Dynamic Systems and Applications 24 (2015) 439-450

NONTRIVIAL SOLUTIONS FOR NEUMANN BOUNDARY VALUE PROBLEM OF SECOND ORDER IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS IN ORDERED BANACH SPACES

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ABSTRACT. This paper is devoted to study the existence of nontrivial solutions for second order Neumann boundary value problem with impulse effects in ordered Banach spaces. Under more general conditions of non-compactness measure and partial ordering, the existence of nontrivial solutions is obtained by employing the fixed point index theory of condensing mapping.

AMS (MOS) Subject Classification. 34A37; 34G20; 34K30.

1. Introduction

The theory of impulsive differential equations describes processes which experience a sudden change their state at certain moments. Processes with such a character arise naturally and often, especially in phenomena studied in physics, chemical, biological, population and dynamics, engineering and economics. The theory of impulsive differential equations has been emerging as an important area of investigation in the last few decades; see [1–8] and the references therein.

In this paper, we use the fixed point index theory of condensing mapping to discuss the existence of solutions to the Neumann boundary value problem (BVP) of second order nonlinear impulsive integro-differential equations of Fredholm type in an ordered Banach space E

$$\begin{cases} -u''(t) + Mu(t) = f(t, u(t), (Su)(t)), & t \in J,\, t \neq t_k, \\ -\Delta u'|_{t=t_k} = I_k(u(t_k)), & k = 1, 2, \dots, m, \\ u'(0) = u'(1) = \theta, \end{cases} \tag{1.1}$$

where $M > 0$ is a constant, $f \in C(J \times E \times E, E)$, $J = [0, 1]$; $0 < t_1 < t_2 < \cdots < t_m < 1$; $I_k \in C(E, E)$ is an impulsive function, $k = 1, 2, \dots, m$; θ denotes the zero element of E; and

$$(Su)(t) = \int_0^1 H(t, s)u(s)ds \tag{1.2}$$

Received June 17, 2014

is a Fredholm integral operator with integral kernel $H \in C(J \times J, \mathbb{R}^+)$; $\Delta u'|_{t=t_k}$ denotes the jump of $u'(t)$ at $t = t_k$, i.e., $\Delta u'|_{t=t_k} = u'(t_k^+) - u'(t_k^-)$, where $u'(t_k^+)$ and $u'(t_k^-)$ represent the right and left limits of $u'(t)$ at $t = t_k$, respectively. Let $PC(J, E) = \{u : J \to E \mid u(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1, 2, \ldots, m\}$. Evidently $PC(J, E)$ is a Banach space with the norm $\|u\|PC = \sup_{t\in J} \|u(t)\|$.

Let $J' = J\backslash\{t_1, t_2, \ldots, t_m\}$, $PC^1(J, E) = \{u \in PC(J, E) \cap C^1(J', E) \mid u'(t_k^+)$ and $u'(t_k^-)$ exist, $k = 1, 2, \ldots, m\}$. For $u \in PC^1(J, E)$, it is easy to see that the left derivative $u'_-(t_k)$ of $u(t)$ at $t = t_k$ exists and $u'_-(t_k) = u'(t_k^-)$, and set $u'(t_k) = u'(t_k^-)$, then $u' \in PC(J, E)$. Therefore, $PC^1(J, E) = \{u : J \to E \mid u \in PC(J, E), u' \in PC(J, E)\}$. Evidently, $PC^1(J, E)$ is also a Banach space with the norm $\|u\|PC^1 = \max\{\|u\|PC, \|u'\|PC\}$. If $u \in PC^1(J, E) \bigcap C^2(J', E)$ satisfy all the equalities of (1.1), we call u a solution of BVP (1.1).

Neumann boundary value problem were studied extensively, see [9–12] and the references therein. Jiang and Liu [9] applied Krasnoselskii's fixed point theorem to establish the existence of positive solution for Neumann boundary value problem in real space $\mathbb{R}$

$$\begin{cases} -u''(t) + Mu(t) = f(t, u(t)), & t \in J,\, t \neq t_k, \\ u'(0) = u'(1) = 0. \end{cases} \tag{1.3}$$

They proved that BVP (1.3) exists one positive solution provided $0 < M < \pi^2/4$ and one of the following conditions holds:

(I) $\lim_{x\to 0} \max_{t\in J} \frac{f(t,x)}{x} = 0$ and $\lim_{x\to\infty} \max_{t\in J} \frac{f(t,x)}{x} = +\infty$;
(II) $\lim_{x\to 0} \max_{t\in J} \frac{f(t,x)}{x} = +\infty$ and $\lim_{x\to\infty} \max_{t\in J} \frac{f(t,x)}{x} = 0$.

