\eta(t), \quad t\in[-\tau,0], \\ \gamma u^{(n-2)}(t)+\delta u^{(n-1)}(t)=\xi(t), \quad t\in[1,1+\theta]. \end{cases} \tag{1.3}
$$

In [3], by using the Guo Krasnoselskii fixed point theorem on cones, Li *et al.* established the positive solutions for the following fractional functional differential equation

$$
\begin{cases} D^{\beta}[p(t)D^{\alpha}u(t)]+f(t,u(t-\tau),u(t+\theta))=0, \quad t\in(0,1), \\ D^{\alpha}u(0)=D^{\alpha}u(1)=(D^{\alpha}u(0))''=0, \\ au(t)-bu'(t)=\eta(t), \quad t\in[-\tau,0], \\ cu(t)+du'(t)=\xi(t), \quad t\in[1,1+\theta], \end{cases} \tag{1.4}
$$

where $1<\alpha\leq 2$, $2<\beta\leq 3$, D^{α} and D^{β} are the Caputo fractional derivatives.

We notice that all the results in the papers mentioned above are obtained by means of fixed point theorems on cones. Motivated by these papers, but taking completely different technique from [3, 4, 5], we will consider the functional fractional boundary value problem (1.1). Here, we will use the monotone iterative technique to establish the existence results of positive solutions for the fractional BVP (1.1). We not only get the existence results of positive solutions, but also construct two iterative schemes for approximating the solutions. Furthermore, the technique does not require the existence of upper and lower solutions. To the author's knowledge, few works were done in the literature concerning the existence of positive solutions for boundary value problems of fractional functional differential equations with p-Laplacian operator by means of the monotone iterative method. Therefore, the aim of this paper is to fill this gap.

The plan of this paper is as follows. In section 2, we give some definitions and lemmas that are used throughout the paper. In section 3, we establish our main results by using the monotone iterative technique. Finally, in section 4, an example is worked out to demonstrate the applicability of our main result.

2. PRELIMINARIES

In this section, we present some definitions and lemmas which are useful for the proof of our main result.

Definition 2.1 ([6, 7]). The Riemann Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ for a continuous function $h : (0, \infty) \to \mathbb{R}$ is defined by

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds, \tag{2.1}$$

where $\Gamma(.)$ is the Euler Gamma function, provided that the integral exists.

Definition 2.2 ([6, 7]). If $h \in C^n[0,1]$, then the Caputo fractional derivative of order α is defined by

$$ {}^cD^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds = I^{n-\alpha} h^{(n)}(t), \quad n-1 < \alpha < n, \tag{2.2}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Remark 2.3. If $\alpha = n \in \mathbb{N}_0$, then the Caputo derivative coincides with a conventional n-th order derivative of the function $h(t)$.

Lemma 2.4 ([6, 7]). *Let $n = [\alpha]+1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$. If $y(t) \in C^n[0,1]$, then*

$$(I^{\alpha c}D^\alpha y)(t) = y(t) - \sum_{i=0}^{n-1} \frac{y^i(0)}{i!} t^i.$$

Lemma 2.5 ([6, 7]). *Let $\alpha > 0$ and $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$. If $h(t) \in C[0,1]$, then the homogeneous fractional differential equation*

$$ {}^cD^\alpha h(t) = 0$$

has a solution

$$h(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1},$$

where $c_i \in \mathbb{R}$, $(i = 1, 2, \ldots, n)$.

Lemma 2.6 ([3]). *If $1 < \alpha \leq 2$ and $f \in \mathcal{C}([0,1] \times [0,+\infty) \times [0,+\infty), \mathbb{R}^+)$, then the boundary value problem for fractional functional differential equation*

$$\begin{cases} {}^cD^\alpha u(t) + f(t, u(t-\tau), u(t+\theta)) = 0, & t \in (0,1), \\ au(t) - bu'(t) = \eta(t), & t \in [-\tau, 0], \\ cu(t) + du'(t) = \xi(t), & t \in [1, 1+\theta] \end{cases} \tag{2.3}$$

is equivalent to the integral equation

$$u(t) = \begin{cases} u(-\tau, t), & t \in [-\tau, 0], \\ \int_0^1 G(t,s) f(s, u(s-\tau), u(s+\theta)) ds, & t \in [0,1], \\ u(\theta, t), & t \in [1, 1+\theta]. \end{cases} \tag{2.4}$$

