THE NARROWING SET-VALUED
STOCHASTIC INTEGRAL EQUATIONS

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ABSTRACT. We analyze set-valued stochastic integral equations whose solutions are mappings with values in the hyperspace of subsets of square integrable random vectors space. In this paper we give a new formulation of these equations resulting in a new property of solutions. Namely, the diameter of the solution values will be a nonincreasing function. Hence we call these equations “narrowing”. We prove a result on existence and uniqueness of the solution to the narrowing set-valued stochastic integral equations. We establish a boundedness type result for the solution and an error of an approximate solution. Also the continuous dependence of the solution with respect to data of the equation is shown.

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1. INTRODUCTION

Set-valued mappings appear naturally in many branches of science, such as economics, biomathematics, physics, game theory, artificial intelligence (see e.g. [7, 13] and references therein). There is a huge interest in this area due to many applications in control theory and optimization (see e.g. [4]–[6] and references therein). Also, thinking about mathematical models of dynamical systems with incomplete information or systems with velocities that are not uniquely determined, one often focuses on set-valued differential equations [25].

Set-valued differential equations with solutions taking on values in compact and convex subsets of the Euclidean space were introduced in [9, 10, 12] and studied since then by many authors. Their range includes, for instance, existence of solutions [1], stability [2, 3, 8, 16, 17, 19, 41, 43], equations involving causal operators [14, 16, 23], monotone iterative technique [15], variation of constants formula [18], equations on time scales [20, 28, 43], periodic solutions [21], monotone flows [27], equations with second type Hukuhara derivative [29]–[32], quasilinearization [42].
On the other hand a new mathematical apparatus in a form of set-valued stochastic integral equations [33]–[40] generalizes the deterministic set-valued differential equations to a stochastic extent. The papers we have been doing so far use a formulation of the set-valued stochastic integral equations in such a form, nevertheless - very natural, which causes that the diameter of the solution values is a nondecreasing function. We show this in Theorem 3.1 of the current paper, where we consider the set-valued stochastic integral equations with solutions being set-valued mappings taking on values in the hyperspace of nonempty closed bounded and convex subsets of square integrable random vectors space. This paper presents a study of set-valued stochastic equations with solutions having a nonincreasing diameter of their values. This is shown in Theorem 3.2 later on. Accordingly to the new property of solutions we call these equations “narrowing”. Consequently the equations studied earlier are called the “widening” set-valued stochastic integral equations. We show that the theory of the narrowing set-valued stochastic integral equations is well-posed. Existence of unique solution is shown together with the stabilities of the solution under small changes of the equation parameters. As distinct from the usual analysis, to obtain existence of solutions we impose existence of some Hukuhara differences, see condition (H4). This condition is necessary and cannot be omitted. It is not used in the previously studied widening equations, because in their analysis there is no need to use the Hukuhara differences. The existence of solutions is obtained under Lipschitz condition with an integrable stochastic process instead of the Lipschitz constant and a condition of boundedness by an integrable stochastic process.

The paper is organized as follows. In Section 2 we summarize some preliminary facts and properties on the Hukuhara difference, set-valued stochastic processes and set-valued stochastic trajectory integrals. The main results are presented in Section 3. We introduce the notion of the narrowing set-valued stochastic integral equation. Then, the existence and uniqueness of solutions is proven. Also, the continuous dependence of solutions with respect to data of the equation is shown. We indicate that under conditions (H1)–(H3), considered in Section 3, the similar results can be obtained for the widening set-valued stochastic integral equations.

2. PRELIMINARIES

Let $X$ be a separable Banach space, and let $K^b_c(X)$ be the hyperspace of all nonempty closed bounded and convex subsets of $X$. The Hausdorff metric $H_X$ in $K^b_c(X)$ is defined by

$$H_X(A, B) := \max \left\{ \sup_{a \in A} \text{dist}_X(a, B), \sup_{b \in B} \text{dist}_X(b, A) \right\},$$

where $\text{dist}_X(a, B) := \inf_{b \in B} \| a - b \|_X$ and $\| \cdot \|_X$ denotes a norm in $X$. It is known (see [22]) that $(K^b_c(X), H_X)$ is a complete metric space. The addition and the multiplication by
reals are defined as usual, i.e. for $A, B \in \mathcal{K}_c^b(\mathcal{X})$ and $\mu \in \mathbb{R}$ we have $A + B := \{a + b : a \in A, b \in B\}$, $\mu A := \{\mu a : a \in A\}$. The Hukuhara difference of $A, B \in \mathcal{K}_c^b(\mathcal{X})$ is defined as the set $A \ominus B \in \mathcal{K}_c^b(\mathcal{X})$ such that $(A \ominus B) + B = A$. If $A \ominus B$ exists, it is unique.

For the metric $H_X$ and $A, B, C, D \in \mathcal{K}_c^b(\mathcal{X})$ and $\mu \in \mathbb{R}$ one has

\begin{align*}
\text{(P1)} \quad H_X(A + B, C + D) &\leq H_X(A, C) + H_X(B, D), \\
\text{(P2)} \quad H_X(\mu A, \mu B) &= |\mu| H_X(A, B), \\
\text{(P3)} \quad H_X(A + C, B + C) &= H_X(A, B), \\
\text{(P4)} \quad \text{if } A \ominus B \text{ exists then } H_X(A \ominus B, \{0\}) &= H_X(A, B), \\
\text{(P5)} \quad \text{if } A \ominus B \text{ and } A \ominus C \text{ exist then } H_X(A \ominus B, A \ominus C) &= H_X(B, C), \\
\text{(P6)} \quad \text{if } A \ominus B \text{ and } C \ominus D \text{ exist then } H_X(A \ominus B, C \ominus D) &\leq H_X(A, C) + H_X(B, D).
\end{align*}

Also, it is known [22] that the family of nonempty, closed and convex subsets of a separable and reflexive Banach space $\mathcal{X}$ supplied with the Mosco topology $\tau_{M_\mathcal{X}}$ is a Polish topological space. The Mosco topology is metrizable and weaker than the topology $\tau_{H_X}$ generated by the Hausdorff metric $H_X$.

Let $(U, \mathcal{U}, \mu)$ be a measure space. Recall that a set-valued mapping $F: U \to \mathcal{K}_c^b(\mathcal{X})$ is said to be $\mathcal{U}$-measurable (or set-valued random variable) if it satisfies:

$$\{u \in U : F(u) \cap O \neq \emptyset\} \in \mathcal{U} \text{ for every open set } O \subset \mathcal{X}.$$ 

A set-valued random variable $F$ is said to be $L^p$-integrally bounded ($p \geq 1$), if $u \mapsto H_X(F(u), \{0\})$ belongs to $L^p(U, \mathcal{U}, \mu; \mathbb{R})$.

