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# ON THE CONVERGENCE OF SOLUTIONS TO NABLA DYNAMIC EQUATIONS ON TIME SCALES

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**ABSTRACT.** This paper deals with the convergence of solutions to nabla dynamic equations $x^{\nabla} = f(t,x)$ on time scales $\{\mathbb{T}_n\}_{n=1}^{\infty}$ when this sequence converges to the time scale $\mathbb{T}$. The convergent rate of solutions is evaluated when $f$ satisfies the Lipschitz condition in both variables. A new approach to the approximation of dynamic equations on time scales is derived by a general view, especially the implicit Euler method for differential equations. Some examples are given to illustrate results.

**Keywords.** Nabla dynamic equations, time scales, the convergence of solutions.

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## 1. Introduction

Many scientific disciplines, for instance in physics, chemistry, biology, and economics, are described by ordinary differential equations (ODEs). Thus, finding solutions of ODEs is important both in theory and practice. However, almost ODEs can not be solved analytically. Therefore, it is necessary to find a numeric approximation to the solutions in science and engineering. The Euler methods is very well-known because it is simple and useful to perform this, see [4, 9, 11, 18]. For solving the stiff initial value problem

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \\ x(t_0) = x_0, \end{cases} \tag{1.1}$$

at each step $[t_{m-1}, t_m]$, the implicit Euler approximation of (1.1) is

$$x_m = x_{m-1} + hf(t_m, x_m), \tag{1.2}$$

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where $t_m = t_{m-1} + h$ and $x_m$ is the approximative value of $x(t)$ at $t = t_m$. The quantity $e_m := |x(t_m) - x_m|$ is called the error of this method after $m-1$ time steps which characterizes the difference between the approximative solution and the exact solution. The interested problem is how the error $e_m$ can be estimasted when the mesh step $h$ tends to zero. We have known that $e_m$ tends to zero as $h$ tends to zero with some added assumptions on $f$. Further, it has been shown that the implicit Euler method is more stable than explicit one (see [9, 11, 14]).

On the other hand, in recent years, the theory of the analysis on time scales has received a lot of attentions, see [1, 2, 7, 12, 13, 15, 16] in order to unify the continuous and discrete analysis. By using the notation of the analysis on time scales, equations (1.1) and (1.2) can be rewritten under the form

$$\begin{cases} x^{\nabla}(t) = f(t, x(t)), \\ x(t_0) = x_0, \end{cases} \tag{1.3}$$

with the time $t$ belongs to the time scales $\mathbb{T} = \mathbb{R}$ or $\mathbb{T}_h = h\mathbb{Z}$. Thus, using the implicit Euler method means we consider equation (1.1) on the time $\mathbb{T}_h$, which is "close" to $\mathbb{T} = \mathbb{R}$ in some sense by view of analysis on time scale. Then the problem of the error estimation above can be restated as follows: How do the solutions of (1.3) on $\mathbb{T}_h$ converge to the solution of (1.3) on $\mathbb{T} = \mathbb{R}$ as the mesh step $h$ tends to zero? In case of positive answer, what is the convergent rate of the error $e_m$?

Following this idea in a more general context, in this article, we will consider the behavior of solutions of equation (1.3) on time scales $\{\mathbb{T}_n\}_{n=1}^{\infty}$ when $\mathbb{T}_n$ tends to $\mathbb{T}$ by the Hausdorff distance. Under assumption that $f(t, x)$ satisfies the Lipschitz condition in the variable $x$, we will prove that

$$x_n(t) \to x(t) \text{ as } \mathbb{T}_n \to \mathbb{T}, \tag{1.4}$$

where $\{x_n(t)\}_{n=1}^{\infty}, x(t)$ are solutions of equation (1.3) on time scales $\{\mathbb{T}_n\}_{n=1}^{\infty}, \mathbb{T}$, respectively. Moreover, if $f$ satisfies the Lipschitz condition in both variables $t$ and $x$ then the convergent rate of solutions is estimated as a same degree as the Hausdorff distance between $\mathbb{T}_n$ and $\mathbb{T}$, i.e.,

$$\|x_n(t) - x(t)\| \leqslant C_2 d_H(\mathbb{T}, \mathbb{T}_n), \text{ for all } t \in \mathbb{T} \cap \mathbb{T}_n : t_0 \leqslant t \leqslant T. \tag{1.5}$$

By using these results, we obtain the convergence of the implicit Euler method as a consequence. It can be considered as a new and general approach to the convergence problems of the approximative solutions.

This paper is organized as follows. Section 2 summarizes some preliminary results on time scales. In Section 3, we study the convergence of solutions of nabla dynamic equations on time scales. The main results of the paper are derived here. In Section 4, we give some illustrating examples and show the convergence of the implicit Euler method. The last section deals with some conclusions.

## 2. **Preliminaries**

Let $\mathbb{T}$ be a closed subset of $\mathbb{R}$, endowed with the topology inherited from the standard topology on $\mathbb{R}$. Let $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, $\mu(t) = \sigma(t) - t$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, $\nu(t) = t - \rho(t)$ (supplemented by $\sup \emptyset = \inf \mathbb{T}$, $\inf \emptyset = \sup \mathbb{T}$). A point $t \in \mathbb{T}$ is said to be *right-dense* if $\sigma(t) = t$, *right-scattered* if $\sigma(t) > t$, *left-dense* if $\rho(t) = t$, *left-scattered* if $\rho(t) < t$ and *isolated* if $t$ is simultaneously right-scattered and left-scattered.

A function $f$ defined on $\mathbb{T}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left dense points. Similarly, $f$ is *ld-continuous* if it is continuous at every left-dense point and if the right-sided limit exists in every right-dense point. It is easy to see that a function is continuous if and only if it is both $rd$-continuous and $ld$-continuous. A function $f$ from $\mathbb{T}$ to $\mathbb{R}$ is *regressive* (*respectively positively regressive*) if $1 - \nu(t)f(t) \neq 0$ (respectively $1 - \nu(t)f(t) > 0$) for every $t \in \mathbb{T}$.

**Definition 2.1** (Nabla Derivative)**.** A function $f : \mathbb{T} \to \mathbb{R}^d$ is called *nabla differentiable* at $t$ if there exists a vector $f^\nabla(t)$ such that for all $\epsilon > 0$

$$\|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)\| \leqslant \epsilon|\rho(t) - s|$$

for all $s \in (t - \delta, t + \delta) \cap \mathbb{T}$ and for some $\delta > 0$. The vector $f^\nabla(t)$ is called the *nabla derivative* of $f$ at $t$.