Here

$$u(-\tau, t) = e^{(a/b)t}\Big(\frac{1}{b}\int_t^0 e^{-(a/b)s}\eta(s)ds + u(0)\Big), \quad t \in [-\tau, 0], \tag{2.5}$$

$$u(\theta, t) = \begin{cases} e^{-(c/d)t}(\frac{1}{d}\int_1^t e^{(c/d)s}\xi(s)ds + e^{c/d}u(1)), & t \in [1, 1+\theta], \quad d \neq 0, \\ \frac{\xi(t)}{c}, & t \in [1, 1+\theta], \quad d = 0, \end{cases} \tag{2.6}$$

and

(2.7)

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} -(t-s)^{(\alpha-1)} + \frac{b+at}{\rho}(c(1-s)^{(\alpha-1)} + d(\alpha-1)(1-s)^{(\alpha-2)}), & s \le t, \\ \frac{b+at}{\rho}(c(1-s)^{(\alpha-1)} + d(\alpha-1)(1-s)^{(\alpha-2)}), & t \le s, \end{cases}$$

where $\rho = bc + ac + ad$.

Lemma 2.7. *If* $1 < \alpha \le 2$, $0 < \beta \le 1$ *and* $f \in \mathcal{C}([0,1] \times [0,+\infty) \times [0,+\infty), \mathbb{R}^+)$, *then the boundary value problem for fractional functional differential equation*

$$\begin{cases} {}^cD^\beta[\phi_p({}^cD^\alpha u(t))] + f(t, u(t-\tau), u(t+\theta)) = 0, \quad t \in (0,1), \\ {}^cD^\alpha u(0) = 0, \\ au(t) - bu'(t) = \eta(t), \quad t \in [-\tau, 0], \\ cu(t) + du'(t) = \xi(t), \quad t \in [1, 1+\theta] \end{cases} \tag{2.8}$$

is equivalent to the integral equation

$$u(t) = \begin{cases} u(-\tau, t), & t \in [-\tau, 0], \\ \int_0^1 G(t,s)\phi_q(I^\beta f(s, u(s-\tau), u(s+\theta)))ds, & t \in [0,1], \\ u(\theta, t), & t \in [1, 1+\theta], \end{cases} \tag{2.9}$$

where $u(-\tau, t)$, $u(\theta, t)$ *and* $G(t,s)$ *are defined by* (2.5)–(2.7) *respectively.*

Proof. For any $f \in \mathcal{C}([0,1] \times [0,+\infty) \times [0,+\infty), \mathbb{R}^+)$, by Lemma 2.4, we have

$$\phi_p({}^cD^\alpha u(t)) = -I^\beta f(t, u(t-\tau), u(t+\theta)) + c_0, \quad c_0 \in \mathbb{R}.$$

Using the boundary condition ${}^cD^\alpha u(0) = 0$, we get $c_0 = 0$. Hence, we obtain

$$ {}^cD^\alpha u(t) + \phi_q(I^\beta f(t, u(t-\tau), u(t+\theta))) = 0. \tag{2.10}$$

By means of Lemma 2.6, a solution of (2.10) with the boundary conditions of (2.8) can be expressed as

$$u(t) = \begin{cases} u(-\tau, t), & t \in [-\tau, 0], \\ \int_0^1 G(t,s)\phi_q(I^\beta f(s, u(s-\tau), u(s+\theta)))ds, & t \in [0,1], \\ u(\theta, t), & t \in [1, 1+\theta], \end{cases} \tag{2.11}$$

in which $u(-\tau, t)$, $u(\theta, t)$ and $G(t,s)$ are given by (2.5)-(2.7) respectively. □

Now, we will present the properties of the Green's function:

Lemma 2.8 ([5]). *The function $G(t,s)$ given by (2.7) verifies the following properties:*

(i) *$G(t,s)$ is continuous on $[0,1] \times [0,1)$.*
(ii) *For $b > \frac{2-\alpha}{\alpha-1}a$, we get $G(t,s) > 0$ for $t, s \in (0,1)$.*
(iii) *$G(t,s) \le G(s,s)$ for $t, s \in (0,1)$.*

