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A GENERALIZED HALANAY-TYPE INEQUALITY ON TIME SCALES

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ABSTRACT. In this paper, we obtain a Halanay-type inequality on time scales. By means of the obtained inequality, we get a new exponential stability condition for linear delay dynamic equations on time scales. An example is given to illustrate the results.

Keywords: Time scale; Delay dynamic equation; Inequality; Exponential stability

1. Introduction and preliminaries

The stability analysis of dynamical systems using differential and difference inequalities has attracted a great deal of attention in the existing literature (see [1–12] and the references therein). For stability analysis of the delay differential equation

$$x'(t) = -px(t) + qx(t-\tau), \quad \tau > 0,$$

in [3], Halanay proved the following result.

Lemma 1 (see [4]).

$$f'(t) \le -\alpha f(t) + \beta \sup_{s\in[t-\tau,t]} f(s), \qquad \text{for } t \ge t_0,$$

and $\alpha > \beta > 0$*, then there exist* $\gamma > 0$ *and* $K > 0$ *such that*

$$f(t) \le Ke^{-\gamma(t-t_0)}, \qquad \text{for } t \ge t_0.$$

In 2000, Mohamad and Gopalsamy gave the next lemmas.

Lemma 2 (see Theorem 2.1 of [1]). *Let* $x(\cdot)$ *be a nonnegative function satisfying*

$$\frac{dx(t)}{dt} \le -a(t)x(t) + b(t)\left(\sup_{s\in[t-\tau(t),t]} x(s)\right), \quad t > t_0, \tag{1.1}$$

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$$x(s) = |\varphi(s)| \quad \text{for } s \in [t_0 - \tau^*, t_0]; \tag{1.2}$$

where $\tau(t)$ denotes a nonnegative, continuous and bounded function defined for $t \in \mathbb{R}$ and $\tau* = \sup_{t\in\mathbb{R}} \tau(t)$; $\varphi(s)$ is continuous and defined for $s \in [t_0 - \tau^*, t_0]$; $a(t)$ and $b(t)$, $t \in \mathbb{R}$, denote nonnegative, continuous bounded functions. Suppose

$$a(t) - b(t) \geq \sigma, \quad t \in \mathbb{R}, \tag{1.3}$$

where $\sigma = \inf_{t\in\mathbb{R}}(a(t) - b(t)) > 0$. Then there exists a positive number $\widetilde{\mu}$ such that

$$x(t) \leq \left(\sup_{t\in[t_0-\tau^*,t_0]} x(s) \right) e^{-\widetilde{\mu}(t-t_0)}, \quad t > t_0. \tag{1.4}$$

Lemma 3 (see Theorem 2.2 of [1])**.** *Let $x(\cdot)$ be a nonnegative function satisfying*

$$\frac{dx(t)}{dt} \leq -a(t)x(t) + b(t) \int_0^\infty K(s)x(t-s)ds, \quad t > t_0, \tag{1.5}$$

$$x(s) = |\varphi(s)| \quad \text{for } s \in (-\infty, t_0], \tag{1.6}$$

where $\varphi(s)$ defined for $s \in (-\infty, t_0]$ is continuous and $\sup_{s\in(-\infty,t_0]} x(s) = M > 0$, $a(t)$ and $b(t)$ are defined for $t \in \mathbb{R}$ and denote nonnegative, continuous bounded functions; the delay kernel $K(\cdot)$ is assumed to satisfy the following properties

$$K : [0, \infty) \mapsto [0, \infty) \quad \text{and} \quad \int_0^\infty K(s)e^{\mu s}ds < \infty, \tag{1.7}$$

for some positive number μ. Suppose further that

$$a(t) - b(t) \int_0^\infty K(s)ds \geq \sigma, \quad t \in \mathbb{R}, \tag{1.8}$$

where $\sigma = \inf_{t\in\mathbb{R}}(a(t) - b(t) \int_0^\infty K(s)ds) > 0$. Then there exists a positive number $\widetilde{\mu}$ such that

$$x(t) \leq \left(\sup_{s\in(-\infty,t_0]} x(s) \right) e^{-\widetilde{\mu}(t-t_0)}, \quad t > t_0. \tag{1.9}$$

In this paper, we extend Lemma 2 and Lemma 3 to time scales. As an application, we obtain a new exponential stability condition for linear delay dynamic equations on time scales. We remark also that in the paper [18], the authors extended the Halanay-type inequality to higher dimensional systems on time scales.