If $\mathbb{T} = \mathbb{R}$ then the nabla derivative is $f^{'}(t)$ from continuous calculus; if $\mathbb{T} = \mathbb{Z}$ then the nabla derivative is the backward difference, $\nabla f(t) = f(t) - f(t-1)$, from discrete calculus.

Let $f$ be a ld-continuous function and $a, b \in \mathbb{T}$. Then, the Riemann integral $\int_a^b f(s)\nabla s$ exists (see, e.g., [6, 7, 10]). In case $b \notin \mathbb{T}$, writing $\int_a^b f(s)\nabla s$ means $\int_a^{\overline{b}} f(s)\nabla s$, where $\overline{b} = \max\{t < b : t \in \mathbb{T}\}$.

Consider the dynamic equation on the time scale $\mathbb{T}$

$$\begin{cases} x^\nabla(t) = f(t, x), \\ x(t_0) = x_0, \end{cases} \tag{2.1}$$

where $f : \mathbb{T} \times \mathbb{R}^d \to \mathbb{R}^d$. If $f$ is ld-continuous and satisfies the Lipschitz condition in the variable $x$ with a positively regressive Lipschitz coefficient then the problem (2.1) has a unique solution. For the existence, uniqueness and extensibility of solution of equation (2.1) we refer to [5, 7].

For any regressive $ld$-continuous functions $p(\cdot)$ from $\mathbb{T}$ to $\mathbb{R}$, the solution of the dynamic equation $x^\nabla = p(t)x$, with the initial condition $x(s) = 1$, defines a so-called exponential function. We denote this exponential function by $\widehat{e}_p(\mathbb{T}; t, s)$. For the properties of exponential function $\widehat{e}_p(\mathbb{T}; t, s)$ the interested reader can see [1] and [7]. To simplify notations, we write $\widehat{e}_p(\mathbb{T}; t, s)$ for $\widehat{e}_p(t, s)$ if there is no confusion. It is known that for any positively regressive number $\alpha$, we have the estimate

$$0 < \widehat{e}_\alpha(t, t_0) \leqslant e^{C_0\alpha(t-t_0)}, \tag{2.2}$$

where $C_0$ is a constant depending on the bounds of $\nu$ (see [1, 2, 7]).

It is easy to see that if $f(t, x)$ is a continuous function in $(t, x)$ then $x(t)$ is a solution to (2.1) if and only if

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,\nabla s.$$

**Lemma 2.2** (Gronwall-Bellman lemma, see [7, 10])**.** *Let $x(t)$ be a continuous function and $k > 0, x_0 \in \mathbb{R}$. Assume that $x(t)$ satisfies the inequality*

$$x(t) \leqslant x_0 + k\int_{t_0}^{t} x(s)\nabla s, \ \textit{for all } t \in \mathbb{T}, t \geqslant t_0, \tag{2.3}$$

*where $k$ is positively regressive. Then, the relation*

$$x(t) \leqslant x_0\widehat{e}_k(t, t_0) \ \textit{for all } t \in \mathbb{T}, t \geqslant t_0 \tag{2.4}$$

*holds.*

Fix $t_0 \in \mathbb{R}$. Let $\mathcal{T} = \mathcal{T}(t_0)$ be the set of all time scales with bounded graininess such that $t_0 \in \mathbb{T}$ for all $\mathbb{T} \in \mathcal{T}$. We endow $\mathcal{T}$ with the Hausdorff distance, that is Hausdorff distance between two time scales $\mathbb{T}_1$ and $\mathbb{T}_2$ defined by

$$d_H(\mathbb{T}_1, \mathbb{T}_2) := \max\{\sup_{t_1\in\mathbb{T}_1} d(t_1, \mathbb{T}_2), \sup_{t_2\in\mathbb{T}_2} d(t_2, \mathbb{T}_1)\}, \tag{2.5}$$

where

$$d(t_1, \mathbb{T}_2) = \inf_{t_2\in\mathbb{T}_2} |t_1 - t_2| \text{ and } d(t_2, \mathbb{T}_1) = \inf_{t_1\in\mathbb{T}_1} |t_2 - t_1|.$$

For properties of the Hausdorff distance, we refer interested readers to [3, 8, 17].

## 3. Convergence of solutions

In this section, we consider the dynamic equation (2.1) on the sequence $\{\mathbb{T}_n\}_{n\in\mathbb{N}}$ of time scales satisfying:

$$\lim_{n\to\infty} \mathbb{T}_n = \mathbb{T},$$

by the Hausdorff distance and $t_0 \in \mathbb{T}_n$ for any $n \in \mathbb{N}$. We define the time scale

$$\widehat{\mathbb{T}} = \overline{\cup_{n\in\mathbb{N}}\mathbb{T}_n \cup \mathbb{T}}. \tag{3.1}$$

Assume that $f$ is continuous on $\widehat{\mathbb{T}}$ and satisfies the Lipschitz condition in the variable $x$, that is there exists a constant $k > 0$ such that

$$\|f(t,x) - f(t,y)\| \leqslant k\|x-y\|, \quad \text{for all } t \in \widehat{\mathbb{T}} : t_0 \leqslant t \leqslant T \text{ and } x, y \in \mathbb{R}^d, \tag{3.2}$$

where $k$ is positively regressive. By these assumptions, the initial value problems (IVPs)

$$x_n^{\nabla}(t) = f(t, x_n(t)),\ t \in \mathbb{T}_n, \quad x_n(t_0) = x_0, \quad n = 1, 2, \ldots \tag{3.3}$$

and

$$x^{\nabla}(t) = f(t, x(t)),\ t \in \mathbb{T}, \quad x(t_0) = x_0, \tag{3.4}$$

have a unique solution $x_n(t)$ defined on $\mathbb{T}_n$ (respectively solution $x(t)$ defined on $\mathbb{T}$). It is clear that the solutions of the IVPs (3.3) and (3.4) are given by

$$x_n(t) = x_0 + \int_{t_0}^{t} f(s, x_n(s))\nabla_n s \tag{3.5}$$

and

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\nabla s, \tag{3.6}$$

respectively, where $\int f \nabla_n s$ denotes the integral on time scale $\mathbb{T}_n$.

The following lemma gives the uniformly bounded property of solutions of the IVPs (3.3) and (3.4) on different time scales.