Throughout this paper, let $x_0(t)$ be a solution of the BVP (1.1) with $f \equiv 0$, then it satisfies

$$x_0(t) = \begin{cases} x_0(-\tau, t), & t \in [-\tau, 0], \\ 0, & t \in [0,1], \\ x_0(\theta, t), & t \in [1, 1+\theta], \end{cases} \tag{2.12}$$

where

$$x_0(-\tau, t) = \frac{e^{(a/b)t}}{b} \int_t^0 e^{-(a/b)s}\eta(s)ds, \quad t \in [-\tau, 0]$$

and

$$x_0(\theta, t) = \begin{cases} e^{-(c/d)t}\frac{1}{d}\int_1^t e^{(c/d)s}\xi(s)ds, & t \in [1, 1+\theta], \quad d \neq 0, \\ \frac{\xi(t)}{c}, & t \in [1, 1+\theta], \quad d = 0. \end{cases}$$

Assume that $u(t)$ is a solution of the BVP (1.1) and $x(t) = u(t) - x_0(t)$. Since $x(t) \equiv u(t)$ for $0 \le t \le 1$, $x(t)$ verifies

$$x(t) = \begin{cases} x(-\tau, t), \quad t \in [-\tau, 0], \\ \int_0^1 G(t,s)\phi_q(I^\beta f(s, (x+x_0)(s-\tau), (x+x_0)(s+\theta)))ds, \quad t \in [0,1], \\ x(\theta, t), \quad t \in [1, 1+\theta], \end{cases} \tag{2.13}$$

where

$$(x+x_0)(s-\tau) = x(s-\tau) + x_0(s-\tau),$$
$$(x+x_0)(s+\theta) = x(s+\theta) + x_0(s+\theta),$$

$$x(-\tau, t) = e^{(a/b)t}x(0), \quad t \in [-\tau, 0]$$

and

$$x(\theta,t)=\begin{cases} e^{-(c/d)(t-1)}x(1), & t\in[1,1+\theta], \quad d\neq 0,\\ 0, & t\in[1,1+\theta], \quad d=0.\end{cases}$$

Let $B=\mathcal{C}[-\tau,1+\theta]$ be endowed with the norm $\|x\|=\max_{t\in[-\tau,1+\theta]}|x(t)|$, then it is clear that B is a Banach space. Define a cone $K\subset B$ as follows:

$$K=\{x\in B : x(t)\geq 0 \ \text{ for any } t\in[-\tau,1+\theta]\}.$$

Consider the operator $T:K\to K$

(2.14)

$$Tx(t)=\begin{cases} e^{at/b}\int_0^1 G(0,s)\phi_q(I^\beta f(s,(x+x_0)(s-\tau),(x+x_0)(s+\theta)))ds, \quad t\in[-\tau,0],\\ \int_0^1 G(t,s)\phi_q(I^\beta f(s,(x+x_0)(s-\tau),(x+x_0)(s+\theta)))ds, \quad t\in[0,1],\\ Ax(t), \quad t\in[1,1+\theta],\end{cases}$$

in which

$$Ax(t)=\begin{cases} e^{-(c/d)(t-1)}\int_0^1 G(1,s)\phi_q(I^\beta f(s,(x+x_0)(s-\tau),(x+x_0)(s+\theta)))ds,\\ \qquad\qquad t\in[1,1+\theta], \quad d\neq 0;\\ 0, \qquad\qquad t\in[1,1+\theta], \quad d=0.\end{cases}$$

It is easy to see that u is a positive solution of the BVP (1.1) if and only if $x=u-x_0$ is a nontrivial fixed point of T, where x_0 is given by (2.12).

Lemma 2.9. *Assume that (A1) and (A2) hold. Then $T:K\to K$ is completely continuous.*

Proof. From the definition of T, it is obvious that $Tx(t)\geq 0$ for $t\in[-\tau,1+\theta]$, i.e., $Tx\in K$, $\forall x\in K$. Also, using the Arzela Ascoli theorem and the standard arguments, one can easily show that $T:K\to K$ is completely continuous operator. □

Remark 2.10. Note that for any $t\in[-\tau,0]$ and $t\in[1,1+\theta]$, $Tx(t)\leq Tx(0)$ and $Tx(t)\leq Tx(1)$ hold respectively. So,

$$\|Tx\|=\|Tx\|_{[0,1]}=\max_{t\in[0,1]}|Tx(t)|.$$

To guarantee the existence of positive solutions, we will assume the following condition:

- **(H1)** There exists $\delta>0$ such that $0\leq u_1\leq u_2\leq \delta+\|x_0\|_{[-\tau,0]}$, $0\leq v_1\leq v_2\leq \delta+\|x_0\|_{[1,1+\theta]}$ and $t\in[0,1]$ imply $f(t,u_1,v_1)\leq f(t,u_2,v_2)$.