In [10], Lin et al. obtained the existence of multiple positive solutions of BVP (1.3) by using the Krasnoselskii's fixed point theorem. Sun and Li [11] obtained the existence of three positive solutions for BVP (1.3) via the Leggett-Williams fixed point theorem. Yao [12] obtained the existence of nontrivial sign-changing solutions of BVP (1.3) by applying the fixed point theorem of increasing operator on the order interval.

Recently, Lin and Jiang [8] studied the Dirichlet boundary value problems of second order impulsive differential equation in $\mathbb{R}$

$$\begin{cases} -u''(t) = f(t, u(t)), & t \in J,\, t \neq t_k, \\ -\Delta u'|_{t=t_k} = I_k(u(t_k)), & k = 1, 2, \ldots, m, \\ u(0) = u(1) = 0. \end{cases} \tag{1.4}$$

They applied a fixed point index theorem in cones to establish the existence of multiple positive solutions of BVP (1.4).

However, all these results mentioned above are in real spaces $\mathbb{R}$. In this paper, we will deal with the existence of positive solutions and negative solutions of BVP (1.1) in abstract Banach spaces by applying the fixed point index theory of condensing mapping. The positive solution of BVP (1.1) is that: $u(t) > \theta$, $t \in J$; the negative solution is that: $u(t) < \theta$, $t \in J$. As far as we know, no works yet exist for the existence of nontrivial solutions to BVP (1.1) in Banach spaces by applying the fixed point index theory of condensing mapping.

2. **Preliminaries**

Let E be an ordered Banach space with the norm $\|\cdot\|$ and partial order "$\leq$", whose positive cone $K = \{x \in E \mid x \geq \theta\}$ is normal with normal constant 1. Let $C(J,E)$ denote the Banach space of all continuous E-value functions on interval J with the norm $\|u\|C = \max_{t\in J} \|u(t)\|$. Evidently, $C(J,E)$ is also an ordered Banach space induced by the convex cone $C(J,K) = \{u \in C(J,E) \mid u(t) \geq \theta, t \in J\}$, and $C(J,K)$ is also a normal cone.

Evidently, $PC(J,E)$ is also an ordered Banach space with the partial order "$\leq$" induced by the positive cone $PC(J,K) = \{u \in PC(J,E) \mid u(t) \geq \theta, t \in J\}$. $PC(J,K)$ is also normal with the same normal constant 1. Since no confusion may occur, we denote by $\alpha(\cdot)$ the Kuratowski measure of noncompactness on both the bounded sets of E and $PC(J,E)$. For the details of the definition and properties of the measure of noncompactness, we refer to the monograph [13].

The following lemmas will be used in the prove of our main results.

Lemma 2.1 ([13])**.** *Let $B \subset PC(J,E)$ be bounded and equicontinuous. Then $\alpha(B(t))$ is continuous on J, and*

$$\alpha(B) = \max_{t\in J} \alpha(B(t)) = \alpha(B(J)). \tag{2.1}$$

Lemma 2.2 ([13])**.** *Let E be a Banach space. Assume that Ω is a bounded closed and convex set in E, and $Q : \Omega \to \Omega$ is condensing. Then Q has a fixed point in Ω.*

Lemma 2.3 ([14])**.** *Let $B = \{u_n\} \subset PC(J,E)$ be a bounded and countable set. Then $\alpha(B(t))$ is Lebesgue integral on J, and*

$$\alpha\Big(\Big\{\int_J u_n(t)dt \mid n \in \mathbb{N}\Big\}\Big) \leq 2\int_J \alpha(B(t))dt. \tag{2.2}$$

Lemma 2.4 ([15, 16])**.** *Let $D \subset E$ be bounded. Then there exists a countable set $D_0 \subset D$, such that $\alpha(D) \leq 2\alpha(D_0)$.*

Let G is Green function of boundary value problem $-x'' + Mx = 0$, $x'(0) = x'(1) = 0$, then

$$G(t,s) = \begin{cases} \frac{\cosh(m(1-t))\cosh(ms)}{m\sinh m}, & 0 \leq s \leq t \leq 1, \\ \frac{\cosh(mt)\cosh(m(1-s))}{m\sinh m}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{2.3}$$

where $m=\sqrt{M}$, $\cosh x=\frac{e^x+e^{-x}}{2}$, $\sinh x=\frac{e^x-e^{-x}}{2}$, and it is easy to see that $G(t,s)\leq G(s,s)$, $0\leq t,s\leq 1$.