**Lemma 3.1.** *Let $x_{\mathbb{S}}(t)$ be the solution to the dynamic equation*

$$x^{\nabla}(t) = f(t, x(t)),\ t \in \mathbb{S}, \quad x(t_0) = x_0.$$

*Then, for any $T > t_0$ one has*

$$\sup_{\mathbb{S} \in \mathfrak{T}; \mathbb{S} \subset \widehat{\mathbb{T}}} \ \sup_{t \in \mathbb{S}: t_0 \leqslant t \leqslant T} \|x_{\mathbb{S}}(t)\| < \infty. \tag{3.7}$$

*Proof.* Let $\mathbb{S} \in \mathfrak{T}; \mathbb{S} \subset \widehat{\mathbb{T}}$. For any $t \in \mathbb{S}$, we have

$$\begin{aligned}
\|x_{\mathbb{S}}(t)\| &= \left\| x_0 + \int_{t_0}^{t} f(s, x_{\mathbb{S}}(s))\nabla s \right\| \\
&\leqslant \|x_0\| + \left\| \int_{t_0}^{t} f(s,0)\nabla s \right\| + \left\| \int_{t_0}^{t} f(s, x_{\mathbb{S}}(s))\nabla s - \int_{t_0}^{t} f(s,0)\nabla s \right\| \\
&\leqslant \|x_0\| + \int_{t_0}^{t} \|f(s,0)\|\nabla s + \int_{t_0}^{t} \|f(s, x_{\mathbb{S}}(s)) - f(s,0)\|\nabla s.
\end{aligned} \tag{3.8}$$

By virtue of continuity of $f$ on $\widehat{\mathbb{T}}$, one has $C = \sup_{s \in \widehat{\mathbb{T}}; t_0 \leqslant s \leqslant T} \|f(s,0)\| \leqslant \infty$. Hence,

$$\int_{t_0}^{t} \|f(s,0)\|\nabla s \leqslant C(t - t_0) \leqslant C(T - t_0).$$

Moreover, since $f$ satisfies the Lipschitz condition (3.2),

$$\int_{t_0}^{t} \|f(s, x_{\mathbb{S}}(s)) - f(s, 0)\| \nabla s \leqslant k \int_{t_0}^{t} \|x_{\mathbb{S}}(s)\| \nabla s.$$

Therefore,

$$\|x_{\mathbb{S}}(t)\| \leqslant \|x_0\| + C(T - t_0) + k \int_{t_0}^{t} \|x_{\mathbb{S}}(s)\| \nabla s.$$

By using the Gronwall-Bellman lemma, we get

$$\|x_{\mathbb{S}}(t)\| \leqslant \big(\|x_0\| + C(T - t_0)\big) \widehat{e}_k(\mathbb{S}; t, t_0),$$

where $\widehat{e}_k(\mathbb{S}; t, t_0)$ is exponential function defined on $\mathbb{S}$. Thus, by (2.2), we obtain

$$\sup_{\mathbb{S} \in \widehat{\mathbb{T}}} \sup_{t \in \mathbb{S}: t_0 \leqslant t \leqslant T} \|x_{\mathbb{S}}(t)\| \leqslant \big(\|x_0\| + C(T - t_0)\big) e^{C_0 k (T - t_0)} < \infty.$$

The proof is complete. □

Let $n \in \mathbb{N}$, we denote by $\rho_n$ the backward jump operator on the time scale $\mathbb{T}_n$. For any $t \in \mathbb{T}$, there exists a unique $s \in \mathbb{T}_n$, say $s = \gamma^{\mathbb{T}, \mathbb{T}_n}(t)$, such that either $s = t$ or $t \in (\rho_n(s), s)$. It is easy to check that the function $\gamma^{\mathbb{T}, \mathbb{T}_n}(t)$ is $ld$-continuous on $\mathbb{T}$. Also, there exists $t_n^* = t_n^*(t) \in \mathbb{T}_n$ satisfying

$$|t - t_n^*| = d(t, \mathbb{T}_n) = \inf\{|t - s| : s \in \mathbb{T}_n\}. \tag{3.9}$$

We choose $t_n^* = \gamma^{\mathbb{T}, \mathbb{T}_n}(t)$ if $|t - \gamma^{\mathbb{T}, \mathbb{T}_n}(t)| = d(t, \mathbb{T}_n)$, otherwise $t_n^* = \rho_n(\gamma^{\mathbb{T}, \mathbb{T}_n}(t))$. Define

$$f_n(t, x) = f(\gamma^{\mathbb{T}, \mathbb{T}_n}(t), x), \quad t \in \mathbb{T}; x \in \mathbb{R}^d, \tag{3.10}$$

$$\widetilde{x}_n(t) = x(\gamma^{\mathbb{T}, \mathbb{T}_n}(t)), \quad t \in \mathbb{T}. \tag{3.11}$$

Assume that $\mathbb{T}_n \subset \mathbb{T}$. Then, by the definition of Riemann integral on time scales, we have

$$\int_{t_0}^{t} f(s, x(s)) \nabla_n s = \int_{t_0}^{t} f_n(s, \widetilde{x}_n(s)) \nabla s,$$

for any $t \in \mathbb{T}_n$ (see, e.g. [6, 7]).

Since $d_H(\mathbb{T}, \mathbb{T}_n) \to 0$ as $n \to \infty$, we can assume that $t_n^*(t) < T + 1$ when $t \leqslant T$. By Lemma 3.1, $A = \sup_{\mathbb{S} \in \widehat{\mathbb{T}}} \sup_{t \in \mathbb{S}: t_0 \leqslant t \leqslant T+1} \|x_{\mathbb{S}}(t)\| < \infty$, and hence let

$$M = \sup\{\|f(t, x)\| : t_0 \leqslant t \leqslant T + 1, \|x\| < A\}.$$

Now, we need the following lemmas for proving the convergence of the solution sequence $\{x_n(t)\}$ of the IVPs (3.3) when $\mathbb{T}_n$ tends to $\mathbb{T}$.

**Lemma 3.2.** *Let $x_n(t)$, $n = 1, 2, \dots$ be solutions to the* IVPs (3.3) *and $x(t)$ be the solution to the* IVP (3.4). *Assume that $\mathbb{T}_n \subset \mathbb{T}$. Then,*

$$\|x(t) - x_n(t)\| \leqslant \delta_T^{(n)} \widehat{e}_k(\mathbb{T}_n; t, t_0), \quad t \in \mathbb{T}_n : t_0 \leqslant t \leqslant T, \tag{3.12}$$

*and*

$$\|x(t) - x_n(t_n^*)\| \leqslant \delta_{T+1}^{(n)} \widehat{e}_k(\mathbb{T}_n; t_n^*, t_0) + M d_H(\mathbb{T}, \mathbb{T}_n), \quad t \in \mathbb{T} : t_0 \leqslant t \leqslant T, \tag{3.13}$$

*where $t_n^*$ is defined by* (3.9) *and*

$$\delta_t^{(n)} = \int_{t_0}^{t} \|f(s,x(s)) - f_n(s,\widetilde{x}_n(s))\|\nabla s. \tag{3.14}$$