Lemma 2.11. *Suppose that (A1), (A2) and (H1) hold. Then for any $x_1,x_2\in\overline{K}_\delta$ with $x_1(t)\leq x_2(t)$, $t\in[-\tau,1+\theta]$ implies $(Tx_1)(t)\leq(Tx_2)(t)$.*

Proof. Let $x_1,\ x_2 \in \overline{K_\delta}$. Then, for any $v \in [0,1]$, we have

(2.15)
$$\begin{aligned}
&0 \le (x_1+x_0)(v-\tau) \le (x_2+x_0)(v-\tau) \le \|x_2\| + \|x_0\|_{[-\tau,0]} \le \delta + \|x_0\|_{[-\tau,0]},\\
&0 \le (x_1+x_0)(v+\theta) \le (x_2+x_0)(v+\theta) \le \|x_2\| + \|x_0\|_{[1,1+\theta]} \le \delta + \|x_0\|_{[1,1+\theta]}.
\end{aligned}$$

It follows from (2.15) and (H1) that

$$f(v,(x_1+x_0)(v-\tau),(x_1+x_0)(v+\theta)) \le f(v,(x_2+x_0)(v-\tau),(x_2+x_0)(v+\theta)),\ \ v \in [0,1],$$

thus we have

$$I^\beta f(v,(x_1+x_0)(v-\tau),(x_1+x_0)(v+\theta)) \le I^\beta f(v,(x_2+x_0)(v-\tau),(x_2+x_0)(v+\theta)).$$

Since ϕ_q is increasing on $\mathbb{R}$, we derive that

$$\phi_q(I^\beta f(v,(x_1+x_0)(v-\tau),(x_1+x_0)(v+\theta))) \le \phi_q(I^\beta f(v,(x_2+x_0)(v-\tau),(x_2+x_0)(v+\theta))),$$

so, we obtain

(2.16)
$$\begin{aligned}
&(Ax_1)(t)-(Ax_2)(t)\\
&= \begin{cases} e^{-\frac{c(t-1)}{d}} \int_0^1 G(1,s)\Big(\phi_q(I^\beta f(s,(x_1+x_0)(s-\tau),(x_1+x_0)(s+\theta)))\\ -\phi_q(I^\beta f(s,(x_2+x_0)(s-\tau),(x_2+x_0)(s+\theta)))\Big)ds, \quad t \in [1,1+\theta], \quad d \neq 0,\\ 0, \quad t \in [1,1+\theta], \quad d = 0. \end{cases}
\end{aligned}$$

Hence for any $t \in [-\tau, 1+\theta]$, by (2.14) and (2.16) we have

$$\begin{aligned}
&(Tx_1)(t)-(Tx_2)(t)\\
&= \begin{cases} e^{\frac{at}{b}} \displaystyle\int_0^1 G(0,s)\Big(\phi_q(I^\beta f(s,(x_1+x_0)(s-\tau),(x_1+x_0)(s+\theta)))\\ -\phi_q(I^\beta f(s,(x_2+x_0)(s-\tau),(x_2+x_0)(s+\theta)))\Big)ds, \quad t \in [-\tau,0],\\ \int_0^1 G(t,s)\Big(\phi_q(I^\beta f(s,(x_1+x_0)(s-\tau),(x_1+x_0)(s+\theta)))\\ -\phi_q(I^\beta f(s,(x_2+x_0)(s-\tau),(x_2+x_0)(s+\theta)))\Big)ds, \quad t \in [0,1],\\ (Ax_1)(t)-(Ax_2)(t), \quad t \in [1,1+\theta] \end{cases}\\
&\le 0.
\end{aligned}$$

Therefore, $(Tx_1)(t) \le (Tx_2)(t)$ is satisfied for $t \in [-\tau, 1+\theta]$. The proof is completed. □

3. MAIN RESULT

In this section, we obtain the existence of positive solutions and its monotone iterative scheme for the fractional BVP (1.1).