Define $I := [0, T]$, where $T < \infty$. Let $(\Omega, \mathcal{A}, \{\mathcal{A}_t\}_{t \in I}, P)$ be a complete filtered probability space satisfying usual hypotheses, i.e. $\{\mathcal{A}_t\}_{t \in I}$ is an increasing and right continuous family of sub-$\sigma$-algebras of $\mathcal{A}$ and $\mathcal{A}_0$ contains all $P$-null sets. Let $\{B(t)\}_{t \in I}$ be an $\{\mathcal{A}_t\}$-Brownian motion. Let $\mathcal{N}$ denote the $\sigma$-algebra of the nonanticipating elements in $I \times \Omega$, i.e.

$$\mathcal{N} = \{A \in \beta_I \otimes \mathcal{A} : A^t \in \mathcal{A}_t \text{ for every } t \in I\},$$

where $\beta_I$ is the Borel $\sigma$-algebra of subsets of $I$ and $A^t = \{\omega : (t, \omega) \in A\}$. A $d$-dimensional stochastic process $f: I \times \Omega \to \mathbb{R}^d$ is called nonanticipating if $f(\cdot, \cdot)$ is $\mathcal{N}$-measurable.

By $\lambda$ we denote the Lebesgue measure on $(I, \beta_I)$. Consider the space

$$L^2_N(\lambda \times P) := L^2(I \times \Omega, \mathcal{N}, \lambda \times P; \mathbb{R}^d).$$

Then for every $f \in L^2_N(\lambda \times P)$ and $\tau, t \in I$, $\tau < t$ the Itô stochastic integral $\int_\tau^t f(s)dB(s)$ exists and one has $\int_\tau^t f(s)dB(s) \in L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d) \subset L^2(\Omega, \mathcal{A}, P; \mathbb{R}^d)$. 
Let $F : I \times \Omega \to K^b_c(\mathbb{R}^d)$ be a set-valued stochastic process, i.e. a family of $\mathcal{A}$-measurable set-valued mappings $F(t, \cdot) : \Omega \to K^b_c(\mathbb{R}^d) , t \in I$. We call $F$ nonanticipating if $F(\cdot, \cdot)$ is an $\mathcal{N}$-measurable set-valued mapping. Let us define the set

$$S^2_N(F, \lambda \times P) := \{ f \in L^2_N(\lambda \times P) : f \in F, \lambda \times P\text{-a.e.} \}.$$

If $F$ is $L^2_N(\lambda \times P)$-integrally bounded, then by the Kuratowski and Ryll-Nardzewski Selection Theorem (see [24]) it follows that $S^2_N(F, \lambda \times P) \neq \emptyset$. Hence for every $\tau, t \in I$, $\tau < t$ we can define the set-valued stochastic Aumann trajectory integral $(S) \int^t_\tau F(s)ds$ as a subset of $L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d)$ in the following way:

$$(S) \int^t_\tau F(s)ds := \left\{ \int^t_\tau f(s)ds : f \in S^2_N(F, \lambda \times P) \right\}.$$

Now we consider the set-valued stochastic Itô trajectory integral. Like in the preceding considerations, let $F : I \times \Omega \to K^b_c(\mathbb{R}^d)$ be a nonanticipating and $L^2_N(\lambda \times P)$-integrally bounded set-valued stochastic process. Then for $\tau, t \in I$, $\tau < t$ we can define the set-valued trajectory Itô stochastic integral

$$(S) \int^t_\tau F(s)dB(s) := \left\{ \int^t_\tau f(s)dB(s) : f \in S^2_N(F, \lambda \times P) \right\}.$$

By this definition we have $\int^t_\tau F(s)dB(s) \subset L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d)$.

**Lemma 2.1.** Let $r \in \mathbb{R}$ and let $F : I \times \Omega \to K^b_c(\mathbb{R}^d)$ be a nonanticipating and $L^2_N(\lambda \times P)$-integrally bounded set-valued stochastic process. Then for $\tau < t$ ($\tau, t \in I$)

$$(S) \int^t_\tau rF(s)ds = r(S) \int^t_\tau F(s)ds \quad \text{and} \quad (S) \int^t_\tau rF(s)dB(s) = r(S) \int^t_\tau F(s)dB(s).$$

In the rest of the paper, for the sake of convenience, we will write $L^2$ instead of $L^2(\Omega, \mathcal{A}, P; \mathbb{R}^d)$ and $L^2_t$ instead of $L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d)$ where $t \in I$. From now on we assume that the $\sigma$-algebra $\mathcal{A}$ is separable with respect to the probability measure $P$.

We have the following properties for the stochastic trajectory integrals (see e.g. [37, 38]).

**Lemma 2.2.** For a nonanticipating and $L^2_N(\lambda \times P)$-integrally bounded set-valued stochastic process $F : I \times \Omega \to K^b_c(\mathbb{R}^d)$ and for every $\tau, a, t \in I$, $\tau \leq a \leq t$ it holds that

$$(S) \int^t_\tau F(s)ds = (S) \int^a_\tau F(s)ds + (S) \int^t_a F(s)ds, \quad \text{and}$$

$$(S) \int^t_\tau F(s)dB(s) = (S) \int^a_\tau F(s)dB(s) + (S) \int^t_a F(s)dB(s).$$
Lemma 2.3. Let \( F, G : I \times \Omega \to \mathcal{K}_c^b(\mathbb{R}^d) \) be nonanticipating and \( L^2_N(\lambda \times P) \)-integrally bounded set-valued stochastic processes. Then for every \( \tau, t \in I, \tau < t \)

\[
H^2_{L^2} \left( (S) \int_{\tau}^{t} F(s)ds, (S) \int_{\tau}^{t} G(s)ds \right) \leq (t - \tau) \int_{[\tau,t] \times \Omega} H^2_{\mathbb{R}^d}(F,G)ds \times dP,
\]

and

\[
H^2_{L^2} \left( (S) \int_{\tau}^{t} F(s)dB(s), (S) \int_{\tau}^{t} G(s)dB(s) \right) \leq \int_{[\tau,t] \times \Omega} H^2_{\mathbb{R}^d}(F,G)ds \times dP.
\]

Lemma 2.4. Let \( F : I \times \Omega \to \mathcal{K}_c^b(\mathbb{R}^d) \) be a nonanticipating and \( L^2_N(\lambda \times P) \)-integrally bounded set-valued stochastic process. Then the mappings

\[
[\tau, T] \ni t \mapsto (S) \int_{\tau}^{t} F(s)ds \in \mathcal{K}_c^b(L^2), \quad [\tau, T] \ni t \mapsto (S) \int_{\tau}^{t} F(s)dB(s) \in \mathcal{K}_c^b(L^2)
\]

are \( H_{L^2} \)-continuous.