For completeness, (see [15] for elementary results for the time scale calculus), we recall some basic results for dynamic equations and the calculus on time scales.

Definition 1. A function $h : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1+\mu(t)h(t) \neq 0$ for all $t \in \mathbb{T}^k$, where $\mu(t) = \sigma(t) - t$. The set of all regressive rd-continuous functions $\varphi : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathfrak{R}$ while the set $\mathfrak{R}^+$ is given by $\mathfrak{R}^+ = \{\varphi \in \mathfrak{R} : 1+\mu(t)\varphi(t) > 0$

for all $t \in \mathbb{T}\}$. Let $\varphi \in \mathfrak{R}$. The exponential function on $\mathbb{T}$ is defined by $e_\varphi(t,s) = \exp\left(\int_s^t \xi_{\mu(r)}(\varphi(r))\Delta r\right)$. Here $\xi_{\mu(s)}$ is the cylinder transformation given by

$$\xi_{\mu(r)}(\varphi(r)) := \begin{cases} \frac{1}{\mu(r)}\mathrm{Log}\,(1+\mu(r)\varphi(r))\,, & \mu(r) > 0, \\ \varphi(r), & \mu(r) = 0. \end{cases}$$

It is well known that (see [15, Theorem 2.48]) if $p \in \mathfrak{R}^+$, then $e_p(t,s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t,s)$ is the unique solution to the initial value problem $y^\Delta = p(t)y,\ y(s) = 1$. Other properties of the exponential function are given in the following lemma.

Lemma 4 (see [2])**.** *Let $p, q \in \mathfrak{R}$. Then*

(i) $e_0(s,t) \equiv 1$ *and* $e_p(t,t) \equiv 1$,
(ii) $e_p(\sigma(t), s) = (1+\mu(t)p(t))\, e_p(t,s)$,
(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$ *where* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$,
(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$,
(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$,
(vi) $\left(\frac{1}{e_p(\cdot,s)}\right)^\Delta = -\frac{p(t)}{e_p^\sigma(\cdot,s)}$.

Lemma 5 (see [2])**.** *For a nonnegative φ with $-\varphi \in \mathfrak{R}^+$, we have the inequalities*

$$1 - \int_s^t \varphi(u) \le e_{-\varphi}(t,s) \le \exp\{-\int_s^t \varphi(u)\} \quad \textit{for all } t \ge s. \tag{1.10}$$

If φ is rd-continuous and nonnegative, then

$$1 + \int_s^t \varphi(u) \le e_{\varphi}(t,s) \le \exp\{\int_s^t \varphi(u)\} \quad \text{for all } t \ge s. \tag{1.11}$$

Remark 1. If $p \in \mathfrak{R}^+$ and $p(r) > 0$ for all $r \in [s,t]_\mathbb{T}$, then

$$e_p(t,r) \le e_p(t,s) \quad \text{and} \quad e_p(a,b) < 1 \text{ for } s \le a < b \le t. \tag{1.12}$$

2. Main Theorem

Theorem 1. *Let $x(\cdot)$ be a nonnegative function satisfying*

$$x^\triangle(t) \le -a(t)x(t) + b(t)\int_0^\infty K(s)x(t-s)\Delta s + c(t) \sup_{s\in[t-\tau(t),t]} x(s), \quad t > t_0, \tag{2.1}$$

$$x(s) = |\varphi(s)| \quad \textit{for } s \in (-\infty, t_0]_\mathbb{T}, \tag{2.2}$$

where $\varphi(s)$ defined for $s \in (-\infty, t_0]_\mathbb{T}$ is rd-continuous, bounded and $\tau(\cdot)$, $a(\cdot)$, $b(\cdot)$, $c(\cdot)$ are defined in $\mathbb{T}$ and are nonnegative, rd-continuous bounded functions. Denote

$$\sup_{t\in\mathbb{T}} \tau(t) = \tau*, \qquad \sup_{s\in(-\infty,t_0]_\mathbb{T}} x(s) = M > 0.$$

The delay kernel $K(\cdot)$ is assumed to satisfy the following properties.