For convenience, let us denote

$$A=\frac{a+b}{\rho[\Gamma(\beta+1)]^{q-1}\Gamma(\alpha)}\int_0^1 (c(1-s)^{\alpha-1}+d(\alpha-1)(1-s)^{\alpha-2})s^{\beta(q-1)}ds.$$

Theorem 3.1. *Assume that* $(A1)$, $(A2)$ *and* $(H1)$ *hold. Suppose also that there exists* $\delta>0$ *such that*

$$\max_{t\in[0,1]} f(t,\delta+\|x_0\|_{[-\tau,0]},\delta+\|x_0\|_{[1,1+\theta]})\le \phi_p(\frac{\delta}{A}).$$

Then the BVP (1.1) *has two positive solutions* $w^*(t)+x_0(t)$ *and* $v^*(t)+x_0(t)$ *satisfying*

$$0\le w^*\le\delta,\quad \lim_{n\to\infty} w_n=\lim_{n\to\infty}T^n w_0=w^*,$$

where

$$w_0(t)=\begin{cases}\dfrac{\delta e^{at/b}}{A[\Gamma(\beta+1)]^{q-1}}\displaystyle\int_0^1 G(0,s)s^{\beta(q-1)}ds, & t\in[-\tau,0],\\ \dfrac{\delta}{A[\Gamma(\beta+1)]^{q-1}}\displaystyle\int_0^1 G(t,s)s^{\beta(q-1)}ds, & t\in[0,1],\\ \dfrac{\delta e^{-(c/d)(t-1)}}{A[\Gamma(\beta+1)]^{q-1}}\displaystyle\int_0^1 G(1,s)s^{\beta(q-1)}ds, & t\in[1,1+\theta],\quad d\neq 0,\\ 0, & t\in[1,1+\theta],\quad d=0,\end{cases}\tag{3.1}$$

and

$$0\le v^*\le\delta,\quad \lim_{n\to\infty} v_n=\lim_{n\to\infty}T^n v_0=v^*,$$

where $v_0(t)=0$, $-\tau\le t\le 1+\theta$.

Proof. Let $x\in\overline{K_\delta}$. Then for any $t\in[0,1]$ we have

$$\begin{aligned}&0\le(x+x_0)(t-\tau)\le\|x\|+\|x_0\|_{[-\tau,0]}\le\delta+\|x_0\|_{[-\tau,0]},\\ &0\le(x+x_0)(t+\theta)\le\|x\|+\|x_0\|_{[1,1+\theta]}\le\delta+\|x_0\|_{[1,1+\theta]}.\end{aligned}\tag{3.2}$$

From (3.2), it follows that

(3.3)

$$0<f(t,(x+x_0)(t-\tau),(x+x_0)(t+\theta))\le f(t,\delta+\|x_0\|_{[-\tau,0]},\delta+\|x_0\|_{[1,1+\theta]})\le\phi_p(\frac{\delta}{A}),$$

thus we have

$$\|Tx\| = \max_{t\in[-\tau,1+\theta]} |Tx(t)|$$

$$= \max_{t\in[0,1]} |Tx(t)|$$

$$= \max_{t\in[0,1]} \left| \int_0^1 G(t,s)\phi_q(\frac{1}{\Gamma(\beta)} \int_0^s (s-z)^{\beta-1} f(z,(x+x_0)(z-\tau),(x+x_0)(z+\theta))dz)ds \right|$$

$$\le \int_0^1 G(s,s)\phi_q(\frac{1}{\Gamma(\beta)} \int_0^s (s-z)^{\beta-1} f(z,(x+x_0)(z-\tau),(x+x_0)(z+\theta))dz)ds$$

$$\le \frac{\delta}{A[\Gamma(\beta+1)]^{q-1}} \int_0^1 G(s,s)s^{\beta(q-1)}ds \le \delta.$$

