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GLOBAL STABILITY AND BOUNDEDNESS OF SOLUTIONS TO DIFFERENTIAL EQUATIONS OF THIRD ORDER WITH MULTIPLE DELAYS

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ABSTRACT. In this paper, the author gives certain sufficient conditions for the global asymptotic stability and boundedness of solutions to a class of functional differential equations of third order with multiple delays. The technique of proofs involve defining an appropriate Lyapunov– Krasovskii functional and applying LaSalle's invariance principle. An example is discussed to illustrate the efficiency of the obtained results. Our results complement and improve some related ones in the literature.

AMS (MOS) Subject Classification. 34K20.

1. PRELIMINARIES

In this paper, we are interested in obtaining sufficient conditions for all solutions of the third order nonlinear functional differential equation with multiple retardations

$$\begin{aligned} (1.1)\qquad x'''(t) \;&=\; \varphi(x(t), x'(t), x''(t)) + \psi(x(t), x'(t)) + \sum_{i=1}^{N} f_i(x(t), x'(t-\tau_i)) \\ &\quad + h(x(t)) + p(t), \end{aligned}$$

to be bounded, and in case $p(t) \equiv 0$, sufficient conditions for the zero solution to be globally asymptotically stable. Our motivation comes partially from a recent paper of El-Nahhas [11] who studied the asymptotic stability of the delay differential equation

$$x'''(t) = ax''(t) + bx'(t) + cx(t) + f(x(t), x'(t-\tau)), \tag{1.2}$$

where a, b and c are negative constants; τ is a constant retardation; $f(0, 0) = 0$.

Functional differential equations of the type (1.1) and (1.2) have been shown to be useful in modeling many phenomena in various fields of science and engineering and in more recent years to problems in biomathematics (see, for example, Cronin-Scanlon [8] and Smith [19]). One special case of equations (1.1) and (1.2) is what is known as the jerky dynamics equation

$$x'''(t) + k_1(x(t), x'(t))x''(t) + k_2(x(t), x'(t), x''(t)) = 0$$

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that has gained some attention in the literature (see, Chlouverakis and Sprott [7], Eichhorn et al. [9], Elhadj and Sprott [10], and Linz [14]). Besides, qualitative properties of solutions of various differential equations of third order with or without delay such as stability, instability, boundedness, uniformly boundedness, oscillation, periodicity of solutions, etc. have been studied by many authors; in this regard, we refer the reader to the monograph by Reissig et al. [17], and the recent papers of Adams et al. [1], Ademola and Arawomo [2], Afuwape and Adesina [3], Bai and Guo [5], Graef and Tunç [12], Ogundare and Okecha [15], Rauch [16], Sadek [18], Tunç [20–27], Zhang and Yu [29] and the references therein.

It should be noted that when we compare equation (1.1) with equation (1.2), it is clear that our equation, equation (1.1), includes and improves the equation studied by El-Nahhas [11], equation (1.2).

One tool to be used here LaSalle's invariance principle. If we consider the delay differential system

$$x' = F(x_t), \quad x_t = x(t+\theta), -r \le \theta \le 0,\, t \ge 0, \tag{1.3}$$

we take $C = C([-r,0],\Re^n)$ to be the space of continuous functions from $[-r,0]$ into $\Re^n$ and ask that $F : C \to \Re^n$ be continuous, (see, also, Krasovskii [13]). Let S be the set of $\phi \in C$ such that $\|\phi\| \ge H$, denote by $S^\bullet$ the set of all functions $\phi \in C$ such that $|\phi(0)| \ge H$, where H is large enough.

Let D be an open (t,x)-set in $\Re_+ \times \Re^{n+1}$ containing the origin, where $\Re_+ = [0,\infty)$.

Consider the real valued function $V(t,x)$ defined on D, $D = \{(t,x) : t \ge 0, \|x\| < d \le +\infty\}$.

Definition 1.1 ([4])**.** A function $V(t,x)$ is known as positive definite on the set D if $V(t,0) = 0$ and there exists a function $a(r)$ such that $a(0) = 0$, $a(r)$ is strictly monotonically increasing in r, and $a(\|x\|) \le V(t,x)$, $(t,x) \in D$.

Definition 1.2 ([4])**.** A function $V(t,x) \ge 0$ is said to be decrescent if there exists a function $b(r)$ such that $b(0) = 0$, $b(r)$ is strictly monotonically increasing in r, and $V(t,x) \le b(\|x\|)$, $(t,x) \in D$.