$$K : [0,\infty)_{\mathbb{T}} \mapsto [0,\infty)_{\mathbb{T}} \text{ and } \forall t \in \mathbb{T}, \quad \int_0^\infty K(s) e_{\lambda_0 p}(t, t-s)\Delta s < \infty, \tag{2.3}$$

for some positive number λ_0, where $p(t)$ is a nonnegative bounded function. Suppose further that

$$\forall t \in \mathbb{T}, \quad a(t) - b(t)\int_0^\infty K(s)\Delta s - c(t) \geq \epsilon = \inf_{t\in\mathbb{T}} \left(a(t) - b(t)\int_0^\infty K(s)\Delta s - c(t) \right) > 0. \tag{2.4}$$

Then there exists a positive number $\overline{\lambda}$ such that

$$x(t) \leq \left(\sup_{s\in(-\infty,t_0]_{\mathbb{T}}} x(s) \right) e_{\ominus\overline{\lambda}p}(t, t_0), \quad t \in (t_0,\infty)_{\mathbb{T}}. \tag{2.5}$$

Proof. By (2.3), we define the binary function $G(t,\lambda)$ by

$$\begin{aligned} G(t,\lambda) &:= -a(t) + \lambda p(t) \\ &\quad + b(t)\int_0^\infty K(s) e_{\lambda p}(t, t-s)\Delta s + c(t) e_{\lambda p}(t, t-\tau^*), \quad t \in \mathbb{T}, \lambda \in [0,\lambda_0]. \end{aligned} \tag{2.6}$$

Since $a(\cdot), b(\cdot), c(\cdot), p(\cdot)$ are nonnegative, rd-continuous, and bounded in $\mathbb{T}$, for fixed $t \in \mathbb{T}$ the binary function $G(t,\lambda)$ is continuous for $\lambda \in [0,\lambda_0]$.

$$\text{By (2.3)}, \exists A(\lambda) > 0, \text{ such that } \sup_{t\in\mathbb{T}} \int_0^\infty K(s) e_{\lambda p}(t, t-s)\Delta s = A(\lambda), \text{ for } \lambda \in [0,\lambda_0]. \tag{2.7}$$

$$\text{By (1.10)}, \exists B(\lambda) > 0, \quad \forall t \in \mathbb{T}, \; e_{\lambda p}(t, t-\tau^*) \leq e^{\lambda p^* \tau^*} \Rightarrow \sup_{t\in\mathbb{T}} e_{\lambda p}(t, t-\tau^*) = B(\lambda), \tag{2.8}$$

where $p^* = \sup_{t\in\mathbb{T}} p(t)$.

From (2.6), (2.7), (2.8) and the boundedness of $a(t), b(t), c(t)$, we can define

$$F(\lambda) = \sup_{t\in\mathbb{T}} G(t,\lambda) \quad \text{for } \lambda \in [0,\lambda_0]. \tag{2.9}$$

Clearly, $F(\lambda)$ is continuous for $\lambda \in [0,\lambda_0]$. Using (2.4), we have

$$\begin{aligned} F(0) = \sup_{t\in\mathbb{T}} G(t,0) &= \sup_{t\in\mathbb{T}} \left(-a(t) + b(t)\int_0^\infty K(s)\Delta s + c(t) \right) \\ &= -\inf_{t\in\mathbb{T}} \left(a(t) - b(t)\int_0^\infty K(s)\Delta s - c(t) \right) \leq -\epsilon < 0. \end{aligned} \tag{2.10}$$

From (2.10) and continuity, there exists $\delta_1 > 0$, such that for $0 \leq \lambda < \delta_1$, we have $F(\lambda) < -\frac{\epsilon}{2}$. In particular, we have

$$F\left(\frac{\delta_1}{2}\right) < -\frac{\epsilon}{2} < 0. \tag{2.11}$$

Set $\overline{\lambda} = \frac{\delta_1}{2}$. From (2.9) and (2.11), it follows that

(2.12)
$$\forall t \in \mathbb{T}, \quad -a(t) + \overline{\lambda} p(t) + b(t) \int_0^\infty K(s) e_{\overline{\lambda}p}(t, t-s) \Delta s + c(t) e_{\overline{\lambda}p}(t, t-\tau^*) \le -\frac{\epsilon}{2} < 0.$$

Now we define

$$\overline{x}(t) = \begin{cases} x(t) e_{\overline{\lambda}p}(t, t_0), & t > t_0, \\ x(t), & t \le t_0. \end{cases} \tag{2.13}$$

Let $\delta > 1$ be arbitrary. We have from (2.2) and (2.13) that $\overline{x}(t) < \delta M$ for $t \in (-\infty, t_0]_{\mathbb{T}}$. We claim

$$\overline{x}(t) < \delta M \quad \text{for } t > t_0. \tag{2.14}$$

Let $t_1 = \sup\{t | \overline{x}(t) < \delta M\} > t_0$. We will show $t_1 = \infty$.