3. SET-VALUED STOCHASTIC INTEGRAL EQUATIONS

Let \( F, G : I \times \Omega \times \mathcal{K}_c^b(L^2) \to \mathcal{K}_c^b(\mathbb{R}^d) \) and \( X_0 \in \mathcal{K}_c^b(L_0^2) \) be given. By a set-valued stochastic integral equation we mean the following relation in the metric space \((\mathcal{K}_c^b(L^2), H_{L^2})\):

\[
X(t) = X_0 + (S) \int_{0}^{t} F(s, X(s))ds + (S) \int_{0}^{t} G(s, X(s))dB(s) \quad \text{for} \quad t \in I.
\]

(3.1)

By a global solution to (3.1) we mean an \( H_{L^2} \)-continuous set-valued mapping \( X : I \to \mathcal{K}_c^b(L^2) \) that satisfies (3.1). A global solution \( X : I \to \mathcal{K}_c^b(L^2) \) to (3.1) is unique if \( X(t) = Y(t) \) for every \( t \in I \), where \( Y : I \to \mathcal{K}_c^b(L^2) \) is any solution of (3.1).

Let \( J := [0, \bar{T}] \subset I = [0, T] \), where \( \bar{T} < T \). A set-valued mapping \( X : J \to \mathcal{K}_c^b(L^2) \) is said to be a local solution to (3.1) if it is \( H_{L^2} \)-continuous and satisfies (3.1) for \( t \in J \).

The uniqueness of the local solution is defined in an obvious way.

Theorem 3.1. Suppose that \( X : I \to \mathcal{K}_c^b(L^2) \) is a global solution to (3.1). Then the function \( t \mapsto \text{diam}X(t) \) is nondecreasing.

Proof. Since \( X \) is a solution to (3.1) and Lemma 2.2 holds, we can write for \( \tau < t \) \((\tau, t \in I)\) that

\[
X(t) = X(\tau) + (S) \int_{\tau}^{t} F(s, X(s))ds + (S) \int_{\tau}^{t} G(s, X(s))dB(s).
\]

Let \( x \) be a fixed point from the set \((S) \int_{\tau}^{t} F(s, X(s))ds + (S) \int_{\tau}^{t} G(s, X(s))dB(s)\). Then we have \( X(t) \supset X(\tau) + \{x\} \) which implies that \( \text{diam}X(t) \geq \text{diam}X(\tau) \).

The same property is true for local solutions to (3.1) and it convinces that we can call equations of type (3.1) as the widening set-valued stochastic integral equations.
In order to consider the mappings \( X : I \to \mathcal{K}_c^b(L^2) \) with the nonincreasing diameter function \( \text{diam} X(\cdot) \) as some solutions to the set-valued stochastic integral equations, it is necessary to change the form of the equation. Below we present such a variation of formulation of the set-valued stochastic integral equations. Namely, we will consider the equations of the following form:

\[ (3.2) \quad X(t) + (S) \int_0^t (-1) F(s, X(s)) ds + (S) \int_0^t (-1) G(s, X(s)) dB(s) = X_0 \text{ for } t \in I. \]

It is easy to see that in the case of singleton-valued \( X_0 \), \( F \) and \( G \) the equations (3.1) and (3.2) coincide. Both of them generalize the classical single-valued stochastic differential equations.

Note that the equation (3.2) can be rewritten as

\[ (3.3) \quad X(t) = X_0 \ominus \left[ (S) \int_0^t (-1) F(s, X(s)) ds + (S) \int_0^t (-1) G(s, X(s)) dB(s) \right] \]

for \( t \in I \). We will call the equations (3.2) as the narrowing set-valued stochastic integral equations because, as we prove below, their solutions \( X \) possess property that \( \text{diam} X(\cdot) \) is nonincreasing. The global and local solutions to (3.2) and their uniqueness are defined like for the global and local solutions to (3.1).

**Theorem 3.2.** Let \( X : I \to \mathcal{K}_c^b(L^2) \) be a global solution to (3.2). Then the function \( t \mapsto \text{diam} X(t) \) is nonincreasing.

**Proof.** Note that due to Lemma 2.2 we have

\[ X(t) + (S) \int_\tau^t (-1) F(s, X(s)) ds + (S) \int_\tau^t (-1) G(s, X(s)) dB(s) \]

for \( \tau < t \). The Hukuhara difference above exists because \( X \) is a solution to (3.2) and it equals \( X(\tau) \). Choosing any point \( x \) that belongs to the set \( (S) \int_0^\tau (-1) F(s, X(s)) ds + (S) \int_0^\tau (-1) G(s, X(s)) dB(s) \) we get

\[ X(t) + \{ x \} \subset X(\tau). \]

As a consequence, the inequality \( \text{diam} X(t) \leq \text{diam} X(\tau) \) follows easily. \( \square \)

A similar assertion holds true for the local solutions to (3.2).

Below we present an existence and uniqueness theorem. It will be achieved with the following conditions imposed on the data of the equation.

Assume that \( X_0 \in \mathcal{K}_c^b(L^0_0) \) and \( F, G : I \times \Omega \times \mathcal{K}_c^b(L^2) \to \mathcal{K}_c^b(\mathbb{R}^d) \) satisfy

(H1) the set-valued mappings \( F(\cdot, \cdot, \cdot), G(\cdot, \cdot, \cdot) : I \times \Omega \times \mathcal{K}_c^b(L^2) \to \mathcal{K}_c^b(\mathbb{R}^d) \) are \( \mathcal{N} \times \beta(\tau_{M_{L^2}}) \)-measurable, where \( \beta(\tau_{M_{L^2}}) \) is the Borel \( \sigma \)-algebra induced by the Mosco topology \( \tau_{M_{L^2}}, \)
(H2) there exists $K_F \in L^2(I \times \Omega, \beta_I \otimes A, \lambda \times P; \mathbb{R})$ such that $\lambda \times P$-a.e. for every $A, B \in \mathcal{K}_b^b(L^2)$

$$H^2_{\mathbb{R}^d}(F(t, \omega, A), F(t, \omega, B)) \leq K_F(t, \omega)H^2_{L_2}(A, B),$$

and there exists $K_G \in L^2(I \times \Omega, \beta_I \otimes A, \lambda \times P; \mathbb{R})$ such that $\lambda \times P$-a.e. for every $A, B \in \mathcal{K}_b^b(L^2)$

$$H^2_{\mathbb{R}^d}(G(t, \omega, A), G(t, \omega, B)) \leq K_G(t, \omega)H^2_{L_2}(A, B),$$

(H3) there exists $C_F \in L^1(I \times \Omega, \beta_I \otimes A, \lambda \times P; \mathbb{R})$ such that $\lambda \times P$-a.e.

$$H^2_{\mathbb{R}^d}(F(t, \omega, \{\Theta\}), \{\theta\}) \leq C_F(t, \omega),$$

and there exists $C_G \in L^1(I \times \Omega, \beta_I \otimes A, \lambda \times P; \mathbb{R})$ such that $\lambda \times P$-a.e.

$$H^2_{\mathbb{R}^d}(G(t, \omega, \{\Theta\}), \{\theta\}) \leq C_G(t, \omega),$$

where $\Theta, \theta$ denote the zero elements in $L^2$ and $\mathbb{R}^d$, respectively,

(H4) there exists $\tilde{T} \in (0, T]$ such that the sequence $\{X_n\}_{n=0}^\infty$ described by

$$X_0(t) = X_0, \ t \in J = [0, \tilde{T}],$$

and for $n = 1, 2, \ldots$ and for $t \in J$

$$X_n(t) = X_0 \cap \left[ (S) \int_0^t (-1)F(s, X_{n-1}(s))ds + (S) \int_0^t (-1)G(s, X_{n-1}(s))dB(s) \right]$$

can be defined, i.e. the Hukuhara differences do exist.