By simply calculation, we know that

$$\frac{1}{m\sinh m}\leq G(t,s)\leq\frac{\cosh m}{m\sinh m}. \tag{2.4}$$

For the convenience, set $\overline{H}=\max_{t,s\in J}H(t,s)$, $\underline{H}=\min_{t,s\in J}H(t,s)$, and assume that $\overline{H}>0$ throughout the paper, where H is integral kernel of the Fredholm integral operator S defined by (1.2).

To prove our main results, for any $h\in PC(J,E)$, we consider the linear impulsive differential equation with Neumann boundary condition in E

$$\begin{cases} -u''(t)+Mu(t)=h(t), & t\in J',\\ -\Delta u'|_{t=t_k}=y_k, & k=1,2,\ldots,m,\\ u'(0)=u'(1)=\theta, \end{cases} \tag{2.5}$$

where $y_k\in E$, $k=1,2,\ldots,m$.

Lemma 2.5. *If $u\in PC^1(J,E)$ is a solution of the following impulsive integral equation*

$$u(t)=\int_0^1 G(t,s)h(s)ds+\sum_{k=1}^m G(t,t_k)y_k, \tag{2.6}$$

then $u\in PC^1(J,E)\cap C^2(J',E)$ is a unique solution of problem (2.5).

Proof. If $u\in PC^1(J,E)$ is a solution of impulsive integral equation (2.6). Direct differentiation of (2.6) implies for $t\neq t_k$,

$$u'(t)=\int_0^1 G'_t(t,s)h(s)ds+\sum_{k=1}^m G'_t(t,t_k)y_k, \tag{2.7}$$

where

$$G'_t(t,s)=\begin{cases} \frac{\cosh(m(1-t))\sinh(ms)}{\sinh m}, & 0\leq s\leq t\leq 1,\\ -\frac{\cosh(mt)\sinh(m(1-s))}{\sinh m}, & 0\leq t\leq s\leq 1. \end{cases} \tag{2.8}$$

Evidently

$$-\Delta u'|_{t=t_k}=y_k,\ \ k=1,2,\ldots,m, \tag{2.9}$$

and

$$-u''(t)+Mu(t)=h(t). \tag{2.10}$$

Hence $u\in PC^1(J,E)\cap C^2(J',E)$, $-\Delta u'|_{t=t_k}=y_k$, and it is easy to verify that $u'(0)=u'(1)=\theta$. $\square$

Now, we define an operator $A : PC(J,E) \to PC(J,E)$ as follows

$$(Au)(t) = \int_0^1 G(t,s)f(s,u(s),(Su)(s))ds + \sum_{k=1}^{m} G(t,t_k)I_k(u(t_k)). \tag{2.11}$$

Lemma 2.6. *Suppose the following condition is satisfied:*

(H_0) *For any $R > 0$, $f(J \times \overline{B}(\theta,R) \times \overline{B}(\theta,R))$ is bounded, and exist constants L, $\overline{L} > 0$, $L_k > 0$ $(k = 1,2,\dots,m)$ with $\frac{4(L+\overline{L}\cdot\overline{H})}{M} + \frac{2\cosh m}{m\sinh m}\sum_{k=1}^{m} L_k < 1$, such that for $\forall\, t \in J$ and D, $\overline{D} \subset \overline{B}(\theta,R)$*

$$\alpha(f(t,D,\overline{D})) \le L\alpha(D) + \overline{L}\alpha(\overline{D}), \quad \alpha(I_k(D)) \le L_k\alpha(D), \tag{2.12}$$

then the operator $A : PC(J,E) \to PC(J,E)$ is condensing.