Suppose $t_1 < \infty$. Clearly we have $\overline{x}(t_1) \le \delta M$. Then we have two cases:
Case (1). Suppose $\overline{x}(t_1) = \delta M$ for $t_1 > t_0$

$$\overline{x}(t) < \delta M \quad \text{for all } t < t_1 \text{ and } \overline{x}^\Delta(t_1) \ge 0. \tag{2.15}$$

We have from (2.13), (2.15), (2.1), (2.12), (1.12) and Lemma 4

$$\begin{aligned}
\overline{x}^\Delta(t_1) &= \left(x(t) e_{\overline{\lambda}p}(t, t_0)\right)^\Delta \Big|_{t=t_1} = x^\Delta(t_1) e_{\overline{\lambda}p}(\sigma(t_1), t_0) + x(t_1) \overline{\lambda} p(t_1) e_{\overline{\lambda}p}(t_1, t_0) \\
&\le \left(-a(t_1) x(t_1) + b(t_1) \int_0^\infty K(s) x(t_1 - s) \Delta s + c(t_1) \sup_{s \in [t_1 - \tau(t_1), t_1]} x(s) \right) \\
&\quad \cdot \left(1 + \overline{\lambda} \mu(t_1) p(t_1)\right) e_{\overline{\lambda}p}(t_1, t_0) + x(t_1) \overline{\lambda} p(t_1) e_{\overline{\lambda}p}(t_1, t_0) \\
&= \left(1 + \overline{\lambda} \mu(t_1) p(t_1)\right) \left(-a(t_1) + \overline{\lambda} p(t_1)\right) \delta M \\
&\quad + \left(1 + \overline{\lambda} \mu(t_1) p(t_1)\right) \left(b(t_1) \int_0^\infty K(s) \overline{x}(t_1 - s) e_{\overline{\lambda}p}(t_1, t_1 - s) \Delta s \right. \\
&\quad \left. + c(t_1) \sup_{s \in [t_1 - \tau(t_1), t_1]} \overline{x}(s) e_{\overline{\lambda}p}(t_1, s) \right) - \overline{\lambda}^2 \mu(t_1) p^2(t_1) \delta M \\
&\le \Big(1 + \overline{\lambda} \mu(t_1) p(t_1)\Big) \Big(- a(t_1) + \overline{\lambda} p(t_1) + b(t_1) \int_0^\infty K(s) e_{\overline{\lambda}p}(t_1, t_1 - s) \Delta s \\
&\quad + c(t_1) e_{\overline{\lambda}p}(t_1, t_1 - \tau^*) \Big) \delta M - \overline{\lambda}^2 \mu(t_1) p^2(t_1) \delta M \\
&\le -\frac{\epsilon}{2} \Big(1 + \overline{\lambda} \mu(t_1) p(t_1)\Big) \delta M - \overline{\lambda}^2 \mu(t_1) p^2(t_1) \delta M < 0, \qquad (2.16)
\end{aligned}$$

which contradicts (2.15).
Case (2). Suppose $\overline{x}(t_1) < \delta M$. In this case, t_1 must be right-scattered, for otherwise if t_1 is right-dense, then we have $\overline{x}(t) < \delta M$, for $t \in (-\infty, t_1]_{\mathbb{T}}$. Therefore, there exists $\varepsilon(> 0)$ sufficiently small so that $\overline{x}(t) < \delta M$, for $t \in (\infty, t_1 + \varepsilon]_{\mathbb{T}}$. This contradicts the definition of t_1. Hence, since t_1 is right-scattered, we have

$$\overline{x}(\sigma(t_1)) > \delta M \quad \text{and } \overline{x}(t) < \delta M \text{ for all } t \le t_1 < \sigma(t_1). \tag{2.17}$$