*Proof.* For any $t \in \mathbb{T}_n, t \leqslant T$ we have

$$\begin{aligned}
x(t) - x_n(t) &= \int_{t_0}^{t} f(s,x(s))\nabla s - \int_{t_0}^{t} f(s,x_n(s))\nabla_n s \\
&= \int_{t_0}^{t} f(s,x(s))\nabla s - \int_{t_0}^{t} f(s,x(s))\nabla_n s + \int_{t_0}^{t} [f(s,x(s)) - f(s,x_n(s))]\nabla_n s \\
&= \int_{t_0}^{t} [f(s,x(s)) - f_n(s,\widetilde{x}_n(s))]\nabla s + \int_{t_0}^{t} [f(s,x(s)) - f(s,x_n(s))]\nabla_n s.
\end{aligned}$$

By virtue of Lipschitz condition

$$\|f(s,x(s)) - f(s,x_n(s))\| \leqslant k\|x(s) - x_n(s)\|,$$

it follows that

$$\begin{aligned}
\|x(t) - x_n(t)\| &\leqslant \int_{t_0}^{t} \|f(s,x(s)) - f_n(s,\widetilde{x}_n(s))\|\nabla s + k\int_{t_0}^{t} \|x(s) - x_n(s)\|\nabla_n s \\
&\leqslant \delta_T^{(n)} + k\int_{t_0}^{t} \|x(s) - x_n(s)\|\nabla_n s.
\end{aligned}$$

By using Gronwall-Bellman lemma, we obtain (3.12).

If $t \in \mathbb{T}, t \leqslant T$ then

$$\|x(t) - x_n(t_n^*)\| \leqslant \|x(t) - x(t_n^*)\| + \|x(t_n^*) - x_n(t_n^*)\|.$$

Since $t_n^* \leqslant T+1$ and $\sup_{t_0 \leqslant t \leqslant T+1} \|x(t))\| \leqslant A$,

$$\|x(t_n^*) - x_n(t_n^*)\| \leqslant \delta_{T+1}^{(n)} \widehat{e}_k(\mathbb{T}_n; t_n^*, t_0), \quad t \in \mathbb{T} : t_0 \leqslant t \leqslant T.$$

Further,

$$\|x(t) - x(t_n^*)\| = \left\| \int_{t}^{t_n^*} f(s,x(s))\nabla s \right\| \leqslant M|t - t_n^*| \leqslant M d_H(\mathbb{T},\mathbb{T}_n).$$

Summing up, (3.13) holds. The proof is complete. □

**Lemma 3.3.** *Assume that $\mathbb{T}_n \subset \mathbb{T}$. For each $\epsilon$ and $T \in \mathbb{T}$, there exists $\theta = \theta(\epsilon, T)$ such that if $d_H(\mathbb{T},\mathbb{T}_n) < \theta$ then*

$$\delta_T^{(n)} \leqslant (T - t_0)\epsilon + \frac{2M(T - t_0)}{\theta} d_H(\mathbb{T},\mathbb{T}_n), \tag{3.15}$$

*where $\delta_T^{(n)}$ is defined by* (3.14).

*Proof.* Since $f$ is continuous, $f$ is uniformly continuous on $[t_0, T] \times B(0, A)$ where $B(0, A)$ is the ball with the center 0 and radius $A$. Therefore, for each $\epsilon$, there exists $\delta = \delta(\epsilon)$ such that if $|t_1 - t_2| + \|x_1 - x_2\| < \delta$ then

$$\|f(t_1, x_1) - f(t_2, x_2)\| \leqslant \epsilon \text{ on } [t_0, T] \times B(0, A).$$

We choose $\theta = \theta(\epsilon) = \dfrac{\delta(\epsilon)}{M+1}$. If $\gamma^{\mathbb{T},\mathbb{T}_n}(t) - t < \theta$ then

$$\|x(t) - \widetilde{x}_n(t)\| = \| \int_t^{\gamma^{\mathbb{T},\mathbb{T}_n}(t)} f(s, x(s))\nabla s\| \leqslant M(\gamma^{\mathbb{T},\mathbb{T}_n}(t) - t) < M\theta,$$

and

$$|t - \gamma^{\mathbb{T},\mathbb{T}_n}(t)| + \|x(t) - \widetilde{x}_n(t)\| < (M+1)\theta = \delta.$$

This implies that if $\gamma^{\mathbb{T},\mathbb{T}_n}(t) - t < \theta$ then

$$\|f(t, x(t)) - f_n(t, \widetilde{x}_n(t))\| < \epsilon.$$

We see that the number of values $s \in \mathbb{T}_n$ satisfying $t_0 \leqslant s \leqslant T$ and

$$\{t \in \mathbb{T} : \rho_n(s) < t < s, s - t \geqslant \theta\} \neq \emptyset,$$

is less than or equal to $[\frac{T-t_0}{\theta}]$. Assume that these values are $s_1 < s_2 < \cdots < s_r$ with $r \leqslant [\frac{T-t_0}{\theta}]$. In case $d_H(\mathbb{T}, \mathbb{T}_n) < \theta$, we see that if $t \in \mathbb{T}$ such that $\rho_n(s_i) < t < s_i, s_i - t \geqslant \theta$ then

$$t - \rho_n(s_i) = d(t, \mathbb{T}_n) \leqslant d_H(\mathbb{T}, \mathbb{T}_n).$$

Let

$$\tau_i = \max\{t \in \mathbb{T} : \rho_n(s_i) < t < s_i, s_i - t \geqslant \theta\}; i = \overline{1, r}.$$

It is clear $\tau_i - \rho_n(s_i) \leqslant d_H(\mathbb{T}, \mathbb{T}_n)$. Further,

$$\begin{aligned}
\delta_T^{(n)} &= \int_{t_0}^{T} \|f(s, x(s)) - f_n(s, \widetilde{x}_n(s))\|\nabla s \\
&= \int_{t_0}^{\rho_n(s_1)} \|f(s, x(s)) - f_n(s, \widetilde{x}_n(s))\|\nabla s + \sum_{i=1}^{r} \int_{\rho_n(s_i)}^{\tau_i} \|f(s, x(s)) - f_n(s, \widetilde{x}_n(s))\|\nabla s \\
&+ \sum_{i=1}^{r-1} \int_{\tau_i}^{\rho_n(s_{i+1})} \|f(s, x(s)) - f_n(s, \widetilde{x}_n(s))\|\nabla s + \int_{\tau_r}^{T} \|f(s, x(s)) - f_n(s, \widetilde{x}_n(s))\|\nabla s.
\end{aligned}$$