Proof. By the definition of operator A, we know that A maps the bounded set of $PC(J,E)$ to bounded and equicontinuous set. For any bounded and noncompactness set $B \subset PC(J,E)$, let $R = \sup\{\|u\|_{PC} \mid u \in B\}$, for any $t \in J$, $B(t) \in \overline{B}(\theta,R)$. By Lemma 2.4, there exist a countable set $B_1 = \{u_n\} \subset B$, such that

$$\alpha(A(B)) \le 2\alpha(A(B_1)). \tag{2.13}$$

For $\forall\, t \in J$, by Lemma 2.3 and assumption (H_0), we have

$$\begin{aligned}
\alpha(A(B_1(t))) &= \alpha\Big(\Big\{\int_0^1 G(t,s)f(s,u_n(s),(Su_n)(s))ds \\
&\qquad + \sum_{k=1}^{m} G(t,t_k)I_k(u_n(t_k)) | n = 1,2,\dots\Big\}\Big) \\
&\le 2\int_0^1 G(t,s)\alpha(f(s,B_1(s),(SB_1)(s)))ds \\
&\qquad + \sum_{k=1}^{m} G(t,t_k)\alpha(I_k(B_1(t_k))) \\
&\le 2\int_0^1 G(t,s)\Big\{L\alpha(B_1(s)) + \overline{L}\alpha\Big(\int_0^1 H(s,\tau)B_1(\tau)d\tau\Big)\Big\}ds \\
&\qquad + \frac{\cosh m}{m\sinh m}\sum_{k=1}^{m} L_k\alpha(B_1(s)) \\
&\le 2(L + \overline{L}\cdot\overline{H})\int_0^1 G(t,s)ds\,\alpha(B_1) + \frac{\cosh m}{m\sinh m}\sum_{k=1}^{m} L_k\alpha(B_1) \\
&= \Big(\frac{2(L+\overline{L}\cdot\overline{H})}{M} + \frac{\cosh m}{m\sinh m}\sum_{k=1}^{m} L_k\Big)\alpha(B_1).
\end{aligned}$$

Since $A(B_1)$ is equicontinuous, by Lemma 2.1, $\alpha(A(B_1)) = \max_{t\in J}\alpha(A(B_1)(t))$. Combining this with (2.13) and the condition (H_0), we have

$$\alpha(A(B)) \le 2\alpha(A(B_1)) \le \Big(\frac{4(L+\overline{L}\cdot\overline{H})}{M} + \frac{2\cosh m}{m\sinh m}\sum_{k=1}^{m} L_k\Big)\alpha(B_1) < \alpha(B). \tag{2.14}$$

Hence, the operator $A : PC(J,E) \to PC(J,E)$ is condensing. $\square$

Let P be a cone in $PC(J,E)$ which is defined as

$$P = \{u \in PC(J,K) | \; u(t) \geq \sigma u(\tau), \, t, \tau \in J\}, \tag{2.15}$$

where $\sigma = \frac{1}{\cosh m}$.

Then we have the following lemma.

Lemma 2.7. *If $f(J \times K \times K) \subset K$, $I_k(K) \subset K$, $k = 1, 2, \ldots, m$, then $A(P) \subset P$.*

Proof. For any $u \in P$ and $t \in J$, by (2.11), we have

$$\begin{aligned}(Au)(\tau) &= \int_0^1 G(\tau,s) f(s,u(s),(Su)(s))ds + \sum_{k=1}^m G(\tau,t_k) I_k(u(t_k)) \\ &\leq \frac{\cosh m}{m \sinh m}\Big[\int_0^1 f(s,u(s),(Su)(s))ds + \sum_{k=1}^m I_k(u(t_k))\Big].\end{aligned}$$

Therefore, from (2.11) and the above inequality, we know that

$$\begin{aligned}(Au)(t) &= \int_0^1 G(t,s) f(s,u(s),(Su)(s))ds + \sum_{k=1}^m G(t,t_k) I_k(u(t_k)) \\ &\geq \frac{1}{m \sinh m}\Big[\int_0^1 f(s,u(s),(Su)(s))ds + \sum_{k=1}^m I_k(u(t_k))\Big] \\ &\geq \sigma (Au)(\tau).\end{aligned}$$

Hence, $Au \in P$, namely $A(P) \subset P$. $\square$

Therefore, if $f(J \times K \times K) \subset K$, $I_k(K) \subset K$, $k = 1, 2, \ldots, m$, then $A : P \to P$ is condensing. And from Lemma 2.2 we know that the positive solution of BVP (1.1) is equivalent to the nonzero fixed point of A in P. For $0 < r < R < \infty$, let

$$\Omega_1 = \{u \in PC(J,E) \mid \|u\|PC < r\}, \quad \Omega_2 = \{u \in PC(J,E) \mid \|u\|PC < R\}.$$