Notice that the conditions (H2) and (H3) formulated with integrable stochastic processes $K_F, K_G, C_F, C_G$ are weaker than the Lipschitz assumptions with constants and boundedness conditions with constants. The condition (H4) cannot be omitted in the studies of equation (3.2). It is motivated by the representation (3.3).

In what follows we present a result on existence of unique solution to (3.2). Due to (H4) this solution can be local or global depending on whether $\tilde{T} < T$ or $\tilde{T} = T$. In its proof, the sequence $\{X_n\}_{n=0}^\infty$ will be exploited. Below we make a discussion that under conditions (H1)–(H4) each $X_n$ is a well-defined $H_{L_2}$-continuous set-valued mapping.

Indeed, starting with $X_0(\cdot)$ we see easily that $X_0(\cdot)$ is well-defined. Now, using (H1), we observe that the set-valued mappings $F(\cdot, \cdot, X_0), G(\cdot, \cdot, X_0): I \times \Omega \to \mathcal{K}_c^b(\mathbb{R}^d)$ are nonanticipating. Due to (H2) and (H3) the following inequalities hold $\lambda \times P$-a.e.

$$H^2_{\mathbb{R}^d}(F(t, \omega, X_0), \{\theta\}) \leq 2K_F(t, \omega)H^2_{L_2}(X_0, \{\Theta\}) + 2C_F(t, \omega),$$

$$H^2_{\mathbb{R}^d}(G(t, \omega, X_0), \{\theta\}) \leq 2K_G(t, \omega)H^2_{L_2}(X_0, \{\Theta\}) + 2C_G(t, \omega).$$

Hence we can infer that $F(\cdot, \cdot, X_0)$ and $G(\cdot, \cdot, X_0)$ are $L_{\mathcal{K}}^2(\lambda \times P)$-integrally bounded. This allows us to claim that the set-valued stochastic trajectory integrals in formulation of $X_1(t)$ are well-defined and belong to $\mathcal{K}_c^b(L^2_{\mathcal{K}})$. Since $X_0 \in \mathcal{K}_c^b(L^2_0) \subset \mathcal{K}_c^b(L^2_{\mathcal{K}})$
and it is assumed that the Hukuhara differences in (H4) exist, we obtain that $X_1(t) \in K^b_c(L^2_t)$ for every $t \in J$. Lemma 2.4 allows us to infer that the mapping $t \mapsto X_1(t)$ is $H_{L^2}$-continuous. Since the Mosco topology $\tau_{M_{L^2}}$ is is weaker than the topology generated by the Hausdorff metric $H_{L^2}$, the mapping $t \mapsto X_1(t)$ is continuous with respect to topology $\tau_{M_{L^2}}$ as well. Hence the set-valued mappings $(t, \omega) \mapsto F(t, \omega, X_1(t))$ and $(t, \omega) \mapsto G(t, \omega, X_1(t))$ are nonanticipating. Since

$$H^2_{\mathbb{R}^d}(F(t, \omega, X_1(t)), \{\theta\}) \leq 2K_F(t, \omega) \sup_{t \in J} H^2_{L^2}(X_1(t), \{\Theta\}) + 2C_F(t, \omega),$$

$$H^2_{\mathbb{R}^d}(G(t, \omega, X_1(t)), \{\theta\}) \leq 2K_G(t, \omega) \sup_{t \in J} H^2_{L^2}(X_1(t), \{\Theta\}) + 2C_G(t, \omega)$$

and $\sup_{t \in J} H^2_{L^2}(X_1(t), \{\Theta\}) < \infty$, we get that $(t, \omega) \mapsto F(t, \omega, X_1(t))$ and $(t, \omega) \mapsto G(t, \omega, X_1(t))$ are $L^2_N(\lambda \times P)$-integrally bounded. Now we are able to state that $X_2$ is well-defined and $H_{L^2}$-continuous. Proceeding recursively one can see that every $X_n$ is well-defined and $H_{L^2}$-continuous.

**Theorem 3.3.** Let $X_0 \in K^b_c(L^2_0)$ and $F, G : I \times \Omega \times K^b_c(L^2) \to K^b_c(\mathbb{R}^d)$ satisfy conditions (H1)–(H4). Then equation (3.2) has a unique local (or global) solution.

**Proof.** It is clear that using Lemma 2.4 we obtain that each $X_n$ is continuous with respect to the metric $H_{L^2}$. We shall show that $\{X_n\}^\infty_{n=0}$ is a Cauchy sequence in the space $C(J, K^b_c(L^2))$ endowed with a supremum metric.

Due to property (P6), Lemma 2.1, (P3) and (P1) we have for $t \in J$

$$H^2_{L^2}(X_1(t), X_0(t))$$

$$= H^2_{L^2} \left( (S) \int_0^t F(s, X_0)ds + (S) \int_0^t G(s, X_0)dB(s), \{\Theta\} \right)$$

$$\leq 2H^2_{L^2} \left( (S) \int_0^t F(s, X_0)ds, \{\Theta\} \right) + 2H^2_{L^2} \left( (S) \int_0^t G(s, X_0)dB(s), \{\Theta\} \right).$$