Since $\|f(s, x(s)) - f_n(s, \widetilde{x}_n(s))\| < \epsilon$ for all $s \in [t_0, \rho_n(s_1)] \cup (\tau_i, \rho_n(s_{i+1})] \cup (\tau_r, T]$

$$\int_{t_0}^{\rho(s_1)} \|f(s, x(s)) - f_n(s, \widetilde{x}_n(s))\|\nabla s \leqslant \epsilon(\rho(s_1) - t_0) \leqslant \epsilon(\tau_1 - t_0),$$

$$\int_{\tau_i}^{\rho_n(s_{i+1})} \|f(s, x(s)) - f_n(s, \widetilde{x}_n(s))\|\nabla s \leqslant \epsilon(\rho_n(s_{i+1}) - \tau_i) \leqslant \epsilon(\tau_{i+1} - \tau_i),$$

$$\int_{\tau_r}^{T} \|f(s, x(s)) - f_n(s, \widetilde{x}_n(s))\|\nabla s \leqslant \epsilon(\rho(s_1) - t_0) \leqslant \epsilon(T - \tau_r),$$

On the other hand, for $i = 1, 2, \ldots, r$ we have

$$\int_{\rho_n(s_i)}^{\tau_i} \|f(s, x(s)) - f_n(s, \widetilde{x}_n(s))\|\nabla s \leqslant 2M(\tau_i - \rho_n(s_i)) \leqslant 2Md_H(\mathbb{T}, \mathbb{T}_n).$$

Thus, we obtain

$$\begin{aligned}\delta_T^{(n)} &\leqslant \epsilon(\tau_1 - t_0) + \epsilon(T - \tau_r) + \sum_{i=1}^{r-1} \epsilon(\tau_{i+1} - \tau_i) + \sum_{i=1}^{r} 2Md_H(\mathbb{T}, \mathbb{T}_n) \\ &= \epsilon(T - t_0) + 2rMd_H(\mathbb{T}, \mathbb{T}_n) \leqslant \epsilon(T - t_0) + \frac{2M(T - t_0)}{\theta} d_H(\mathbb{T}, \mathbb{T}_n).\end{aligned}$$

The proof is complete. □

We are now derive the convergence theorem for the IVPs (3.3) and (3.4).

**Theorem 3.4.** *Let the sequence of time scales* $\{\mathbb{T}_n\}_{n=1}^{\infty}$ *satisfy* $\lim_{n\to\infty} \mathbb{T}_n = \mathbb{T}$. *Let* $x_n(t)$, $n = 1, 2, \ldots$ *be the solutions to the* IVPs (3.3) *and* $x(t)$ *be the solution to the* IVP (3.4). *Then, for any* $T > t_0$ *we have*

$$\lim_{n\to\infty} \sup_{t\in\mathbb{T}; t_0 \leqslant t \leqslant T} \|x(t) - x_n(t_n^*)\| = 0, \tag{3.16}$$

*where* $t_n^*$ *is defined by* (3.9).

*Proof.* Firstly, we assume that $\mathbb{T}_n \subset \mathbb{T}$ for all $n \in \mathbb{N}$. From Lemma 3.2, it follows that

$$\begin{aligned}\|x(t) - x_n(t_n^*)\| &\leqslant \delta_{T+1}^{(n)} \widehat{e}_k(\mathbb{T}_n; t_n^*, t_0) + Md_H(\mathbb{T}_n, \mathbb{T}) \leqslant \delta_{T+1}^{(n)} e^{C_0 k(t_n^* - t_0)} + Md_H(\mathbb{T}_n, \mathbb{T}) \\ &\leqslant \delta_{T+1}^{(n)} e^{C_0 k(T+1-t_0)} + Md_H(\mathbb{T}_n, \mathbb{T}),\end{aligned}$$

for any $t_0 \leqslant t \leqslant T$. By Lemma 3.3, we get $\lim_{n\to\infty} \delta_{T+1}^{(n)} = 0$. Therefore, (3.16) holds. In the general case, we put

$$\widehat{\mathbb{T}}_n = \mathbb{T}_n \cup \mathbb{T}.$$

Then, it is easy to see that

$$d_H(\mathbb{T}, \mathbb{T}_n) = \max\{d_H(\widehat{\mathbb{T}}_n, \mathbb{T}), d_H(\widehat{\mathbb{T}}_n, \mathbb{T}_n)\}. \tag{3.17}$$

Let $\widehat{x}_n(t)$ be the solution to equation (2.1) on time scale $\widehat{\mathbb{T}}_n$. For $t \in \mathbb{T}$, we have

$$\|x(t) - x_n(t_n^*)\| \leqslant \|\widehat{x}_n(t) - x(t)\| + \|\widehat{x}_n(t) - x_n(t_n^*)\|.$$

Since $\mathbb{T} \subset \widehat{\mathbb{T}}_n$ and $\mathbb{T}_n \subset \widehat{\mathbb{T}}_n$, we can apply Lemma 3.2 to obtain

$$\|\widehat{x}_n(t) - x(t)\| \leqslant \widehat{\delta}_T^{(n1)} e^{C_0 k(T-t_0)}, \quad \|\widehat{x}_n(t) - x_n(t_n^*)\| \leqslant \widehat{\delta}_{T+1}^{(n2)} e^{C_0 k(T+1-t_0)} + Md_H(\widehat{\mathbb{T}}_n, \mathbb{T}_n),$$

where

$$\begin{aligned}\widehat{\delta}_T^{(n1)} &= \int_{t_0}^{T} \|f(s, \widehat{x}_n(s)) - f(\gamma^{\widehat{\mathbb{T}}_n, \mathbb{T}}(s), \widehat{x}_n(\gamma^{\widehat{\mathbb{T}}_n, \mathbb{T}}(s)))\| \nabla_{\widehat{\mathbb{T}}_n} s, \\ \widehat{\delta}_{T+1}^{(n2)} &= \int_{t_0}^{T+1} \|f(s, \widehat{x}_n(s)) - f(\gamma^{\widehat{\mathbb{T}}_n, \mathbb{T}_n}(s), \widehat{x}_n(\gamma^{\widehat{\mathbb{T}}_n, \mathbb{T}_n}(s)))\| \nabla_{\widehat{\mathbb{T}}_n} s.\end{aligned} \tag{3.18}$$

By Lemma 3.1, Lemma 3.3 and equality (3.17), we imply that $\widehat{\delta}_T^{(n1)} \to 0, \widehat{\delta}_{T+1}^{(n2)} \to 0$ as $n \to \infty$. Thus, (3.16) holds. The proof is complete. □

For estimating the convergent rate, we need the following lemma.