We say that $V : C \to \Re$ is a Lyapunov function on a set $G \subset C$ relative to F if V is continuous on $\bar{G}$, the closure of G, $V \ge 0$, V is positive definite, V' is defined on G, and $V' \le 0$ on G.

The following form of LaSalle's invariance principle can be found in Smith [19, Theorem 5.17]. Here, ω denotes the omega limit set of a solution.

Theorem 1.3 ([19])**.** *If V is a Lyapunov function on G and $x_t(\phi)$ is a bounded solution such that $x_t(\phi) \in G$ for $t \ge 0$, then $\omega(\phi) \ne 0$ is contained in the largest invariant subset of $E \equiv \{\phi \in \bar{G} : V'(\phi) = 0\}$.*

Theorem 1.4 ([19]). *Suppose that there exists a continuous Lyapunov functional $V(\phi)$ defined for all $\phi \in S^{\bullet}$, which satisfies the following conditions;*

(i) $a(|\phi(0)|) \le V(\phi) \le b_1(|\phi(0)|) + b_2(\|\phi\|)$, *where* $a(r), b_1(r), b_2(r) \in CI$, *(CI denotes the families of continuous increasing functions), and are positive for* $r > H$ *and* $a(r) - b_2(r) \to \infty$ *as* $r \to \infty$*;*

(ii) $V'(\phi) \le 0$.

Then the solutions of equation (1.3) *are uniformly bounded.*

2. MAIN RESULTS

We consider the nonlinear third order differential equation with multiple delays

$$
\begin{aligned}
x'''(t) &= \varphi(x(t), x'(t), x''(t)) + \psi(x(t), x'(t)) + \sum_{i=1}^{N} f_i(x(t), x'(t-\tau_i)) \\
&\quad + h(x(t)) + p(t),
\end{aligned} \tag{2.1}
$$

where $\Re = (-\infty, \infty)$, $\Re_+ = [0, \infty)$, $\varphi : \Re^3 \to \Re$, $\psi : \Re^2 \to \Re$, $f_i : \Re^2 \to \Re$, $i = 1, 2, \ldots, N$, $h : \Re \to \Re$, and $p : \Re_+ \to \Re$ are continuous with $\varphi(x(t), x'(t), 0) = 0$, $\psi(x(t), 0) = 0$, $f_i(0, 0) = 0$, $h(0) = 0$ and τ_i (> 0) are constants. The continuity of the functions φ, ψ, f_i, h and p guarantees the existence of the solutions, and we assume that φ, ψ, f_i and h satisfy local Lipschitz conditions so that we have uniqueness of solutions to initial value problems as well, and the functions ψ, f_i and h are differentiable.

We can write equation (2.1) as the system

$$
\begin{aligned}
x'(t) &= y(t), \\
y'(t) &= z(t), \\
z'(t) &= \varphi(x(t), y(t), z(t)) + \psi(x(t), y(t)) \\
&\quad + \sum_{i=1}^{N} f_i(x(t), y(t-\tau_i)) + h(x(t)) + p(t).
\end{aligned}
$$

Let

$$\sum_{i=1}^{N} g_i(x(t), y(t-\tau_i)) = \sum_{i=1}^{N} f_i(x(t), y(t-\tau_i)) + h(x(t)).$$

Hence, we have

$$
\begin{aligned}
x'(t) &= y(t), \\
y'(t) &= z(t), \\
z'(t) &= \varphi(x(t), y(t), z(t)) + \psi(x(t), y(t)) \\
&\quad + \sum_{i=1}^{N} g_i(x(t), y(t-\tau_i)) + p(t),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
x'(t) &= y(t),\\
y'(t) &= z(t),\\
z'(t) &= \varphi(x(t),y(t),z(t))+\psi(x(t),y(t))\\
&\quad+\sum_{i=1}^{N} g_i(x(t),y(t))\\
&\quad-\sum_{i=1}^{N}\int_{-\tau_i}^{0} g_{i_y}(x(t),y(t+\sigma))z(t+\sigma)d\sigma+p(t), \qquad (2.2)
\end{aligned}
$$

where

$$g_{i_y}=\frac{\partial g_i}{\partial y}.$$

Assume that there are constants $P_i>0$, $a<0$, $\tau_i>0$ and $\alpha_i>0$, $i=1,2,\dots,N$, such that the following assumptions hold:

(A1) $v(t+\tau_i)[\sum_{i=1}^{N} f_i(x(t),0)+h'(x(t))x(t)]\le 0$;
(A2) $\sum_{i=1}^{N}\int_0^{v(t+\tau_i)}[f_i(x(t),u)du-f_i(x(t),0)]v(t+\tau_i)<0$ for $v(t+\tau_i)\neq 0$;
(A3) $v(t+\tau_i)[h'(x)v(t+\tau_i)+\sum_{i=1}^{N}\int_0^{v(t+\tau_i)} f_{ix}(x(t),u)du]\ge 0$;
(A4) $|f_{iv}(x,v)|<P_i<\infty$;
(A5) $a+\tau_i\alpha_i<0$;
(A6) $4\alpha_i(a+\tau_i\alpha_i)+\tau_iP_i^2<0$.

Define the functions $H_i(x,y)$, $i=1,2,\dots,N$, by

$$H_i(x,y)=-\int_0^y g_i(x,u)du-\frac{1}{2}by^2,\quad (x,y)\in\Omega_0, b\in\Re, b<0,$$

$$\Omega_0=\{(x(t),y(t)):(x(t),y(t+\tau_i))\in\Omega,\ t\ge 0\},$$

and Ω_0 is a domain of the two dimensional Euclidean space $\Re^2$.

Lemma 2.1. *Assume that*

(B1) $\psi(x,0)=0$, $\frac{\psi(x,y)}{y}\le bN$ *for* $x,y\neq 0$, $b\in\Re$, $b<0$, *and* $\frac{\partial\psi}{\partial x}=\psi_x(x,y)\ge 0$ *for* x,y;
(B2) $yg_i(x,0)\le 0$ *for* x, y; $\int_0^y g_i(x,u)du-g_i(x,0)y<0$ *for* x, $y\neq 0$.

Then the functions $H_i(x,y)=L_ix^2+2M_ixy+N_iy^2$ *are positive definite and decrescent, where*

$$L_i=L_i(x,y)=\frac{1}{x^2}\left[-\int_0^y g_i(x,u)du+\int_0^y g_i(x,0)du\right],$$

$$M_i=M_i(x)=\frac{1}{2x}g_i(x,0),\ \textit{and}\ \ N_i=N_i(x,y)=-\frac{1}{2b}y^2.$$

Proof. By noting the assumptions of Lemma 2.1, it follows that

$$L_i = \frac{1}{x^2}\left[-\int_0^y g_i(x,u)du + \int_0^y g_i(x,0)du\right] > 0,$$

$$2M_i xy = -y g_i(x,0) \geq 0$$

and

$$N_i y^2 = -\frac{1}{2} b y^2 \geq 0.$$

Then, we can conclude that

$$H_i(x,y) \geq K_i(x^2+y^2),$$

where $K_i = \min\{[\inf L_i(x,y)] \text{ for all } x,y \in \Omega_0, N_i\}$, $K_i > 0$. This means that $H_i(x,y)$ are positive definite.

It is also clear that the quadratic forms $H_i(x,y)$ can be rearranged as

$$H_i(x,y) = X^T[T_i(x,y)]X,$$

where

$$X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad X^T \text{ is transpose of } X,$$

$$T_i(x,y) = \begin{bmatrix} \frac{1}{x^2}\left[-\int_0^y g_i(x,u)du + \int_0^y g_i(x,0)du\right] & -\frac{1}{2x} g_i(x,0) \\ -\frac{1}{2x} g_i(x,0) & -\frac{1}{2} b \end{bmatrix}.$$

Let $\lambda_{1i}(x,y)$ and $\lambda_{2i}(x,y)$ denote the characteristic roots of the matrices $T_i(x,y)$. Then, it is clear that

$$H_i(x,y) \leq L_i^{\frac{1}{2}}(x^2+y^2), \tag{2.3}$$

where $L_i = \sup[\lambda_{1i}^2(x,y) + \lambda_{2i}^2(x,y)]$ for all $x,y \in \Omega_0$, and $L_i > 0$. Thus, the functions $H_i(x,y) = L_i x^2 + 2M_i xy + N_i y^2$ are decrescent. This completes the proof of Lemma 2.1. □