Then

$$\partial\Omega_1 = \{u \in PC(J,E) \mid \|u\|PC = r\}, \quad \partial\Omega_2 = \{u \in PC(J,E) \mid \|u\|PC = R\}.$$

Hence, the fixed point of A in $P \cap (\Omega_2 \backslash \overline{\Omega}_1)$ is the positive solution of BVP (1.1). In this paper, we will seek the nonzero fixed point of A in $P \cap (\Omega_2 \backslash \overline{\Omega}_1)$ by applying the fixed point index theory of condensing mapping.

Let E be a Banach space, and let P be a closed convex cone of E. Assume that Ω is a bounded open subset of E and let $\partial\Omega$ be its boundary, $P \cap \overline{\Omega} \neq \emptyset$. Let $\Phi : P \cap \overline{\Omega} \to P$ be a condensing mapping. If $\Phi u \neq u$ for every $u \in P \cap \partial\Omega$, then the fixed point index $i(\Phi, P \cap \Omega, P)$ is defined. If $i(\Phi, P \cap \Omega, P) \neq 0$, then Φ has a fixed point in $P \cap \Omega$.

The following two lemmas are needed in our argument.

Lemma 2.8 ([17]). *Let $\Omega \subset E$ be a bounded open set, $\theta \in \Omega$, $\Phi : P \cap \overline{\Omega} \to P$ be a condensing mapping, if Φ satisfy*

$$u \neq \lambda \Phi u, \qquad \forall\, u \in P \cap \partial\Omega,\ 0 < \lambda \leq 1.$$

Then $i(\Phi, P \cap \Omega, P) = 1$.

Lemma 2.9 ([17]). *Let $\Omega \subset E$ be a bounded open set, $\theta \in \Omega$, $\Phi : P \cap \overline{\Omega} \to P$ be a condensing mapping. If there exist $v_0 \in P$, $v_0 \neq \theta$, such that*

$$u - \Phi u \neq \tau v_0, \qquad \forall\, u \in P \cap \partial\Omega,\ \tau \geq 0.$$

Then $i(\Phi, P \cap \Omega, P) = 0$.

3. Main results

Theorem 3.1. *Assume that $f : J \times E \times E \to E$ is continuous, $I_k : E \to E$ is continuous, and the condition (H_0) is satisfied. If $f(J \times K \times K) \subset K$, $I_k(K) \subset K$, $k = 1, 2, \dots, m$, and f and I_k satisfy the following conditions:*

- (H_1) *There exist constants a, $b > 0$ with $a + b\overline{H} < M$, $\delta > 0$ and $q_k > 0$ with $\sum_{k=1}^{m} q_k < \sigma(M - a - b\overline{H})$ such that for any x, $y \in K_\delta$*

$$f(t, x, y) \leq ax + by, \quad I_k(x) \leq q_k x,\ k = 1, 2, \dots, m,$$

- (H_2) *There exist c, $d > 0$ with $c + d\underline{H} > M$ and $h_0 \in PC(J, K)$ such that for any x, $y \in K$*

$$f(t, x, y) \geq cx + dy - h_0(t).$$

Then the BVP (1.1) has at least one positive solution.

Proof. Let $0 < r < \min\{\delta, \frac{\delta}{\overline{H}}\}$, where δ is the constant in (H_1). For $\forall\, t \in J$, $u \in P \cap \partial\Omega_1$, we have $\|u(t)\|PC \leq \|u\|PC = r < \delta$, $\|(Su)(t)\|PC \leq \|Su\|PC \leq \overline{H}r < \delta$. Hence by the condition (H_1), we know that

$$f(t, u(t), (Su)(t)) \leq au(t) + b(Su)(t) \quad I_k(u(t_k)) \leq q_k u(t_k),\ k = 1, 2, \dots, m. \tag{3.1}$$

Next, we show that the operator A satisfy

$$u \neq \lambda A u, \quad \forall\, u \in P \cap \partial\Omega_1,\ 0 < \lambda \leq 1. \tag{3.2}$$