Now by Lemma 2.3

$$H^2_{L^2}(X_1(t), X_0(t))$$

$$\leq 2t \int_{[0,t] \times \Omega} H^2_{\mathbb{R}^d}(F(s, X_0), \{\theta\})ds \times dP + 2 \int_{[0,t] \times \Omega} H^2_{\mathbb{R}^d}(G(s, X_0), \{\theta\})ds \times dP$$

$$\leq 4t \int_{[0,t] \times \Omega} H^2_{\mathbb{R}^d}(F(s, X_0), F(s, \{\Theta\}))ds \times dP$$

$$+ 4t \int_{[0,t] \times \Omega} H^2_{\mathbb{R}^d}(F(s, \{\Theta\}), \{\theta\})ds \times dP$$

$$+ 4 \int_{[0,t] \times \Omega} H^2_{\mathbb{R}^d}(G(s, X_0), G(s, \{\Theta\}))ds \times dP$$

$$+ 4 \int_{[0,t] \times \Omega} H^2_{\mathbb{R}^d}(G(s, \{\Theta\}), \{\theta\})ds \times dP$$
and by the assumptions (H2) and (H3)

\[
H_{L^2}^2(X_1(t), X_0(t)) \leq (4t + 4) H_{L^2}^2(X_0, \{\Theta\}) \int_{[0,t] \times \Omega} (K_F(s) + K_G(s)) ds \times dP
\]

\[
+ (4t + 4) \int_{[0,t] \times \Omega} (C_F(s) + C_G(s)) ds \times dP
\]

\[
\leq M_1,
\]

where

\[
M_1 = (4\tilde{T} + 4) \left[ H_{L^2}^2(X_0, \{\Theta\}) \int_{J \times \Omega} (K_F(s) + K_G(s)) ds \times dP \right. \\
+ \left. \int_{J \times \Omega} (C_F(s) + C_G(s)) ds \times dP \right] < \infty.
\]

For \(n \geq 2\) we have

\[
H_{L^2}^2(X_n(t), X_{n-1}(t)) \leq 2t \int_{[0,t] \times \Omega} K_F(s) H_{L^2}^2(X_{n-1}(s), X_{n-2}(s)) ds \times dP
\]

\[
+ 2 \int_{[0,t] \times \Omega} K_G(s) H_{L^2}^2(X_{n-1}(s), X_{n-2}(s)) ds \times dP
\]

\[
\leq \left[ 2t \left( \int_{[0,t] \times \Omega} K_F^2(s) ds \times dP \right)^{1/2} + 2 \left( \int_{[0,t] \times \Omega} K_G^2(s) ds \times dP \right)^{1/2} \right]
\]

\[
\times \left( \int_{[0,t] \times \Omega} H_{L^2}^4(X_{n-1}(s), X_{n-2}(s)) ds \times dP \right)^{1/2}.
\]

Hence

\[
H_{L^2}^4(X_n(t), X_{n-1}(t)) \leq M_2 \int_0^t H_{L^2}^4(X_{n-1}(s), X_{n-2}(s)) ds,
\]

where

\[
M_2 = 8\tilde{T}^2 \int_{J \times \Omega} K_F^2(s) ds \times dP + 8 \int_{J \times \Omega} K_G^2(s) ds \times dP.
\]

This allows us to infer that

\[
H_{L^2}(X_n(t), X_{n-1}(t)) \leq \left( \frac{M_1 (M_2 t)^{n-1}}{(n-1)!} \right)^{1/4}
\]

and

\[
\sup_{t \in J} H_{L^2}(X_n(t), X_{n-1}(t)) \leq \left( \frac{M_1 (M_2 \tilde{T})^{n-1}}{(n-1)!} \right)^{1/4}.
\]

Consequently for \(m < n\)

\[
\sup_{t \in J} H_{L^2}(X_n(t), X_m(t)) \leq \sum_{k=m}^{n-1} \left( \frac{M_1 (M_2 \tilde{T})^k}{k!} \right)^{1/4}.
\]
Now it is clear that \( \{X_n\} \) is a Cauchy sequence in \( C(J, \mathcal{K}_c^b(L^2)) \). Thus there exists \( X \in C(J, \mathcal{K}_c^b(L^2)) \) such that

\[
\sup_{t \in J} H_{L^2}(X_n(t), X(t)) \to 0 \text{ as } n \to \infty.
\]

In the sequel we shall show that \( X \) is a solution to (3.2). Observe that for every fixed \( t \in J \) we have

\[
H_{L^2}^2\left( X(t), X_0 \ominus \left[ (S) \int_0^t (-1)F(s, X(s))ds + (S) \int_0^t (-1)G(s, X(s))dB(s) \right]\right) \\
\leq 2H_{L^2}^2(X(t), X_n(t)) + 2R_n(t),
\]

where

\[
R_n(t) = H_{L^2}^2\left( X_0 \ominus \left[ (S) \int_0^t (-1)F(s, X_{n-1}(s))ds + (S) \int_0^t (-1)G(s, X_{n-1}(s))dB(s) \right]\right).
\]

Note that

\[
R_n(t) \leq 2H_{L^2}^2 \left( (S) \int_0^t F(s, X_{n-1}(s))ds, (S) \int_0^t F(s, X(s))ds \right) \\
+ 2H_{L^2}^2 \left( (S) \int_0^t G(s, X_{n-1}(s))dB(s), (S) \int_0^t G(s, X(s))dB(s) \right) \\
\leq \left[ 2t \left( \int_{[0,t] \times \Omega} K^2_F(s)ds \times dP \right)^{1/2} + 2 \left( \int_{[0,t] \times \Omega} K^2_G(s)ds \times dP \right)^{1/2} \right] \\
\times \left( \int_{[0,t] \times \Omega} H_{L^2}^4(X_{n-1}(s), X(s))ds \times dP \right)^{1/2}.
\]

Hence

\[
R_n(t) \leq M_2 \int_0^t H_{L^2}^4(X_{n-1}(s), X(s))ds \\
\leq M_2 \bar{T} \left( \sup_{t \in J} H_{L^2}(X_{n-1}(t), X(t)) \right)^4.
\]

Since \( \sup_{t \in J} H_{L^2}(X_{n-1}(t), X(t)) \xrightarrow{n \to \infty} 0 \), we have \( R_n(t) \xrightarrow{n \to \infty} 0 \) for every \( t \in J \). Now it is easy to see that

\[
H_{L^2}\left( X(t), X_0 \ominus \left[ (S) \int_0^t (-1)F(s, X(s))ds + (S) \int_0^t (-1)G(s, X(s))dB(s) \right]\right) = 0
\]

for every \( t \in J \). This means that \( X \) is a solution to (3.2).

Suppose that \( X: J \to \mathcal{K}_c^b(L^2) \) and \( Y: J \to \mathcal{K}_c^b(L^2) \) are two solutions to (3.2). Then it can be verified that for \( t \in J \)

\[
H_{L^2}^4(X(t), Y(t)) \leq M_2 \int_0^t H_{L^2}^4(X(s), Y(s))ds.
\]
Thus, after application of the Gronwall inequality, we can infer that

\[ H_{L^2}^4(X(t), Y(t)) = 0 \text{ for every } t \in J \]

which implies that \( X(t) = Y(t) \) for every \( t \in J \). Hence the uniqueness of \( X \) is proven.

The sequence \( \{X_n\} \) defined in (H4) converges to the solution \( X : J \to \mathcal{K}^b_c(L^2) \) to (3.2). Hence it can derive some approximate solutions to (3.2). An estimation of an error between the \( n \)th approximation \( X_n \) and the exact solution \( X \) is a subject of the next result.