Theorem 2.2. *Assume that* $p(t) \equiv 0$, *conditions* (A1)–(A6) *hold, and*

$$h(0) = 0, h'(x) \leq c, \quad for \;\; x, c \in \Re, c < 0;$$

$$\psi(x,0) = 0, \frac{\partial \psi}{\partial x} = \psi_x(x,y) \geq 0, \frac{\psi(x,y)}{y} \leq bN \quad for \;\; x, y \neq 0, b \in \Re, b < 0;$$

$$\varphi(x,y,0) = 0, \frac{\varphi(x,y,z)}{z} \leq aN \quad for \;\; x,y,z \neq 0, a \in \Re, a < 0.$$

Then, the zero solution of equation (2.1) *is globally asymptotically stable.*

Proof. We define the Lyapunov–Krasovskii functional $V(t) = V(x(t), y(t), z(t))$ by

$$V(t) = -\sum_{i=1}^{N}\int_0^y g_i(x,u)du - \int_0^y \psi(x,u)du + \frac{1}{2}z^2 + \sum_{i=1}^{N}\alpha_i \int_{-\tau_i}^{0}\left[\int_\theta^0 z^2(\sigma)d\sigma\right]d\theta, \tag{2.4}$$

where

$$\sum_{i=1}^{N} g_i(x, y(t-\tau_i)) = \sum_{i=1}^{N} f_i(x, y(t-\tau_i)) + h(x),$$
$$(x,y) \in \Omega_0, z = \{z(t) : z(t) = y'(t), \quad t \geq 0\}.$$

and $\alpha_i(>0)$ are certain positive constants.

Consider the terms

$$-\int_0^y g_1(x,u)du - \int_0^y g_2(x,u)du - \cdots - \int_0^y g_N(x,u)du - \int_0^y \psi(x,u)du,$$

which are included in (2.4).

It is clear that

$$\begin{aligned}
&-\int_0^y g_1(x,u)du - \int_0^y g_2(x,u)du - \cdots - \int_0^y g_N(x,u)du - \int_0^y \psi(x,u)du\\
&= -\int_0^y g_1(x,u)du - \int_0^y g_2(x,u)du - \cdots - \int_0^y g_N(x,u)du - \int_0^y \frac{\psi(x,u)}{u}udu\\
&\geq -\int_0^y g_1(x,u)du - \int_0^y g_2(x,u)du - \cdots - \int_0^y g_N(x,u)du - \int_0^y bNudu\\
&\geq -\int_0^y g_1(x,u)du - \int_0^y g_2(x,u)du - \cdots - \int_0^y g_N(x,u)du - \frac{1}{2}(bN)y^2\\
&= -\int_0^y g_1(x,u)du - \frac{1}{2}by^2 - \int_0^y g_2(x,u)du - \frac{1}{2}by^2 \cdots - \int_0^y g_N(x,u)du - \frac{1}{2}by^2
\end{aligned}$$

by (A1).

Then, it is obvious that

$$-\int_0^y g_i(x,u)du - \frac{1}{N}\int_0^y \psi(x,u)du \geq -\int_0^y g_i(x,u)du - \frac{1}{2}by^2.$$

Hence,

$$V(t) \geq \sum_{i=1}^{N}\left[-\int_0^y g_i(x,u)du - \frac{1}{2}by^2\right] + \frac{1}{2}z^2 + \sum_{i=1}^{N}\alpha_i\int_{-\tau_i}^{0}\left[\int_\theta^0 z^2(\sigma)d\sigma\right]d\theta. \tag{2.5}$$

Therefore, in view of the result of Lemma 2.1, the Lyapunov–Krasovskii functional $V(t)$, and (2.5), it is clear that $V(t) \geq 0$ for all $t \geq 0$. Also note that $V(t) = 0$ implies that $y = z = 0$. Then, $x = \xi$, $y = z = 0$. Substituting these estimates in (2.2), we get $h(\xi) = 0$, which necessarily implies that $\xi = 0$ since $h(0) = 0$. Thus, $V(t) = 0$ implies that $x = y = z = 0$. Therefore, V is positive definite. Next, we show that $V(x,y,z) \to \infty$ as $|x| + |y| + |z| \to \infty$. The existence of this estimate is clear when we consider the above discussion, (2.3) and (2.4).