Hence, we get $T\overline{K_\delta} \subset \overline{K_\delta}$. Let

$$w_0(t) = \begin{cases} \dfrac{\delta e^{at/b}}{A[\Gamma(\beta+1)]^{q-1}} \displaystyle\int_0^1 G(0,s)s^{\beta(q-1)}ds, & t \in [-\tau,0], \\ \dfrac{\delta}{A[\Gamma(\beta+1)]^{q-1}} \displaystyle\int_0^1 G(t,s)s^{\beta(q-1)}ds, & t \in [0,1], \\ \dfrac{\delta e^{-(c/d)(t-1)}}{A[\Gamma(\beta+1)]^{q-1}} \displaystyle\int_0^1 G(1,s)s^{\beta(q-1)}ds, & t \in [1,1+\theta], \quad d \neq 0, \\ 0, & t \in [1,1+\theta], \quad d = 0, \end{cases} \tag{3.4}$$

then $\|w_0\| \le \delta$ and $w_0(t) \in \overline{K_\delta}$. Let $w_1 = Tw_0$, then $w_1 \in \overline{K_\delta}$. We denote

$$w_{n+1} = Tw_n = T^{n+1}w_0 \quad (n = 0,1,2,\dots). \tag{3.5}$$

Since $T\overline{K_\delta} \subset \overline{K_\delta}$, we get $w_n \in \overline{K_\delta}$ $(n = 0,1,2,\dots)$. By Lemma 2.9, T is compact, we assert that $\{w_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{w_{n_k}\}_{k=1}^{\infty}$ and there exists $w^* \in \overline{K_\delta}$ such that $w_{n_k} \longrightarrow w^*$. From the definition of T, (3.4) and (3.5), we have

$$w_1(t) = (Tw_0)(t)$$

$$= \begin{cases} e^{at/b} \int_0^1 G(0,s)\phi_q(I^\beta f(s,(w_0+x_0)(s-\tau),(w_0+x_0)(s+\theta)))ds, \\ \qquad t \in [-\tau,0]; \\ \int_0^1 G(t,s)\phi_q(I^\beta f(s,(w_0+x_0)(s-\tau),(w_0+x_0)(s+\theta)))ds, \\ \qquad t \in [0,1]; \\ e^{-(c/d)(t-1)} \int_0^1 G(1,s)\phi_q(I^\beta f(s,(w_0+x_0)(s-\tau),(w_0+x_0)(s+\theta)))ds, \\ \qquad t \in [1,1+\theta], \quad d \neq 0; \\ 0, \qquad t \in [1,1+\theta], \quad d = 0 \end{cases}$$

$$\begin{aligned} &\le \begin{cases} \dfrac{\delta e^{at/b}}{A[\Gamma(\beta+1)]^{q-1}} \displaystyle\int_0^1 G(0,s)s^{\beta(q-1)}ds, & t \in [-\tau, 0], \\ \dfrac{\delta}{A[\Gamma(\beta+1)]^{q-1}} \displaystyle\int_0^1 G(t,s)s^{\beta(q-1)}ds, & t \in [0, 1], \\ \dfrac{\delta e^{-c(t-1)/d}}{A[\Gamma(\beta+1)]^{q-1}} \displaystyle\int_0^1 G(1,s)s^{\beta(q-1)}ds, & t \in [1, 1+\theta], \quad d \neq 0, \\ 0, & t \in [1, 1+\theta], \quad d = 0 \end{cases} \\ &= w_0(t), \quad t \in [-\tau, 1+\theta]. \end{aligned}$$

Hence, $w_1(t) \le w_0(t)$. By means of Lemma 2.11, we obtain $Tw_1(t) \le Tw_0(t)$, i.e., $w_2(t) \le w_1(t), \quad t \in [-\tau, 1+\theta]$. Thus, we have

$$w_{n+1}(t) \le w_n(t), \; t \in [-\tau, 1+\theta], \qquad (n = 0, 1, 2, \dots).$$

Therefore, $w_n \longrightarrow w^*$. Let $n \longrightarrow \infty$ in (3.5). Then we get $Tw^* = w^*$ since T is continuous. Evidently, w^* is a fixed point of the operator T, that is $y_1(t) = w^*(t) + x_0(t)$ is a positive solution of the BVP (1.1).