**Lemma 3.5.** *Assume that* $\mathbb{T}_n \subset \mathbb{T}$. *Then, we have*

$$\int_{t_0}^{T} (\gamma^{\mathbb{T},\mathbb{T}_n}(s) - s)\nabla s \leqslant 2(T - t_0)d_H(\mathbb{T}, \mathbb{T}_n).$$

*Proof.* With the value $\theta = d_H(\mathbb{T}, \mathbb{T}_n)$, we follow a similar way as in the proof of Lemma 3.3 to construct the sequence $s_1, s_2, \ldots, s_r$ and the sequence $\tau_1, \tau_2, \ldots, \tau_r$ satisfying

$$\rho_n(s_1) < \tau_1 < s_1 < \cdots < \rho_n(s_r) < \tau_r < s_r.$$

Note that $\gamma^{\mathbb{T},\mathbb{T}_n}(s) - s \leqslant s_i - \rho_n(s_i) = \nu_n(s_i)$ for all $s \in [\rho_n(s_i), \tau_i]$ and $\gamma^{\mathbb{T},\mathbb{T}_n}(s) - s \leqslant d_H(\mathbb{T}, \mathbb{T}_n)$ for all $s \in [t_0, \rho_n(s_1)] \cup (\tau_i, \rho_n(s_{i+1})] \cup (\tau_r, T]$. Therefore,

$$\int_{\rho_n(s_i)}^{\tau_i} (\gamma^{\mathbb{T},\mathbb{T}_n}(s) - s)\nabla s \leqslant \nu_n(s_i)(\tau_i - \rho_n(s_i)) \leqslant \nu_n(s_i)d_H(\mathbb{T}, \mathbb{T}_n).$$

Thus, we get

$$\begin{aligned}\int_{t_0}^{T} (\gamma^{\mathbb{T},\mathbb{T}_n}(s) - s)\nabla s &\leqslant (T - t_0)d_H(\mathbb{T}, \mathbb{T}_n) + \sum_{i=1}^{r} \nu_n(s_i)d_H(\mathbb{T}, \mathbb{T}_n)\\ &\leqslant 2(T - t_0)d_H(\mathbb{T}, \mathbb{T}_n).\end{aligned}$$

The proof is complete. □

Assume further that $f(t, x)$ satisfies the Lipschitz condition in both variables $t$ and $x$, that is

$$\|f(t, x) - f(s, y)\| \leqslant k(|t - s| + \|x - y\|), \quad \text{for all } s, t \in \mathbb{T} \text{ and } x, y \in \mathbb{R}^d. \tag{3.19}$$

We now estimate the convergent rate of approximation.

**Theorem 3.6.** *Assume that assumption* (3.19) *is satisfied. Let* $x_n(t)$, $n = 1, 2, \ldots$ *be solutions of the* IVPs (3.3) *and* $x(t)$ *be the solution of the* IVP (3.4). *If* $t \in \mathbb{T} : t_0 \leqslant t < T$ *then*

$$\|x(t) - x_n(t_n^*)\| \leqslant C_1 d_H(\mathbb{T}, \mathbb{T}_n), \tag{3.20}$$

*where* $C_1 = 2k(2T + 1 - 2t_0)(M + 1)e^{C_0 k(T+1-t_0)} + M$ *and* $t_n^*$ *is defined by* (3.9). *Moreover, if* $t \in \mathbb{T} \cap \mathbb{T}_n : t_0 \leqslant t < T$ *then*

$$\|x(t) - x_n(t)\| \leqslant C_2 d_H(\mathbb{T}, \mathbb{T}_n),$$

*where* $C_2 = 4k(T - t_0)(M + 1)e^{C_0 k(T-t_0)}$.

*Proof.* Let

$$\widehat{\mathbb{T}}_n = \mathbb{T}_n \cup \mathbb{T},$$

and $\widehat{x}_n(t)$ be the solution of equation (2.1) on the time scale $\widehat{\mathbb{T}}_n$. It is showed in Theorem 3.4, for $t \in \mathbb{T} : t_0 \leqslant t \leqslant T$, we have

$$\begin{aligned}\|x(t) - x_n(t_n^*)\| &\leqslant \|\widehat{x}_n(t) - x(t)\| + \|\widehat{x}_n(t) - x_n(t_n^*)\|,\\ &\leqslant (\widehat{\delta}_T^{(n1)} + \widehat{\delta}_{T+1}^{(n2)})e^{C_0 k(T+1-t_0)} + M d_H(\widehat{\mathbb{T}}_n, \mathbb{T}_n),\end{aligned}$$

where $\widehat{\delta}_t^{(n1)}, \widehat{\delta}_t^{(n2)}$ are given by (3.18). Note that if $t \in \mathbb{T} \cap \mathbb{T}_n : t_0 \leqslant t \leqslant T$ then $\|x(t) - x_n(t)\| \leqslant (\widehat{\delta}_T^{(n1)} + \widehat{\delta}_T^{(n2)}) e^{C_0 k(T-t_0)}$.
Since $f(t,x)$ satisfies the Lipschitz condition (3.19),

$$\begin{aligned}\widehat{\delta}_T^{(n1)} &= \int_{t_0}^{T} \|f(s, \widehat{x}_n(s)) - f(\gamma^{\widehat{\mathbb{T}}_n,\mathbb{T}}(s), \widehat{x}_n(\gamma^{\widehat{\mathbb{T}}_n,\mathbb{T}}(s)))\| \nabla_{\widehat{\mathbb{T}}_n} s \\ &\leqslant k \int_{t_0}^{T} (|s - \gamma^{\widehat{\mathbb{T}}_n,\mathbb{T}}(s)| + \|\widehat{x}_n(s) - \widehat{x}_n(\gamma^{\widehat{\mathbb{T}}_n,\mathbb{T}}(s))\|) \nabla_{\widehat{\mathbb{T}}_n} s.\end{aligned}$$

We have

$$\begin{aligned}\|\widehat{x}_n(s) - \widehat{x}_n(\gamma^{\widehat{\mathbb{T}}_n,\mathbb{T}}(s))\| &= \Big\| \int_{\gamma^{\widehat{\mathbb{T}}_n,\mathbb{T}}(s)}^{s} f(u, \widehat{x}_n(\gamma^{\widehat{\mathbb{T}}_n,\mathbb{T}}(u))) \nabla_{\widehat{\mathbb{T}}_n} u \Big\| \\ &\leqslant M |s - \gamma^{\widehat{\mathbb{T}}_n,\mathbb{T}}(s)|.\end{aligned}$$