Differentiating the functional V with respect to t and benefiting from the assumptions of Theorem 2.2, we have

$$\begin{aligned} V'(t) &= -y\sum_{i=1}^{N}\int_0^y g_{ix}(x,u)du + \varphi(x,y,z)z - y\int_0^y \psi_x(x,u)du \\ &\quad -\sum_{i=1}^{N}\int_{-\tau_i}^{0} g_{i_y}(x(t),y(t+\sigma))z(t)z(t+\sigma)d\sigma \\ &\quad +\sum_{i=1}^{N}\alpha_i\int_{-\tau_i}^{0}[z^2(t)-z^2(t+\sigma)]d\sigma \\ &\le -y\sum_{i=1}^{N}\int_0^y g_{ix}(x,u)du + \sum_{i=1}^{N}\int_{-\tau_i}^{0}\left[\left(\frac{a}{\tau_i}+\alpha_i\right)z^2(t)-\alpha_i z^2(t+\sigma)\right]d\sigma \\ &\quad -\sum_{i=1}^{N}\int_{-\tau_i}^{0} g_{i_y}(x(t),y(t+\sigma))z(t)z(t+\sigma)d\sigma, \end{aligned}$$

where

$$g_{i_x}=\frac{\partial g_i}{\partial x},\quad g_{i_y}=\frac{\partial g_i}{\partial y}.$$

In view of the assumption (A3), it follows that $y\sum_{i=1}^{N}\int_0^y g_{ix}(x,u)du \ge 0$. Then

$$\begin{aligned} V'(t) &\le \sum_{i=1}^{N}\int_{-\tau_i}^{0}\left[\left(\frac{a}{\tau_i}+\alpha_i\right)z^2(t)-\alpha_i z^2(t+\sigma)\right]d\sigma \\ &\quad -\sum_{i=1}^{N}\int_{-\tau_i}^{0} g_{i_y}(x(t),y(t+\sigma))z(t)z(t+\sigma)d\sigma. \end{aligned}$$

Consider the terms

$$\begin{aligned} &\sum_{i=1}^{N}\int_{-\tau_i}^{0}\left[\left(\frac{a}{\tau_i}+\alpha_i\right)z^2(t)-\alpha_i z^2(t+\sigma)\right]d\sigma \\ &\quad -\sum_{i=1}^{N}\int_{-\tau_i}^{0} g_{i_y}(x(t),y(t+\sigma))z(t)z(t+\sigma)d\sigma \\ &= \int_{-\tau_1}^{0}\left[\left(\frac{a}{\tau_1}+\alpha_1\right)z^2(t)-\alpha_1 z^2(t+\sigma)\right]d\sigma \\ &\quad -\int_{-\tau_1}^{0} g_{1_y}(x(t),y(t+\sigma))z(t)z(t+\sigma)d\sigma \\ &\quad +\int_{-\tau_2}^{0}\left[\left(\frac{a}{\tau_2}+\alpha_2\right)z^2(t)-\alpha_2 z^2(t+\sigma)\right]d\sigma \\ &\quad -\int_{-\tau_2}^{0} g_{2_y}(x(t),y(t+\sigma))z(t)z(t+\sigma)d\sigma \end{aligned}$$

$$+\cdots+\int_{-\tau_N}^{0}\left[\left(\frac{a}{\tau_N}+\alpha_N\right)z^2(t)-\alpha_N z^2(t+\sigma)\right]d\sigma$$
$$-\int_{-\tau_N}^{0}g_{N_y}(x(t),y(t+\sigma))z(t)z(t+\sigma)d\sigma.$$

By noting the assumption (A4)-(A6), it can be seen that if

$$\begin{aligned}-\alpha_{\mathrm{i}}^2-\frac{a\alpha_i}{\tau_i}-\frac{1}{4}g_{iy}^2(x(t),z(t+\sigma)) &= -\frac{4\tau_i\alpha_{\mathrm{i}}^2+4a\alpha_i+\tau_i g_{iy}^2(x(t),z(t+\sigma))}{\tau_i}\\ &\geq -\frac{4\tau_i\alpha_{\mathrm{i}}^2+4a\alpha_i+\tau_i P_i^2}{\tau_i}>0,\end{aligned}$$