We have from (2.13) and (2.1),

$$
\begin{aligned}
&\frac{\overline{x}(\sigma(t_1))-\overline{x}(t_1)}{\mu(t_1)} = \overline{x}^{\triangle}(t_1) = \Big(x(t)e_{\overline{\lambda}p}(t,t_0)\Big)^{\Delta}\Big|_{t=t_1} \\
&= x^{\Delta}(t_1)e_{\overline{\lambda}p}(\sigma(t_1),t_0) + x(t_1)\overline{\lambda}p(t_1)e_{\overline{\lambda}p}(t_1,t_0) \\
&\le \Big(-a(t_1)x(t_1)+b(t_1)\int_0^{\infty}K(s)x(t_1-s)\Delta s + c(t_1)\sup_{s\in[t_1-\tau(t_1),t_1]}x(s)\Big) \\
&\quad \cdot \Big(1+\overline{\lambda}\mu(t_1)p(t_1)\Big)e_{\overline{\lambda}p}(t_1,t_0) + x(t_1)\overline{\lambda}p(t_1)e_{\overline{\lambda}p}(t_1,t_0) \\
&= \Big(-a(t_1)\overline{x}(t_1)+b(t_1)\int_0^{\infty}K(s)\overline{x}(t_1-s)e_{\overline{\lambda}p}(t_1,t_1-s)\Delta s \\
&\quad + c(t_1)\sup_{s\in[t_1-\tau(t_1),t_1]}\overline{x}(s)e_{\overline{\lambda}p}(t_1,s)\Big) \\
&\quad \cdot \Big(1+\overline{\lambda}\mu(t_1)p(t_1)\Big) + \overline{\lambda}p(t_1)\overline{x}(t_1).
\end{aligned} \tag{2.18}
$$

By (2.18), (2.17), (2.12), (1.12) and $1-a(t)\mu(t)>0,\ t\in\mathbb{T}$, we have

$$
\begin{aligned}
&\delta M < \overline{x}(\sigma(t_1)) \le \Big(1-a(t_1)\mu(t_1)\Big)\Big(1+\overline{\lambda}\mu(t_1)p(t_1)\Big)\overline{x}(t_1) + \mu(t_1)\Big(1+\overline{\lambda}\mu(t_1)p(t_1)\Big) \\
&\cdot \Big[b(t_1)\int_0^{\infty}K(s)\overline{x}(t_1-s)e_{\overline{\lambda}p}(t_1,t_1-s)\Delta s + c(t_1)\sup_{s\in[t_1-\tau(t_1),t_1]}\overline{x}(s)e_{\overline{\lambda}p}(t_1,s)\Big] \\
&< \Big(1-a(t_1)\mu(t_1)\Big)\Big(1+\overline{\lambda}\mu(t_1)p(t_1)\Big)\delta M + \mu(t_1)\Big(1+\overline{\lambda}\mu(t_1)p(t_1)\Big) \\
&\quad \cdot \Big[b(t_1)\int_0^{\infty}K(s)\overline{x}(t_1-s)e_{\overline{\lambda}p}(t_1,t_1-s)\Delta s + c(t_1)\sup_{t_1-\tau^*\le s\le t_1}\overline{x}(s)e_{\overline{\lambda}p}(t_1,t_1-\tau^*)\Big] \\
&\le \Big(1-\overline{\lambda}^2\mu^2(t_1)p^2(t_1)\Big)\delta M - \mu(t_1)\Big(1+\overline{\lambda}\mu(t_1)p(t_1)\Big)\delta M\frac{\epsilon}{2} < \delta M.
\end{aligned} \tag{2.19}
$$

This gives a contradiction.

Hence the claim (2.14) holds. Since $\delta>1$ is arbitrary, by letting $\delta\to 1^+$, we have $\overline{x}(t)\le M$ for $t\in(t_0,\infty)_{\mathbb{T}}$. It then follows from (2.13) and (iii) of Lemma 4 that $x(t)\le Me_{\ominus\overline{\lambda}p}(t,t_0)$ for $t\in(t_0,\infty)_{\mathbb{T}}$, and hence the assertion (2.5) is satisfied. This completes the proof. □