If this is not true, then there exist $u_0 \in P \cap \partial\Omega_1$ and $0 < \lambda_0 \leq 1$, such that $u_0 = \lambda_0 A u_0$. By the definition of operator A, we get that

$$\begin{cases} -u_0''(t) + Mu_0(t) = \lambda_0 f(t, u_0(t), (Su_0)(t)), & t \in J,\ t \neq t_k, \\ -\Delta u_0'|_{t=t_k} = \lambda_0 I_k(u_0(t_k)), & k = 1, 2, \dots, m, \\ u_0'(0) = u_0'(1) = \theta, \end{cases} \tag{3.3}$$

Integrating (3.3) from 0 to 1, use integration by parts in the left sides, we have

$$-\lambda_0 \sum_{k=1}^{m} I_k(u_0(t_k)) + M \int_0^1 u_0(t)dt = \lambda_0 \int_0^1 f(t, u_0(t), (Su_0)(t))dt. \tag{3.4}$$

By (3.1), we get that

$$(M - a - b\overline{H})\sigma u_0(\tau) \le (M - a - b\overline{H}) \int_0^1 u_0(t)dt \le \sum_{k=1}^{m} q_k u_0(t_k). \tag{3.5}$$

Hence, we have

$$\sigma(M - a - b\overline{H}) \le \sum_{k=1}^{m} q_k, \tag{3.6}$$

which is a contradiction. Therefore, (3.2) is satisfied, by Lemma 2.8, we have

$$i(A, P \cap \Omega_1, P) = 1. \tag{3.7}$$

On the other hand, let $v_0(t) \equiv e \in P$, $\|e\|PC = 1$. Next we show that there exists a constant $R > r$, which is large enough, such that

$$u - Au \ne \tau v_0, \quad \forall\ u \in P \cap \partial\Omega_2,\ \tau \ge 0. \tag{3.8}$$

In fact, if there exist $u_0 \in P \cap \partial\Omega_2$ and $\tau_0 \ge 0$, such that $u_0 - Au_0 = \tau_0 v_0$, i.e. $u_0 = Au_0 + \tau_0 v_0$. Then by the definition of operator A, we have

$$\begin{cases} -u_0''(t) + Mu_0(t) = f(t, u_0(t), (Su_0)(t)) + \tau_0 v_0(t), & t \in J,\ t \ne t_k, \\ -\Delta u_0'|_{t=t_k} = I_k(u_0(t_k)), & k = 1, 2, \ldots, m, \\ u_0'(0) = u_0'(1) = \theta, \end{cases} \tag{3.9}$$

Integrating (3.9) from 0 to 1, use integration by parts in the left sides, we have

$$-\sum_{k=1}^{m} I_k(u_0(t_k)) + M \int_0^1 u_0(t)dt = \int_0^1 f(t, u_0(t), (Su_0)(t))dt + \tau_0 \int_0^1 v_0(t)dt. \tag{3.10}$$

By the condition (H_2), we get that

$$\int_0^1 h_0(t)dt \ge (c + d\underline{H} - M) \int_0^1 \sigma u_0(\tau)dt = \sigma(c + d\underline{H} - M)u_0(\tau). \tag{3.11}$$

Therefore, by the normality of cone K, we have

$$\|u_0\|PC \le \frac{\|h_0\|PC}{\sigma(c + d\underline{H} - M)} := \overline{R}. \tag{3.12}$$

If we chose $R > \max\{\delta, \frac{\delta}{\overline{H}}, \overline{R}\}$, then (3.8) is satisfied. By Lemma 2.9, we have

$$i(A, P \cap \Omega_2, P) = 0. \tag{3.13}$$

Hence, from (3.7) and (3.13), we get that

$$i(A, P \cap (\Omega_2 \backslash \overline{\Omega}_1), P) = i(A, P \cap \Omega_2, P) - i(A, P \cap \Omega_1, P) = -1 \tag{3.14}$$

Therefore, A has a fixed point u in $P \cap (\Omega_2 \backslash \overline{\Omega}_1)$, which means $u(t)$ is a positive solution of the BVP (1.1). $\square$

Theorem 3.2. *Assume that* $f : J \times E \times E \to E$ *is continuous,* $I_k : E \to E$ *is continuous, and the condition* (H_0) *is satisfied. If* $f(J \times K \times K) \subset K$, $I_k(K) \subset K$, $k = 1, 2, \ldots, m$, *and* f *and* I_k *satisfy the following conditions:*