that is, $4\alpha_i(a+\tau_i\alpha_i)+\tau_i P_i^2<0$, then the quadratic form

$$\begin{aligned}&\alpha_i z^2(t+\sigma)+g_{i_y}(x(t),y(t+\sigma))z(t+\sigma)z(t)-\left(\frac{a}{\tau_i}+\alpha_i\right)z^2(t)\\ =\ &[z(t+\sigma),z(t)]\begin{bmatrix}\alpha_i & \frac{1}{2}g_{iy}(x(t),z(t+\sigma))\\ \frac{1}{2}g_{iy}(x(t),z(t+\sigma)) & -\left(\frac{a}{\tau_i}+\alpha_i\right)\end{bmatrix}\begin{bmatrix}z(t+\sigma)\\ z(t)\end{bmatrix}\end{aligned}$$

is positive for any $z(t+\sigma)$ and $z(t)$. Hence,

$$\int_{-\tau_i}^{0}\left[\left(\frac{a}{\tau_i}+\alpha_1\right)z^2(t)-\alpha_i z^2(t+\sigma)\right]d\sigma-\int_{-\tau_i}^{0}g_{i_y}(x(t),y(t+\sigma))z(t)z(t+\sigma)d\sigma$$

is negative. Therefore, we can conclude

$$V'(t)<0.$$

We will now apply LaSalle's invariance principle, so consider the set

$$E=\{(x,y,z):V'(x,y,z)=0\}.$$

Observe that $(x,y,z)\in E$ implies $y=z=0$, and substituting this into (2.2) shows that $x=0$. Clearly, the largest invariant set contained in E is $\{(0,0,0)\}$. Therefore, the zero solution of (2.2) is globally asymptotically stable. □

Finally, for the case $p(t)\neq 0$, we prove two theorems.

Theorem 2.3. *In addition to conditions* (A1)–(A6), *assume that* $p\in L^1(0,\infty)$. *Then all solutions of* (2.2) *are bounded.*

Proof. For the case $p(t)\neq 0$, by an easily calculation from V, which is given in (2.4), we can see that

$$V'(t)\leq zp(t).$$

Then, we have

$$V'(t)\leq |z|\ \ |p(t)|\leq (1+z^2)\,|p(t)|\,.$$

From the discussion made for (2.3), it follows that

$$\frac{1}{2}z^2 + \sum_{i=1}^{N} K_i(x^2 + y^2) \leq V(t).$$

Then, we have

$$V'(t) \leq (1 + 2V)\,|p(t)|\,,$$

and an application of Gronwall's inequality [4] bounds V. Thus, all solutions of (2.2) are bounded. □

Remark 2.4. If the assumptions of Theorem 2.3 hold, then, in view of the whole discussion made for the functional V, we can conclude

$$\sum_{i=1}^{N} \alpha_i \int_{-\tau_i}^{0} \left[\int_{\theta}^{0} z^2(\sigma)d\sigma\right] d\theta + \frac{1}{2}z^2 + \sum_{i=1}^{N} K_i(x^2 + y^2) \leq V(t)$$

$$\leq \sum_{i=1}^{N} K_i^{\frac{1}{2}}(x^2 + y^2) + \frac{1}{2}z^2 + \sum_{i=1}^{N} \alpha_i \int_{-\tau_i}^{0} \left[\int_{\theta}^{0} z^2(\sigma)d\sigma\right] d\theta.$$

Hence, we can conclude that all solutions of equation (2.2) are uniformly bounded.

Example 2.5. In case $N = 1$, consider nonlinear delay differential equation of third order

$$\begin{aligned}(2.6)\ x'''(t) \quad &= \quad (a - \exp(-x^2(t) - x'^2(t) - x''^2(t))x''(t) + (b - \exp(-x(t)))x'(t) \\ &\quad\ + cx(t) + \beta x^2(t) - \gamma x'(t-\tau) + \exp(-t)\sin t,\end{aligned}$$

where a, b and c are negative constants, τ is a sufficiently small positive constant.