Remark 2. If in (2.18), we use the formula for the delta-derivative, we get

$$
\begin{aligned}
&\frac{\overline{x}(\sigma(t_1))-\overline{x}(t_1)}{\mu(t_1)} = \overline{x}^{\triangle}(t_1) = \Big(x(t)e_{\overline{\lambda}p}(t,t_0)\Big)^{\Delta}\Big|_{t=t_1} \\
&= x^{\triangle}(t_1)e_{\overline{\lambda}p}(t_1,t_0) + x(\sigma(t_1))\overline{\lambda}p(t_1)e_{\overline{\lambda}p}(t_1,t_0) \\
&\le \Big(-a(t_1)x(t_1)+b(t_1)\int_{t_0}^{\infty}K(s)x(t_1-s)\Delta s + c(t_1)\sup_{s\in[t_1-\tau(t_1),t_1]}x(s)\Big)e_{\overline{\lambda}p}(t_1,t_0) \\
&\quad + x(\sigma(t_1))\overline{\lambda}p(t_1)e_{\overline{\lambda}p}(t_1,t_0)
\end{aligned} \tag{2.20}
$$

$$
\begin{aligned}
&= -a(t_1)\overline{x}(t_1) + b(t_1)\int_{t_0}^{\infty} K(s)\overline{x}(t_1-s)e_{\overline{\lambda}p}(t_1,t_1-s)\Delta s \\
&\quad + c(t_1)\sup_{s\in[t_1-\tau(t_1),t_1]} \overline{x}(s)e_{\overline{\lambda}p}(t_1,s) + \overline{\lambda}p(t_1)e_{\overline{\lambda}p}(t_1,\sigma(t_1))\overline{x}(\sigma(t_1)) \\
&\le -a(t_1)\overline{x}(t_1) + b(t_1)\int_{t_0}^{\infty} K(s)\overline{x}(t_1-s)e_{\overline{\lambda}p}(t_1,t_1-s)\Delta s \\
&\quad + c(t_1)\sup_{t_1-\tau^*\le s\le t_1} \overline{x}(s)e_{\overline{\lambda}p}(t_1,t_1-\tau^*) + \overline{\lambda}p(t_1)e_{\overline{\lambda}p}(t_1,\sigma(t_1))\overline{x}(\sigma(t_1)).
\end{aligned}
$$

By (2.20), (2.12), (1.12) and $1-a(t)\mu(t)>0, t\in\mathbb{T}$, we have

$$
\begin{aligned}
&\Big(1-\overline{\lambda}\mu(t_1)p(t_1)\Big)\delta M < \Big(1-\overline{\lambda}\mu(t_1)p(t_1)e_{\overline{\lambda}p}(t_1,\sigma(t_1))\Big)\overline{x}(\sigma(t_1)) \\
&\le \Big(1-a(t_1)\mu(t_1)\Big)\overline{x}(t_1) + b(t_1)\mu(t_1)\int_{t_0}^{\infty} K(s)\overline{x}(t_1-s)e_{\overline{\lambda}p}(t_1,t_1-s)\Delta s \\
&\quad + c(t_1)\mu(t_1)\sup_{t_1-\tau^*\le s\le t_1} \overline{x}(s)e_{\overline{\lambda}p}(t_1,t_1-\tau^*) \\
&< \Big(1-\overline{\lambda}\mu(t_1)p(t_1)\Big)\delta M - \frac{\epsilon}{2}\mu(t_1)\delta M, \qquad (2.21)
\end{aligned}
$$

as long as $1-\overline{\lambda}\mu(t)p(t)>0$, and so (2.21) leads to a contradiction. So when $\mu(t)=\sigma(t)-t$ is bounded for $t\in\mathbb{T}$, we can choose a sufficiently small positive number $\overline{\lambda}$ satisfying $1-\overline{\lambda}\mu(t)p(t)>0$. This situation is similar to Case (2).

When either $b(t)=0$, $p(t)=1$ or $c(t)=0$, $p(t)=1$ we can obtain the following corollaries, which can be regarded as the extensions of Theorem 2.1, Theorem 2.2 of [1]; respectively.

Corollary 1. *Let $x(\cdot)$ be a nonnegative function satisfying*

$$x^{\triangle}(t) \le -a(t)x(t) + c(t)\sup_{s\in[t-\tau(t),t]} x(s), \quad t>t_0, \tag{2.22}$$

$$x(s) = |\varphi(s)| \quad \textit{for } s\in[t_0-\tau^*,t_0]_{\mathbb{T}}, \tag{2.23}$$

where $\varphi(s)$ defined for $s\in(-\infty,t_0]_{\mathbb{T}}$ is rd-continuous and $\tau(\cdot)$, $a(\cdot)$, $c(\cdot)$ are defined on $\mathbb{T}$ and denote nonnegative, rd-continuous bounded functions and

$$\sup_{t\in\mathbb{T}}\tau(t)=\tau*, \qquad \sup_{s\in(-\infty,t_0]_{\mathbb{T}}} x(s) = M>0.$$