Therefore, by Lemma 3.5, we get

$$\widehat{\delta}_T^{(n1)} \leqslant k(M+1) \int_{t_0}^{T} |s - \gamma^{\widehat{\mathbb{T}}_n,\mathbb{T}}(s)| \nabla_{\widehat{\mathbb{T}}_n} s \leqslant 2k(M+1)(T-t_0) d_H(\widehat{\mathbb{T}}_n, \mathbb{T}).$$

Similarly, we imply that

$$\widehat{\delta}_{T+1}^{(n2)} \leqslant 2k(M+1)(T+1-t_0) d_H(\widehat{\mathbb{T}}_n, \mathbb{T}_n).$$

Thus, we obtain

$$\begin{aligned}\|x(t) - x_n(t_n^*)\| &\leqslant (\widehat{\delta}_T^{(n1)} + \widehat{\delta}_{T+1}^{(n2)}) e^{C_0 k(T-t_0)} \\ &\leqslant 2k(M+1) e^{C_0 k(T+1-t_0)} \Big( (T-t_0) d_H(\widehat{\mathbb{T}}_n, \mathbb{T}) + (T+1-t_0) d_H(\widehat{\mathbb{T}}_n, \mathbb{T}_n) \Big) \\ &\quad + M d_H(\widehat{\mathbb{T}}_n, \mathbb{T}_n) \\ &\leqslant 2k(2T+1-2t_0)(M+1) e^{C_0 k(T+1-t_0)} d_H(\mathbb{T}, \mathbb{T}_n) + M d_H(\mathbb{T}, \mathbb{T}_n) \\ &= C_1 d_H(\mathbb{T}, \mathbb{T}_n),\end{aligned}$$

where $C_1 = 2k(2T+1-2t_0)(M+1)e^{C_0 k(T+1-t_0)} + M$. Similarly, if $t \in \mathbb{T} \cap \mathbb{T}_n : t_0 \leqslant t < T$ then

$$\|x(t) - x_n(t)\| \leqslant C_2 d_H(\mathbb{T}, \mathbb{T}_n),$$

where $C_2 = 4k(T-t_0)(M+1)e^{C_0 k(T-t_0)}$. The proof is complete. □

## 4. **Examples**

**Example 4.1.** Consider the IVP

$$x' = f(t,x), \quad t_0 \leqslant t \leqslant T, \quad x(t_0) = x_0. \tag{4.1}$$

In numerical analysis, approximations to the solution $x(t)$ of (4.1) will be generated at various values, called mesh points, in the interval $[t_0, T]$. For a positive integer $n$, we select a subdivision of the interval $[t_0, T]$

$$t_0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{k_n-1}^{(n)} < t_{k_n}^{(n)} := T, \ k_n \in \mathbb{N}. \tag{4.2}$$

Associating with (4.2), we study a difference equation, called the implicit Euler method [9, 11, 14], as follows

$$x_0^{(n)} = x_0, \quad x_i^{(n)} = x_{i-1}^{(n)} + (t_i^{(n)} - t_{i-1}^{(n)}) f(t_i^{(n)}, x_i^{(n)}), \quad i = 1, 2, \ldots, k_n. \tag{4.3}$$

Let $\mathbb{T} := [t_0, T]$ and $\mathbb{T}_n := \{t_0^{(n)}, t_1^{(n)}, \ldots, t_{k_n-1}^{(n)}, t_{k_n}^{(n)}\}$. Then $\mathbb{T}$ and $\mathbb{T}_n$ are time scales. This leads to that we can rewrite (4.1) and (4.3) as follows

$$x^{\nabla}(t) = f(t, x(t)),\ t \in \mathbb{T}, \quad x(t_0) = x_0,$$

and

$$x_n^{\nabla}(\tau) = f(\tau, x_n(\tau)),\ \tau \in \mathbb{T}_n, \quad x_n(t_0) = x_0,$$

respectively. In this case, it is easy to see that

$$2d_H(\mathbb{T}_n, \mathbb{T}) = h_n := \sup_{1 \leqslant i \leqslant k_n} \{t_i^{(n)} - t_{i-1}^{(n)}\} \ \text{ for all } n \in \mathbb{N}. \tag{4.4}$$

Suppose that $f(t, x)$ is continuous and satisfies Lipschitz condition

$$\|f(t, x_1) - f(t, x_2)\| \leqslant k\|x_1 - x_2\|,\ \text{for all } t \in [t_0, T].$$

By Theorem 3.4 and (4.4), we see that $\lim_{n\to\infty} x_n(\gamma_n^{\mathbb{T}_n,\mathbb{T}}(t)) = \lim_{n\to\infty} x_n(t_n^*(t)) = x(t)$ uniformly in $[t_0, T]$. Hence, we obtain the well-known result for the convergence of implicit Euler method in numerical analysis [9, 11, 14].

Assume further that $f$ satisfies the Lipschitz condition in both variables with constant $k$. That is,

$$\|f(t, x_1) - f(s, x_2)\| \leqslant k(|t - s| + \|x_1 - x_2\|),\ \text{for all } t, s \in [t_0, T],\ x_1, x_2 \in \mathbb{R}^d.$$

By Theorem 3.6, we get an estimation of the convergent rate as well as an error bound for the implicit Euler method as follows

$$\sup_{0 \leqslant i \leqslant k_n} \|x(t_i^{(n)}) - x_i^{(n)}\| \leqslant C_3 h_n, \ \text{ for all } n \in \mathbb{N},$$

where $C_3 = 2k(T - t_0)(M + 1)e^{C_0 k(T - t_0)}$.

**Example 4.2** (Approximation of solutions to logistic equations on $\mathbb{R}_+$)**.** Let $\mathbb{T} = [0, \infty)$. We now consider plant population models. Let $x(t)$ be the number of plants of one particular kind at time $t \in \mathbb{T}$ in a certain area. By experiments we know that $x(t)$ grows according to the logistic equation

$$x^{\nabla}(t) = x(t)[1 - 4x(t)],\ t \in \mathbb{T} \text{ and } x(0) = 1 > 0. \tag{4.5}$$

Suppose that we are unable to observe the values of $x(t)$ but $x_n(t)$ with $x_n(t)$ to be the number of plants of one particular kind at time $t \in \mathbb{T}_n$ in a certain area, subjecting to the equation

$$x_n^{\nabla}(t) = x_n(t)[1 - 4x_n(t)],\ t \in \mathbb{T}_n \text{ and } x_n(0) = 1 \ \text{ for all } n \in \mathbb{N}, \tag{4.6}$$

where $\mathbb{T}_n$ is a time scale given by

$$\mathbb{T}_n = \{0\} \cup \bigcup_{k=1}^{\infty} \left[\frac{2k-1}{n}, \frac{2k}{n}\right] \quad \text{for all } n \in \mathbb{N}.$$