**Proposition 3.4.** Let for \( X_0 \in \mathcal{K}^b_c(L^0_0) \) and \( F, G : I \times \Omega \times \mathcal{K}^b_c(L^2) \to \mathcal{K}^b_c(\mathbb{R}^d) \) the conditions (H1)–(H4) be satisfied. Then for every \( n \in \mathbb{N} \) it holds that

\[
\sup_{t \in J} H_{L^2}^2(X_n(t), X(t)) \leq 2^{3/4} \left( M_1 \frac{(M_2 \tilde{T})^n}{n!} \right)^{1/4} \exp\{2M_2 \tilde{T}\},
\]

where the constants \( M_1 \) and \( M_2 \) are defined as in (3.4) and (3.5), respectively.

**Proof.** Proceeding similarly like in the proof of Theorem 3.3 we get for \( t \in J \)

\[
H_{L^2}^2(X_n(t), X(t)) \leq 2t \left( \int_{[0,t] \times \Omega} K^2_F(s)ds \times dP \right)^{1/2} \left( \int_{[0,t] \times \Omega} H_{L^2}^4(X_{n-1}(s), X(s))ds \times dP \right)^{1/2} + 2 \left( \int_{[0,t] \times \Omega} K^2_G(s)ds \times dP \right)^{1/2} \left( \int_{[0,t] \times \Omega} H_{L^2}^4(X_{n-1}(s), X(s))ds \times dP \right)^{1/2}.
\]

Hence

\[
H_{L^2}^4(X_n(t), X(t)) \leq M_2 \int_0^t H_{L^2}^4(X_{n-1}(s), X(s))ds + 8M_2 \int_0^t H_{L^2}^4(X_n(s), X(s))ds + 8M_2 \int_0^t H_{L^2}^4(X_n(s), X(s))ds
\]

and by (3.6) we can write

\[
H_{L^2}^4(X_n(t), X(t)) \leq 8M_1 \frac{(M_2 \tilde{T})^n}{n!} + 8M_2 \int_0^t H_{L^2}^4(X_n(s), X(s))ds.
\]

Thus by the Gronwall inequality

\[
H_{L^2}^4(X_n(t), X(t)) \leq 8M_1 \frac{(M_2 \tilde{T})^n}{n!} \exp\{8M_2 t\} \text{ for every } t \in J
\]

and the assertion follows easily. \(\square\)
Proposition 3.5. Under assumptions of Theorem 3.3 for the solution $X: J \rightarrow \mathcal{K}^b_c(L^2)$ to (3.2) it holds that

$$\sup_{t \in J} H_{L^2}(X(t), \{\Theta\}) \leq M_3 \exp\{8M_2 \bar{T}\},$$

where

$$M_3 = \left[ 8H^4_{L^2}(X_0, \{\Theta\}) + 2^8 \bar{T}^2 \left( \int_{J \times \Omega} C_F(s) ds \times dP \right)^2 + 2^8 \left( \int_{J \times \Omega} C_G(s) ds \times dP \right)^2 \right]^{1/4},$$

and $M_2$ is defined like in (3.5).

Proof. By (P4), the triangle inequality, Lemma 2.1 and (P1) we have for $t \in J$

$$H^4_{L^2}(X(t), \{\Theta\}) \leq H^4_{L^2}(X_0, \{\Theta\}) + H^4_{L^2} \left( (S) \int_0^t F(s, X(s)) ds + (S) \int_0^t G(s, X(s)) dB(s), \{\Theta\} \right).$$

Hence

$$H^4_{L^2}(X(t), \{\Theta\}) \leq 8H^4_{L^2}(X_0, \{\Theta\}) + 8H^4_{L^2} \left( (S) \int_0^t F(s, X(s)) ds + (S) \int_0^t G(s, X(s)) dB(s), \{\Theta\} \right).$$

Thus

$$H^4_{L^2}(X(t), \{\Theta\}) \leq 8H^4_{L^2}(X_0, \{\Theta\}) + 2^6 H^4_{L^2} \left( (S) \int_0^t F(s, X(s)) ds, \{\Theta\} \right) + 2^6 H^4_{L^2} \left( (S) \int_0^t G(s, X(s)) dB(s), \{\Theta\} \right).$$

Due to Lemma 2.3

$$H^4_{L^2}(X(t), \{\Theta\}) \leq 8H^4_{L^2}(X_0, \{\Theta\}) + 2^6 \left[ 2t \int_{[0,t] \times \Omega} H^2_{\mathbb{R}^d}(F(s, X(s)), F(s, \{\Theta\})) ds \times dP \right. + \left. 2t \int_{[0,t] \times \Omega} H^2_{\mathbb{R}^d}(F(s, \{\Theta\}), \{\theta\}) ds \times dP \right]^2 + 2^6 \left[ 2 \int_{[0,t] \times \Omega} H^2_{\mathbb{R}^d}(G(s, X(s)), G(s, \{\Theta\})) ds \times dP \right. + \left. 2 \int_{[0,t] \times \Omega} H^2_{\mathbb{R}^d}(G(s, \{\Theta\}), \{\theta\}) ds \times dP \right]^2.
and due to assumptions (H2) and (H3)

\[ H_{L^2}^4(X(t), \{\Theta\}) \]

\[ \leq 8H_{L^2}^4(X_0, \{\Theta\}) + 2^6 \left[ 2t \int_{[0,t] \times \Omega} K_F(s)H_{L^2}^2(X(s), \{\Theta\})ds \times dP + 2t \int_{[0,t] \times \Omega} C_F(s)ds \times dP \right]^2 \]

Further

\[ H_{L^2}^2(X(t), \{\Theta\}) \]

\[ \leq 8H_{L^2}^4(X_0, \{\Theta\}) + 2^6 \left[ 2 \left( \int_{[0,t] \times \Omega} K_F^2(s)ds \times dP \int_{[0,t] \times \Omega} H_{L^2}^4(X(s), \{\Theta\})ds \times dP \right)^{1/2} \right. \]

\[ + 2t \int_{[0,t] \times \Omega} C_F(s)ds \times dP \right] \]

\[ + 2^6 \left[ 2 \left( \int_{[0,t] \times \Omega} K_G^2(s)ds \times dP \int_{[0,t] \times \Omega} H_{L^2}^4(X(s), \{\Theta\})ds \times dP \right)^{1/2} \right. \]

\[ + 2 \int_{[0,t] \times \Omega} C_G(s)ds \times dP \right] \]