(H_3) *There exist constants* c, $d > 0$ *with* $c + d\underline{H} > M$ *and* $\delta > 0$, *such that for any* x, $y \in K_\delta$

$$f(t, x, y) \geq cx + dy,$$

(H_4) *There exist* a, $b > 0$ *with* $a + b\overline{H} < M$, $q_k > 0$ *with* $\sum_{k=1}^{m} q_k < \sigma(M - a - b\overline{H})$ *and* $h_0 \in PC(J, K)$, *such that for any* x, $y \in K$

$$f(t, x, y) \leq ax + by + h_0(t), \quad I_k(x) \leq q_k x,\ k = 1, 2, \ldots, m.$$

Then the BVP (1.1) *has at least one positive solution.*

Proof. Let $0 < r < \min\{\delta, \frac{\delta}{\overline{H}}\}$, where δ is the constant in (H_3). We show that

$$u - Au \neq \tau v_0, \qquad \forall\, u \in P \cap \partial\Omega_1,\ \tau \geq 0, \tag{3.15}$$

where, $v_0(t) \equiv e \in P$, $\|e\|PC = 1$. If (3.15) is not true, then there exist $u_0 \in P \cap \partial\Omega_1$ and $\tau_0 \geq 0$, such that $u_0 - Au_0 = \tau_0 v_0$, i.e. $u_0 = Au_0 + \tau_0 v_0$. By the definition of operator A, we have

$$\begin{cases} -u_0''(t) + Mu_0(t) = f(t, u_0(t), (Su_0)(t)) + \tau_0 v_0(t), & t \in J,\ t \neq t_k, \\ -\Delta u_0'|_{t=t_k} = I_k(u_0(t_k)), & k = 1, 2, \ldots, m, \\ u_0'(0) = u_0'(1) = \theta. \end{cases} \tag{3.16}$$

Integrating (3.16) from 0 to 1, use integration by parts in the left sides, we have

$$-\sum_{k=1}^{m} I_k(u_0(t_k)) + M\int_0^1 u_0(t)dt = \int_0^1 f(t, u_0(t), (Su_0)(t))dt + \tau_0 \int_0^1 v_0(t)dt. \tag{3.17}$$

By the condition (H_3), we get that

$$M\int_0^1 u_0(t)dt \geq -\sum_{k=1}^{m} I_k(u_0(t_k)) + M\int_0^1 u_0(t)dt \geq (c + d\underline{H})\int_0^1 u_0(t)dt. \tag{3.18}$$

Therefore, we get that

$$(M - c - d\underline{H})\int_0^1 u_0(t)dt \geq 0, \tag{3.19}$$

which is an contradiction. Hence, (3.15) is satisfied, by Lemma 2.9, we have

$$i(A, P \cap \Omega_1, P) = 0. \tag{3.20}$$

On the other hand, We show that there exists a constant $R > r$, which is large enough, such that

$$u \neq \lambda Au, \qquad \forall\, u \in P \cap \partial\Omega_2,\ 0 < \lambda \leq 1. \tag{3.21}$$

In fact, if there exist $u_0 \in P \cap \partial\Omega_2$ and $0 < \lambda_0 \leq 1$, such that $u_0 = \lambda_0 A u_0$. Then by the definition of operator A, we have

$$\begin{cases} -u_0''(t) + Mu_0(t) = \lambda_0 f(t, u_0(t), (Su_0)(t)), & t \in J,\ t \neq t_k, \\ -\Delta u_0'|_{t=t_k} = \lambda_0 I_k(u_0(t_k)), & k = 1, 2, \dots, m, \\ u_0'(0) = u_0'(1) = \theta. \end{cases} \tag{3.22}$$

Integrating (3.22) from 0 to 1, use integration by parts in the left sides, we have

$$-\lambda_0 \sum_{k=1}^{m} I_k(u_0(t_k)) + M \int_0^1 u_0(t)dt = \lambda_0 \int_0^1 f(t, u_0(t), (Su_0)(t))dt. \tag{3.23}$$

By the condition (H_4), we get that

$$\sigma(M-a-b\overline{H})u_0(\tau) - \sum_{k=1}^{m} q_k u_0(t_k) \leq (M-a-b\overline{H}) \int_0^1 u_0(t)dt - \sum_{k=1}^{m} q_k u_0(t_k) \leq \int_0^1 h_0(t)dt. \tag{3.24}$$