This means that we lack the observation, for some reasons, at the times in the intervals $\left(\frac{2k}{n}, \frac{2k+1}{n}\right)$. It is easy to see that $d_H(\mathbb{T}_n, \mathbb{T}) = \frac{1}{2n}$. Hence,

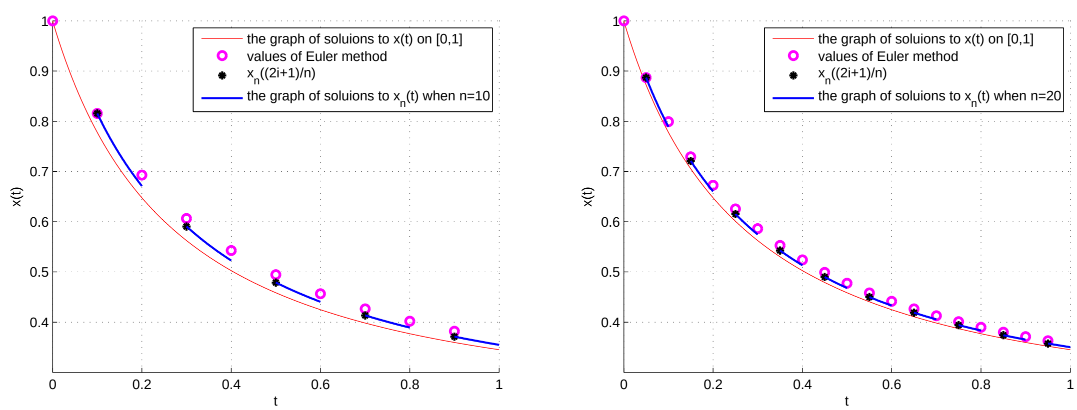
$$\lim_{n\to\infty} \mathbb{T}_n = \mathbb{T}.$$

We have $x(t) = \dfrac{e^t}{1+4(e^t-1)}; t \in \mathbb{T}$ and

$$x_n(0) = 1; x_n\Big(\frac{2k+1}{n}\Big) = x_n\Big(\frac{2k}{n}\Big) + \frac{1}{n}x_n\Big(\frac{2k+1}{n}\Big) - \frac{4}{n}x_n^2\Big(\frac{2k+1}{n}\Big);$$
$$x_n(t) = \frac{x_n(\frac{2k-1}{n})e^{t-\frac{2k-1}{n}}}{1+4x_n(\frac{2k-1}{n})\big(e^{t-\frac{2k-1}{n}}-1\big)}, \ \forall\, t \in \left[\frac{2k-1}{n}, \frac{2k}{n}\right],$$

for all $k = 1, 2, \ldots, n \in \mathbb{N}$. Note that if $f$ satisties the local Lipschitz condition and the solution sequence $\{x_n(t)\}$ is bounded then Theorem 3.4 also holds. Therefore, by this theorem, we imply that $x_n(t) \to x(t)$ as $n \to \infty$.

The discrete graph of solutions $x_n(t)$ and $x(t)$ on the interval $[0, 1]$ is shown in Figure 1.




(a) $x_n(t)$ and $x(t)$ with $n = 10$ on the interval $[0, 1]$ in Example 4.2

(b) $x_n(t)$ and $x(t)$ with $n = 20$ on the interval $[0, 1]$ in Example 4.2

FIGURE 1. The graph of the solution $x_n(t)$ on the time scale $\mathbb{T}_n$

**Example 4.3** (Approximation of solutions to logistic equation on Cantor set)**.** Let $K$ be be the Cantor set in $[0, 1]$. Following the construction of this Cantor set, we define $K_0 = [0, 1]$. We obtain $K_1$ by removing the "middle third" of $K_0$, i.e., the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ from $K_0$. $K_2$ is obtained by removing two "middle thirds of $K_1$, i.e., the two open intervals $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$ from $K_1$. Proceeding in this manner we obtain a sequence of time scales $(K_n)_{n\in\mathbb{N}}$. The Cantor set is defined

$$K = \bigcap_{n=0}^{\infty} K_n.$$

Let $(\mathbb{T}_n)$ be a sequence of time scales, where $\mathbb{T}_n = K_n \cup (K_n + 1)$ and $\mathbb{T} = K \cup (K + 1)$. Consider the dynamic equation (4.5) with $x(0) = 1$. It is known that we are unable to give an explicit formula for solutions as well as a numerical solution to equation (4.5). However, we can use Theorem 3.4 to approximate these solutions.

We illustrate this approximation by Figure 2. It is seen that the graph on $\mathbb{T}_4$ of the equation (4.5) (the green line) is similar to one on $\mathbb{T}_0 = [0, 2]$ (the red line).

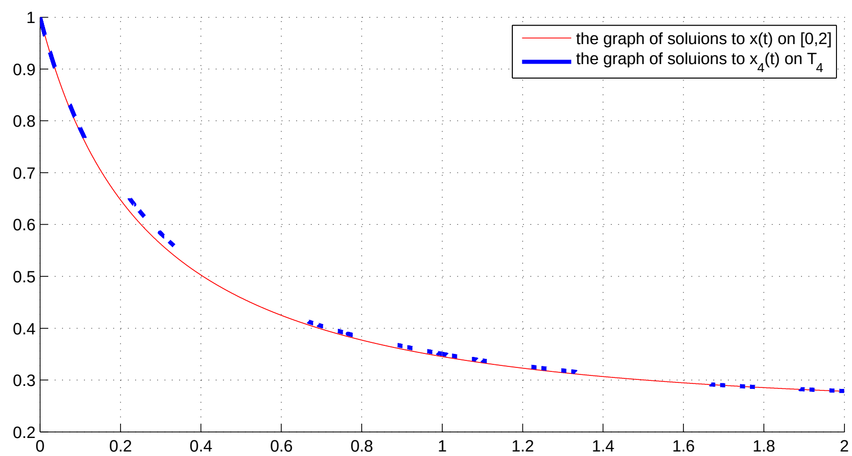



FIGURE 2. The graph of the solution $x_4(t)$ on the time scale $\mathbb{T}_4$

## 5. Conclusion

In this paper, we have proved the convergence of solutions of nabla dynamic equations $x^\nabla(t) = f(t, x)$ on time scales $\{\mathbb{T}_n\}_{n=1}^{\infty}$ when this sequence converges to the time scale $\mathbb{T}$. The convergent rate of solutions is estimated when $f$ satisfies the Lipschitz condition in both variables.

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