Suppose

$$a(t)-c(t)\ge\epsilon=\inf_{t\in\mathbb{T}}\Big(a(t)-c(t)\Big)>0, \quad t\in\mathbb{T}. \tag{2.24}$$

Then there exists a positive number $\overline{\lambda}$ such that

$$x(t)\le\Big(\sup_{s\in[t_0-\tau^*,t_0]_{\mathbb{T}}} x(s)\Big)e_{\ominus\overline{\lambda}}(t,t_0), \quad t\in(t_0,\infty)_{\mathbb{T}}. \tag{2.25}$$

Corollary 2. *Let $x(\cdot)$ be a nonnegative function satisfying*

$$x^{\triangle}(t) \le -a(t)x(t) + b(t)\int_{t_0}^{\infty} K(s)x(t-s)\Delta s, \quad t > t_0, \tag{2.26}$$

$$x(s) = |\varphi(s)| \quad \text{for } s \in (-\infty, t_0]_{\mathbb{T}}, \tag{2.27}$$

where $\varphi(s)$ defined for $s \in (-\infty, t_0]_{\mathbb{T}}$ is rd-continuous and $\tau(\cdot)$, $a(\cdot)$, $b(\cdot)$ are defined on $\mathbb{T}$ and denote nonnegative, rd-continuous bounded functions and

$$\sup_{t\in\mathbb{T}} \tau(t) = \tau *, \qquad \sup_{s\in(-\infty,t_0]_{\mathbb{T}}} x(s) = M > 0.$$

The delay kernel $K(\cdot)$ is assumed to satisfy the following properties,

$$K : [0,\infty)_{\mathbb{T}} \mapsto [0,\infty)_{\mathbb{T}} \text{ and } \forall t \in \mathbb{T}, \quad \int_0^{\infty} K(s)e_{\lambda_0}(t, t-s)\Delta s < \infty, \tag{2.28}$$

for some positive number λ_0. Suppose further that

$$a(t) - b(t)\int_{t_0}^{\infty} K(s)\Delta s \ge \epsilon = \inf_{t\in\mathbb{T}}\Big(a(t) - b(t)\int_{t_0}^{\infty} K(s)\Delta s\Big) > 0, \quad t \in \mathbb{T}. \tag{2.29}$$

Then there exists a positive number $\overline{\lambda}$ such that

$$x(t) \le \Big(\sup_{s\in(-\infty,t_0]_{\mathbb{T}}} x(s)\Big)e_{\ominus\overline{\lambda}}(t, t_0), quadt \in (t_0,\infty)_{\mathbb{T}}. \tag{2.30}$$

3. **Examples**

Consider the delay dynamic equation

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + b(t)\int_0^{\infty} K(s)x(t-s)\Delta s + c(t)x(t-\tau), \qquad t \in [t_0, +\infty)_{\mathbb{T}} \tag{3.1}$$

where $x(t) = \varphi(t)$, for $s \in (-\infty, t_0]_{\mathbb{T}}$, φ is rd-continuous and bounded. $a(t) \ge 0$, $b(t) \ge 0$, $c(t) \ge 0$, for $t \ge t_0$. Suppose that there exists a nonnegative rd-continuous bounded function $p(t)$ such that $\int_{t_0}^{\infty} K(s)e_{\lambda_0 p}(t, t-s)\Delta s < \infty$, for all $t \in \mathbb{T}$ and some positive number λ_0 and $a(t) - b(t)\int_{t_0}^{\infty} K(s)\Delta s - c(t) \ge \epsilon > 0$ for all $t \in \mathbb{T}$ and some positive number ϵ. From (3.1), we have

$$x(t) = x(t_0)e_{-a}(t, t_0) + \int_{t_0}^{t} e_{-a}(t,\sigma(s))\Big[b(s)\int_0^{\infty} K(v)x(s-v)\Delta v + c(s)x(s-\tau)\Big]\Delta s. \tag{3.2}$$

Let the functions $y(t)$ be defined as follows: $y(t) = |x(t)|$, for $t \in (-\infty, t_0]_{\mathbb{T}}$ and

$$y(t) = |x(t_0)|e_{-a}(t, t_0) + \int_{t_0}^{t} e_{-a}(t,\sigma(s))\Big[b(s)\int_0^{\infty} K(v)|x(s-v)|\Delta v + c(s)\sup_{s-\tau\le\theta\le s}|x(\theta)|\Big]\Delta s \tag{3.3}$$

for $t > t_0$. Then we have $|x(t)| \le y(t)$, for all $t \in (-\infty, +\infty)_{\mathbb{T}}$.