Hence

\[ H_{L^2}^4(X(t), \{\Theta\}) \]

\[ \leq 8H_{L^2}^4(X_0, \{\Theta\}) + 2^6 \left[ 4t^2 \int_{[0,t] \times \Omega} K_F^2(s)ds \times dP \int_{[0,t] \times \Omega} H_{L^2}^4(X(s), \{\Theta\})ds \times dP \right. \]

\[ + 4t^2 \left( \int_{[0,t] \times \Omega} C_F(s)ds \times dP \right)^2 \]

\[ + 2^6 \left[ 4 \int_{[0,t] \times \Omega} K_G^2(s)ds \times dP \int_{[0,t] \times \Omega} H_{L^2}^4(X(s), \{\Theta\})ds \times dP \right. \]

\[ + 4 \left( \int_{[0,t] \times \Omega} C_G(s)ds \times dP \right)^2 \]

\[ \leq M_3^4 + 32M_2 \int_0^t H_{L^2}^4(X(s), \{\Theta\})ds. \]

Applying the Gronwall inequality we infer that

\[ H_{L^2}^4(X(t), \{\Theta\}) \leq M_3^4 \exp\{32M_2t\} \]

for every \( t \in J \),

which yields the assertion. \( \square \)

It is worth mentioning that the theory of narrowing set-valued stochastic integral equations will be well-posed, if we show that the solutions to (3.2) do not change
much if the data of the equation have some small changes. Below we present some
studies in this direction.

Consider equation (3.2) and the same equation with another initial value \( \tilde{X}_0 \), i.e.
\[
(3.7) \quad X(t) + (S) \int_0^t (-1)F(s, X(s))ds + (S) \int_0^t (-1)G(s, X(s))dB(s) = \tilde{X}_0 \quad \text{for} \quad t \in I.
\]

Let \( X: J_1 \to \mathcal{K}_c^b(L^2) \) and \( Y: J_2 \to \mathcal{K}_c^b(L^2) \) denote the unique solutions (if they exist) to these equations, respectively, \( J_1 = [0, \tilde{T}_1], J_2 = [0, \tilde{T}_2] \) for some \( \tilde{T}_1, \tilde{T}_2 \in (0, T] \). Let \( J = J_1 \cap J_2 \).

**Theorem 3.6.** Let \( X_0, F, G \) satisfy the conditions (H1)–(H4). Assume also that \( \tilde{X}_0, F, G \) satisfy (H1)–(H4). Then
\[
\sup_{t \in J} H_{L^2}(X(t), Y(t)) \leq 2^{3/4} H_{L^2}(X_0, \tilde{X}_0) \exp\{2\bar{M}_2 \min\{\tilde{T}_1, \tilde{T}_2\}\},
\]
where \( \bar{M}_2 = 8(\min\{\tilde{T}_1, \tilde{T}_2\})^2 \int_{J \times \Omega} K_F^2(s)ds \times dP + 8 \int_{J \times \Omega} K_G^2(s)ds \times dP \).

**Proof.** Observe that for \( t \in J \) we have, accordingly to (P6), Lemma 2.1 and (P2),
\[
\begin{align*}
H_{L^2}^4(X(t), Y(t)) & \leq 8H_{L^2}^4(X_0, \tilde{X}_0) \\
& + 8 \left[ H_{L^2} \left( \int_0^t F(s, X(s))ds, \int_0^t F(s, Y(s))ds \right) \right]^4 \\
& + \bar{M}_2 \left( \int_0^t G(s, X(s))dB(s), \int_0^t G(s, Y(s))dB(s) \right) \\
& \leq 8H_{L^2}^4(X_0, \tilde{X}_0) \\
& + 64H_{L^2}^4 \left( \int_0^t F(s, X(s))ds, \int_0^t F(s, Y(s))ds \right) \\
& + 64H_{L^2}^4 \left( \int_0^t G(s, X(s))dB(s), \int_0^t G(s, Y(s))dB(s) \right).
\end{align*}
\]
Hence by Lemma 2.3 and the assumptions (H2) and (H3) we get
\[
H_{L^2}^4(X(t), Y(t)) \leq 8H_{L^2}^4(X_0, \tilde{X}_0) \\
+ 64t^2 \left( \int_{[0, t] \times \Omega} K_F(s)H_{L^2}^2(X(s), Y(s))ds \times dP \right)^2 \\
+ 64 \left( \int_{[0, t] \times \Omega} K_G(s)H_{L^2}^2(X(s), Y(s))ds \times dP \right)^2.
\]
Thus
\[
H_{L^2}(X(t), Y(t)) \leq 8H_{L^2}^4(X_0, \tilde{X}_0) + 8\bar{M}_2 \int_0^t H_{L^2}^4(X(s), Y(s))ds
\]
and by the Gronwall inequality
\[
H_{L^2}^4(X(t), Y(t)) \leq 8H_{L^2}^4(X_0, \tilde{X}_0) \exp\{8\bar{M}_2t\} \quad \text{for} \quad t \in J.
\]
This leads to
\[ \sup_{t \in J} H_{L^2}(X(t), Y(t)) \leq 2^{3/4} H_{L^2}(X_0, \tilde{X}_0) \exp\{2 \bar{M}_2 \min\{T_1, T_2\}\}. \]

By this assertion the stability of solution to (3.2) with respect to small changes of
initial value follows.

Now, let us consider equation (3.2) and equations (for $n \in \mathbb{N}$)
\begin{equation}
(3.8) \quad X(t) + (S) \int_0^t (-1)F_n(s, X(s))ds + (S) \int_0^t (-1)G_n(s, X(s))dB(s) = X_0
\end{equation}
for $t \in I$, with another coefficients $F_n$ and $G_n$. Let $X, X_n$ denote the unique solutions
(if they exist) to these equations, respectively. Assume that they all are defined on a
common interval $J = [0, T]$ with $T \in (0, T]$.

**Theorem 3.7.** Let $X_0, F, G$ satisfy the conditions (H1)–(H4). Assume also that
$\tilde{X}_0, F_n, G_n$ satisfy (H1)–(H4), in particular the conditions (H2) and (H3) are satisfied with the processes $K_{F_n}, K_{G_n}$ and $C_{F_n}, C_{G_n}$, respectively. Assume that there exist constants $S_F, S_G > 0$ such that for every $n \in \mathbb{N}$
\[ \int_{J \times \Omega} K_{F_n}^2(s)ds \times dP \leq S_F \text{ and } \int_{J \times \Omega} K_{G_n}^2(s)ds \times dP \leq S_G. \]
Suppose that for every $A \in \mathcal{K}_c^b(L^2)$
\[ \int_{J \times \Omega} H_{R^d}^2(F_n(s, A), F(s, A))ds \times dP \to 0 \text{ as } n \to \infty \text{ and } \]
\[ \int_{J \times \Omega} H_{R^d}^2(G_n(s, A), G(s, A))ds \times dP \to 0 \text{ as } n \to \infty. \]
Then for the solution $X : J \to \mathcal{K}_c^b(L^2)$ to (3.2) and the solutions $X_n : J \to \mathcal{K}_c^b(L^2)$ to
(3.8) it holds that
\[ \sup_{t \in J} H_{L^2}(X_n(t), X(t)) \to 0 \text{ as } n \to \infty. \]