It is clear that

$$\varphi(x, y, z) = (a - \exp(-x^2 - y^2 - z^2))z,$$

$$\varphi(x, y, 0) = 0, \frac{\varphi(x, y, z)}{z} = a - \exp(-x^2 - y^2 - z^2) \leq a, z \neq 0,$$

$$h(x) = cx, h(0) = 0, h'(x) = c,$$

$$\psi(x, 0) = 0, \frac{\psi(x, y)}{y} = b - \exp(-xy) \leq b, y \neq 0,$$

$$\frac{\partial\psi}{\partial x} = \psi_x(x, y) = \exp(-xy)y^2 \geq 0,$$

$$f(x(t), v(t)) = \beta x^2(t) - \gamma v(t)$$

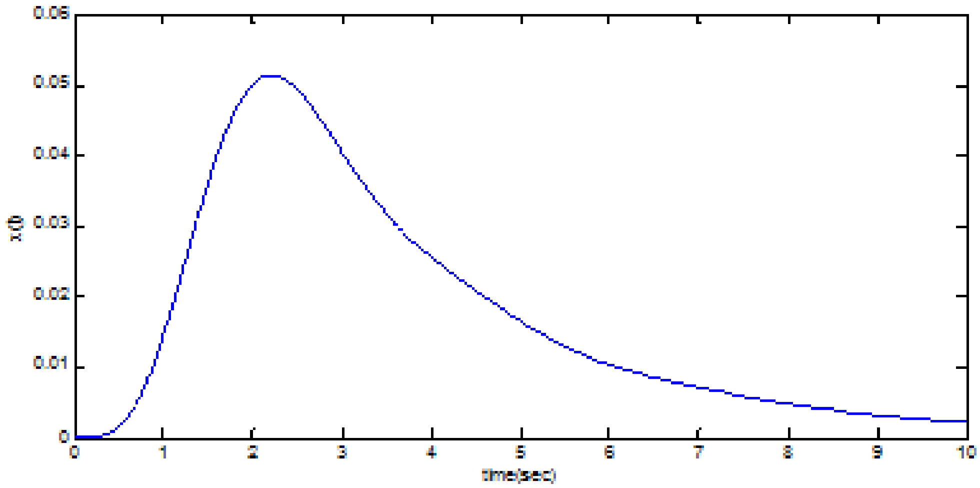

FIGURE 1. Trajectories of $x(t)$ of equation (2.6) in Example 2.5, when $a=-1$, $b=-2$, $c=-3$, $p(t)=0$, $\beta=7$, $\gamma=5$, $\tau=0.0001$.

such that $x(t)$, $v(t)>0$ for $t\geq 0$, β and γ are constants $(0<\beta,\gamma<\infty)$, $\beta x+c<0$, $\alpha=-\frac{a}{2\tau}$, $P=\gamma+1$,

$$\begin{aligned} g(x,v) &= f(x,v)+h(x) \\ &= \beta x^2-\gamma v+cx, \end{aligned}$$

$$vg(x,0)=-\gamma v^2\leq 0,$$

$$\begin{aligned} \int_0^y g(x,u)du-g(x,0)y &= \int_0^y (\beta x^2-\gamma u+cx)du-\beta x^2y-cxy \\ &= -\frac{1}{2}\gamma y^2<0 \quad \text{for} \quad x,y\neq 0, \end{aligned}$$

and

$$p(t)=\exp(-t)\sin t, \quad p\in L^1(0,\infty).$$

It can also be easily shown that all assumptions (A1)–(A6) hold. Thus, all assumptions of Theorem 2.2 and Theorem 2.3 hold. Hence, we conclude that, in case $p(t)\equiv 0$, the zero solution of equation (2.6) is asymptotically stable, and all solutions of equation (2.6) are bounded and uniformly bounded in case $p(t)\neq 0$.

3. DISCUSSION

A kind of functional differential equations of third order with retarded argument has been considered. By defining an appropriate Lyapunov– Krasovskii functional

[13] globally asymptotically stability, boundedness and uniformly boundedness of solutions have been discussed. It is clear that our equation, equation (1.1), includes the equation studied by El-Nahhas [11], equation (1.2). This case is an extension and contribution to El-Nahhas [11]. In [11], the author studied the asymptotic stability of solutions. However, in this paper, we study, the globally asymptotically stability, boundedness and uniformly boundedness of solutions, and the globally asymptotically stability implies the asymptotically stability of solutions, but its inverse is not true. This means that Theorem 2.2 includes and improves the result of El-Nahhas [11], and Theorem 2.3 and Remark 2.4 give additional results to that in [11]. Our results also complement some recent ones in the literature (Adams et al. [1], Ademola and Arawomo [2], Afuwape and Adesina [3], Bai and Guo [5], Ogundare and Okecha [15], Rauch [16], Sadek [18], Zhang and Yu [29]).

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