Let $v_0(t) = 0$, $t \in [-\tau, 1+\theta]$, then $v_0(t) \in \overline{K_\delta}$. Let $v_1 = Tv_0$, then $v_1 \in \overline{K_\delta}$, we denote

$$v_{n+1} = Tv_n = T^{n+1}v_0 \; (n = 0, 1, 2, \dots).$$

Similar to $\{w_n\}_{n=1}^{\infty}$, we claim that $\{v_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ and there exists $v^* \in \overline{K_\delta}$ such that $v_{n_k} \longrightarrow v^*$, which means $v_{n_k}(t) \longrightarrow v^*(t)$, $k \to \infty$, $t \in [-\tau, 1+\theta]$. Since $v_1 = Tv_0 = T0 \in \overline{K_\delta}$, we have

$$v_1(t) = Tv_0(t) = (T0)(t) \ge 0,$$

that is

$$v_2(t) = (Tv_1)(t) \ge (T0)(t) = v_1(t), \; t \in [-\tau, 1+\theta].$$

By induction, it is obvious that

$$v_{n+1}(t) \ge v_n(t), \; t \in [-\tau, 1+\theta] \qquad (n = 0, 1, 2, \dots),$$

so, we have $v_n \longrightarrow v^*$ in norm $\| \,.\, \|$ and $Tv^* = v^*$. Therefore, T has fixed points w^* and v^*, which means that $y_1(t) = w^*(t) + x_0(t)$ and $y_2(t) = v^*(t) + x_0(t)$ are positive solutions of the fractional BVP (1.1). The proof is completed. □

Remark 3.2. It is obvious that $w^* + x_0$ and $v^* + x_0$ are the maximal and minimal solutions of the BVP (1.1). If they coincide then (1.1) has a unique positive solution in K_δ.

Corollary 3.3. *Assume that (A1), (A2) and (H1) hold. Suppose also that there exist* $0 < \delta_1 < \delta_2 < \cdots < \delta_n$ *such that*

$$\max_{t\in[0,1]} f(t, \delta_k + \|x_0\|_{[-\tau,0]}, \delta_k + \|x_0\|_{[1,1+\theta]}) \le \phi_p\left(\frac{\delta_k}{A}\right).$$

Then the BVP (1.1) has $2n$ *positive solutions* $w_k^*(t)+x_0(t)$ *and* $v_k^*(t)+x_0(t)$ *satisfying*

$$0 \le w_k^* \le \delta_k, \quad \lim_{n\to\infty} w_{k_n} = \lim_{n\to\infty} T^n w_{k_0} = w_k^*,$$

where

$$w_{k_0}(t) = \begin{cases} \dfrac{\delta_k e^{at/b}}{A[\Gamma(\beta+1)]^{q-1}} \displaystyle\int_0^1 G(0,s)s^{\beta(q-1)}ds, & t\in[-\tau,0], \\ \dfrac{\delta_k}{A[\Gamma(\beta+1)]^{q-1}} \displaystyle\int_0^1 G(t,s)s^{\beta(q-1)}ds, & t\in[0,1], \\ \dfrac{\delta_k e^{-c(t-1)/d}}{A[\Gamma(\beta+1)]^{q-1}} \displaystyle\int_0^1 G(1,s)s^{\beta(q-1)}ds, & t\in[1,1+\theta], \quad d\neq 0, \\ 0, & t\in[1,1+\theta], \quad d=0, \end{cases} \tag{3.6}$$

and

$$0 \le v_k^* \le \delta_k, \quad \lim_{n\to\infty} v_{k_n} = \lim_{n\to\infty} T^n v_{k_0} = v_k^*,$$

where $v_{k_0}(t) = 0, -\tau \le t \le 1+\theta$.

4. AN EXAMPLE

Consider the following fractional functional boundary-value problem:

$$\begin{cases} D^{1/2}(\phi_2(D^{3/2}u(t))) + f(t, u(t-\frac{1}{6}), u(t+\frac{1}{7})) = 0, \quad t\in(0,1), \\ D^{3/2}u(0) = 0, \\ u'(t) = \sin(\pi t), \ t\in[-\frac{1}{6},0], \\ u(t) = e^{1-t}-1, \ t\in[1,\frac{8}{7}], \end{cases} \tag{4.1}$$

where

$$f(t,u,v) = t+1+\frac{1}{40}(u+v), \quad (t,u,v)\in\left[-\frac{1}{6},\frac{8}{7}\right]\times[0,\infty)\times[0,\infty),$$