Therefore, by the normality of the cone K, we have

$$\|u_0\|PC \leq \frac{\|h_0\|PC}{\sigma(M-a-b\overline{H}) - \sum_{k=1}^{m} q_k} := \overline{R}. \tag{3.25}$$

If we chose $R > \max\{\delta, \frac{\delta}{\underline{H}}, \overline{R}\}$, then (3.21) is satisfied. By Lemma 2.8, we have

$$i(A, P \cap \Omega_2, P) = 1. \tag{3.26}$$

Hence, from (3.20) and (3.26), we get that

$$i(A, P \cap (\Omega_2 \backslash \overline{\Omega}_1), P) = i(A, P \cap \Omega_2, P) - i(A, P \cap \Omega_1, P) = 1. \tag{3.27}$$

Therefore, A has a fixed point u in $P \cap (\Omega_2 \backslash \overline{\Omega}_1)$, which means $u(t)$ is a positive solution of the BVP (1.1). $\square$

Next, we discuss the existence of the negative solutions of the BVP (1.1). The following assumptions are needed:

$(H_1)'$ There exist constants a, $b > 0$ with $a + b\overline{H} < M$, $\delta > 0$ and $q_k > 0$ with $\sum_{k=1}^{m} q_k < \sigma(M-a-b\overline{H})$ such that for any x, $y \in -K_\delta$

$$f(t, x, y) \geq -ax - by, \quad I_k(x) \geq -q_k x,\ k = 1, 2, \dots, m,$$

$(H_2)'$ There exist c, $d > 0$ with $c + d\underline{H} > M$ and $h_0 \in PC(J, K)$ such that for any x, $y \in -K$

$$f(t, x, y) \leq -cx - dy + h_0(t),$$

$(H_3)'$ There exist constants c, $d > 0$ with $c + d\underline{H} > M$ and $\delta > 0$ such that for any x, $y \in -K_\delta$

$$f(t, x, y) \leq -cx - dy,$$

$(H_4)'$ There exist a, $b > 0$ with $a + b\overline{H} < M$, $q_k > 0$ with $\sum_{k=1}^{m} q_k < \sigma(M - a - b\overline{H})$ and $h_0 \in PC(J, K)$ such that for any x, $y \in -K$

$$f(t, x, y) \geq -ax - by - h_0(t), \quad I_k(x) \geq -q_k x,\ k = 1, 2, \ldots, m.$$

Theorem 3.3. *Assume that $f : J \times E \times E \to E$ is continuous, $I_k : E \to E$ is continuous, and the condition (H_0) is satisfied. If $f(J \times \{-K\} \times \{-K\}) \subset -K$, $I_k(-K) \subset -K$, $k = 1, 2, \ldots, m$, and f and I_k satisfy the conditions $(H_1)'$ and $(H_2)'$. Then the BVP (1.1) has at least one negative solution.*

Proof. Let $\overline{f}(t, x, y) = -f(t, -x, -y)$, $\overline{I}_k(x) = -I_k(-x)$, $t \in J$, x, $y \in E$. If f and I_k satisfy the conditions $(H_1)'$ and $(H_2)'$, it is easy to verify that $\overline{f}$ and $\overline{I}_k$ satisfy the conditions (H_1) and (H_2). Hence, by Theorem 3.1, the problem

$$\begin{cases} -u''(t) + Mu(t) = \overline{f}(t, u(t), (Su)(t)), & t \in J,\ t \neq t_k, \\ -\Delta u'|_{t=t_k} = \overline{I}_k(u(t_k)), & k = 1, 2, \ldots, m, \\ u'(0) = u'(1) = \theta \end{cases} \tag{3.28}$$

has a positive solution $\widetilde{u}$. Evidently, $-\widetilde{u}$ is the negative solution of the BVP (1.1). □

Similar with Theorem 3.3, we have the following result:

Theorem 3.4. *Assume that $f : J \times E \times E \to E$ is continuous, $I_k : E \to E$ is continuous, and the condition (H_0) is satisfied. If $f(J \times \{-K\} \times \{-K\}) \subset -K$, $I_k(-K) \subset -K$, $k = 1, 2, \ldots, m$, and f and I_k satisfy the conditions $(H_3)'$ and $(H_4)'$. Then the BVP (1.1) has at least one negative solution.*

Acknowledgments. The authors would like to express sincere thanks to the anonymous referee for his/her carefully reading the manuscript and valuable comments and suggestions.

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