**Proof.** Proceeding similarly as in the previous proofs we obtain for $t \in J$
\[ H_{L^2}^1(X_n(t), X(t)) \leq 8 H_{L^2}^1 \left( \int_0^t F_n(s, X_n(s))ds, \int_0^t F(s, X(s))ds \right) \]
\[ + 8 H_{L^2}^1 \left( \int_0^t G_n(s, X_n(s))dB(s), \int_0^t G(s, X(s))dB(s) \right). \]
Hence
\[ H_{L^2}^1(X_n(t), X(t)) \leq 32 \left[ t \int_{[0,t] \times \Omega} H_{R^d}^2(F_n(s, X_n(s)), F_n(s, X(s)))ds \times dP \right]^2 \]
\[ + 32 \left[ t \int_{[0,t] \times \Omega} H_{R^d}^2(F_n(s, X(s)), F(s, X(s)))ds \times dP \right]^2. \]
Thus by the Gronwall inequality for every $t \in J$

$$H^4_{L^2}(X_n(t), X(t)) \leq \left(32\bar{T}^2 S_F + 32 S_G\right) \int_0^t H^4_{L^2}(X_n(s), X(s))ds.$$
Consequently

\[
\sup_{t \in J} H_{L^2}(X_n(t), X(t)) \leq \left(32T^2 \left[ \int_{J \times \Omega} H^2_{\mathbb{R}^d}(F_n(s, X(s)), F(s, X(s)))ds \times dP \right] \right)^2
+ 32 \left[ \int_{J \times \Omega} H^2_{\mathbb{R}^d}(G_n(s, X(s)), G(s, X(s)))ds \times dP \right]^{1/4}
\times \exp \left\{ 8T(\bar{T}^2S_F + S_G) \right\}.
\]

Since the both the sequences of numbers \( \int_{J \times \Omega} H^2_{\mathbb{R}^d}(F_n(s, X(s)), F(s, X(s)))ds \times dP \) and \( \int_{J \times \Omega} H^2_{\mathbb{R}^d}(G_n(s, X(s)), G(s, X(s)))ds \times dP \) converge to zero by assumptions, we infer that \( \sup_{t \in J} H_{L^2}(X_n(t), X(t)) \) converges to zero as well.

Although the condition (H4) can be thought as a constricting one, it is crucial and indispensable in the studies of narrowing set-valued stochastic integral equations. The difficulties are caused by a requirement of existence of the Hukuhara differences. This condition is satisfied immediately in the case of single-valued and singleton defined data of equation (3.2). In this special case, the Hukuhara differences in (3.3) and (H4) reduce to the usual differences in the space \( L^2 \). But this is not the only case when (H4) is seen to be fulfilled. Notice that \( \mathbb{R} \) can be embedded into \( L^2(\Omega, \mathcal{A}, P; \mathbb{R}) \).

Hence the following deterministic, narrowing set-valued integral equation

\[(3.9) \quad X(t) + \int_0^t (-1)\Psi(s, X(s))ds = X_0 \quad \text{for } t \in I, \]

where \( \Psi: I \times \mathcal{K}_c^b(\mathbb{R}) \to \mathcal{K}_c^b(\mathbb{R}), X_0 \in \mathcal{K}_c^b(\mathbb{R}) \) and the integral is the set-valued Aumann integral, is a particular case of equation (3.2). The theory of deterministic, widening set-valued equations is examined widely in [25]. We shall show that for equation (3.9) the condition of type (H4) is satisfied. Assume that there exists a positive constant \( M \) such that for any \( (t, A) \in I \times \mathcal{K}_c^b(\mathbb{R}) \) it holds \( H_{\mathbb{R}}(\Psi(t, A), \{0\}) \leq M \). We claim that the sequence \( \{X_n\} \) described by \( X_0(t) = X_0 \) and \( X_n(t) = X_0 \ominus \int_0^t (-1)\Psi(s, X_{n-1}(s))ds \) (for \( n \in \mathbb{N} \)) is well defined on the interval \( J = [0, \bar{T}] \), where \( \bar{T} = \text{diam}X_0/(2M) \).

Indeed, for \( t \in J \) we have

\[
\text{diam} \left( (-1) \int_0^t \Psi(s, X_{n-1}(s))ds \right) = \text{diam} \left( \int_0^t \Psi(s, X_{n-1}(s))ds \right)
\leq 2 \int_0^t H_{\mathbb{R}}(\Psi(s, X_{n-1}(s)), \{0\})ds
\leq 2Mt \leq \text{diam}X_0.
\]

In the hyperspace \( \mathcal{K}_c^b(\mathbb{R}) \) we have: if \( \text{diam}A \geq \text{diam}B \) then \( A \ominus B \) exists, where \( A, B \in \mathcal{K}_c^b(\mathbb{R}) \). Hence the Hukuhara difference \( X_0 \ominus \int_0^t (-1)\Psi(s, X_{n-1}(s))ds \) exists for each \( t \in J \) and condition of the type (H4) is fulfilled. The equations of the type (3.9) are well suited (see [30]) in modeling a problem of number of radioactive nuclei in radioactive substances.
The studies presented in this paper treat of the narrowing set-valued stochastic integral equations (3.2) mainly. It is worth mentioning that under conditions (H1)–(H3) (without (H4)), all the above presented assertions can be repeated for the widening set-valued stochastic integral equations (3.1). Moreover, since (3.1) does not involve any condition on existence of Hukuhara differences, each result established for the (local or global) solutions of the narrowing equations can be repeated for global solutions of the widening equations. At this place we rewrite only one and the most important result on the existence and uniqueness of solution to (3.1). All the remaining counterparts of the results can also be proved.

**Proposition 3.8.** Let \( X_0 \in K^h_c(L^2_0) \), and \( F, G: I \times \Omega \times K^h_c(L^2) \to K^h_c(\mathbb{R}^d) \) satisfy conditions (H1)–(H3). Then equation (3.1) possesses a unique global solution \( X: I \to K^h_c(L^2) \).

This assertion can be proved using the sequence of the approximate solutions \( \{X_n\} \) defined as

\[
X_0(t) = X_0, \quad t \in I,
\]

and for \( n = 1, 2, \ldots \)

\[
X_n(t) = X_0 + (S) \int_0^t F(s, X_{n-1}(s))ds + (S) \int_0^t G(s, X_{n-1}(s))dB(s), \quad t \in I.
\]

The condition (H4) is not needed in Proposition 3.8 and does not apply to the remaining results which can be repeated for the widening set-valued stochastic integral equations with global solutions.

**REFERENCES**