and $a=d=0$, $b=c=1$, $p=2$, $q=2$, $\alpha=\frac{3}{2}$, $\beta=\frac{1}{2}$, $\tau=\frac{1}{6}$, $\theta=\frac{1}{7}$. Notice that $\eta(t) = -\sin(\pi t)$, and $\xi(t) = e^{1-t}-1$ are nonnegative functions satisfying $\eta(0) = \xi(1) = 0$. By easy calculation, we evaluate $x_0(t) = \frac{2}{\pi}\sin^2(\frac{\pi}{2}t)$, for $t\in[-\frac{1}{6},0]$ and $x_0(t) = e^{1-t}-1$ for $t\in[1,\frac{8}{7}]$, so $\|x_0\|_{[-\frac{1}{6},0]} = \frac{2}{\pi}\sin^2(\frac{\pi}{12})$, $\|x_0\|_{[1,\frac{8}{7}]} = 0$. Choosing $\delta = 20$, we get $A = \frac{1}{2}$. Moreover, it is obvious that $f(t,u,v)$ satisfies

(1) $f(t,u_1,v_1) \le f(t,u_2,v_2)$ for any $0\le t\le 1, 0\le u_1\le u_2\le 20+\frac{2}{\pi}\sin^2(\frac{\pi}{12})$, $0\le v_1\le v_2\le 20$;
(2) $\max_{0\le t\le 1} f(t,\delta+\|x_0\|_{[-\frac{1}{6},0]}, \delta+\|x_0\|_{[1,\frac{8}{7}]}) = f(1, 20+\frac{2}{\pi}\sin^2(\frac{\pi}{12}), 20) \le \phi_2(\frac{\delta}{A}) \equiv 40.$

Thus, by means of Theorem 3.1, the BVP (4.1) has two positive solutions $w^* + x_0$ and $v^* + x_0$. For $n = 0, 1, 2, \ldots$, the two iterative schemes are as follows:

$$w_0(t) = \begin{cases} \delta, & -\frac{1}{6} \leq t \leq 0, \\ (1-t^2)\delta, & 0 \leq t \leq 1, \\ 0, & 1 \leq t \leq \frac{8}{7}, \end{cases}$$

$$\begin{aligned} &w_{n+1}(t) \\ &= \begin{cases} \frac{2}{\pi}\int_0^1 (1-s)^{1/2}[\int_0^s (s-\tau)^{-1/2}(\tau + 1 + \frac{1}{40}[(w_n + x_0)(\tau - \frac{1}{6}) + (w_n + x_0)(\tau + \frac{1}{7})])d\tau]ds, \\ \qquad t \in [-\frac{1}{6}, 0]; \\ \frac{1}{\sqrt{\pi}}\int_0^1 G(t,s)[\int_0^s (s-\tau)^{-1/2}(\tau + 1 + \frac{1}{40}[(w_n + x_0)(\tau - \frac{1}{6}) + (w_n + x_0)(\tau + \frac{1}{7})])d\tau]ds, \\ \qquad t \in [0,1]; \\ 0, \qquad t \in [1, \frac{8}{7}], \end{cases} \end{aligned}$$

$$v_0(t) = 0,$$

$$\begin{aligned} &v_{n+1}(t) \\ &= \begin{cases} \frac{2}{\pi}\int_0^1 (1-s)^{1/2}[\int_0^s (s-\tau)^{-1/2}(\tau + 1 + \frac{1}{40}[(v_n + x_0)(\tau - \frac{1}{6}) + (v_n + x_0)(\tau + \frac{1}{7})])d\tau]ds, \\ \qquad t \in [-\frac{1}{6}, 0]; \\ \frac{1}{\sqrt{\pi}}\int_0^1 G(t,s)[\int_0^s (s-\tau)^{-1/2}(\tau + 1 + \frac{1}{40}[(v_n + x_0)(\tau - \frac{1}{6}) + (v_n + x_0)(\tau + \frac{1}{7})])d\tau]ds, \\ \qquad t \in [0,1]; \\ 0, \qquad t \in [1, \frac{8}{7}], \end{cases} \end{aligned}$$

in which

$$G(t,s) = \frac{2}{\sqrt{\pi}} \begin{cases} -\sqrt{t-s} + \sqrt{1-s}, & s \leq t, \\ \sqrt{1-s}, & t \leq s. \end{cases}$$

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