By [16, Theorem 5.37], we get that

$$y^{\Delta}(t) = -a(t)\Big[|x(t_0)|e_{-a}(t, t_0) + \int_{t_0}^{t} e_{-a}(t,\sigma(s))\Big[b(s)\int_0^{\infty} K(v)|x(s-v)|\Delta v$$

$$+ c(s) \sup_{s-\tau\leq\theta\leq s} |x(\theta)|\Big]\Delta s\Big] + b(t)\int_0^\infty K(v)|x(t-v)|\Delta v + c(t)\sup_{t-\tau\leq\theta\leq t}|x(\theta)|$$

$$\leq -a(t)y(t) + b(t)\int_0^\infty K(v)|y(t-v)|\Delta v + c(t)\sup_{t-\tau\leq\theta\leq t}|y(\theta)| \tag{3.4}$$

for all $t \in [t_0, \infty)$. Therefore, it follows from main theorem that there exists a positive number $\overline{\lambda}$ such that

$$x(t) \leq \Big(\sup_{s\in(-\infty,t_0]_{\mathbb{T}}} x(s)\Big)e_{\ominus\overline{\lambda}p}(t,t_0), \quad t \in (t_0,\infty)_{\mathbb{T}}. \tag{3.5}$$

In the following, we let $\mathbb{T} = \mathbb{Z}$ and choose some explicit functions for $a(t)$, $b(t)$, $c(t)$, $K(t)$ and $p(t)$. Let

$$a(n) = \frac{n+2}{n+3}, \quad b(n) = \frac{n}{4(n+3)}, \quad c(n) = \frac{1}{2(n+3)}, \quad K(n) = \frac{1}{2^n}, \quad n \in \mathbb{N}.$$

We have

$$a(n) - b(n)\sum_{n=0}^{\infty} K(n) - c(n) = \frac{n+2}{n+3} - \frac{n}{4(n+3)}\sum_{n=0}^{\infty}\frac{1}{2^n} - \frac{1}{2(n+3)} = \frac{1}{2} = \epsilon > 0.$$

Let $p(n) = 1 + n\sin\frac{1}{n+1}$, we have

$$\sum_{j=0}^{\infty} K(j)e_{\lambda p}(n,n-j) \leq \sum_{j=0}^{\infty} K(j)(1+2\lambda)^j e_{\lambda p}(n-j,n-j) = \sum_{j=0}^{\infty}\Big(\frac{1+2\lambda}{2}\Big)^j,$$

where $e_{\lambda p}(n-j,n-j) = 1$. Clearly, as long as

$$\frac{1+2\lambda}{2} < 1 \Rightarrow \overline{\lambda} \in \Big(0, \frac{1}{2}\Big).$$

So

$$\sum_{j=0}^{\infty} K(j)e_{\lambda p}(n,n-j) < \infty,$$

$$x(j) = |\varphi(j)|, \quad \text{for } j \leq n_0.$$

Therefore the conditions (2.1), (2.2), (2.3), (2.4) are satisfied. Then there exists a positive number $\overline{\lambda}$ such that

$$x(n) \leq \Big(\sup_{j\leq n_0}|\varphi(j)|\Big)e_{\ominus\overline{\lambda}p}(n,n_0), \quad n > n_0, \overline{\lambda} \in \Big(0,\frac{1}{2}\Big).$$

It is easy to get that $1 + \ominus\overline{\lambda}p = \frac{1}{2+n\sin\frac{1}{n+1}} \leq \frac{1}{2}$. So $e_{\ominus\overline{\lambda}p}(n,n_0) \leq \left(\frac{1}{2}\right)^{n-n_0}$. Therefore we get that

$$x(n) \leq \Big(\sup_{j\leq n_0}|\varphi(j)|\Big)\Big(\frac{1}{2}\Big)^{n-n_0}, \quad n > n_0, \overline{\lambda} \in (0, \frac{1}{2}).$$

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