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PARAMETRIC p-LAPLACIAN EQUATIONS WITH SUPERLINEAR REACTIONS

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ABSTRACT. We consider a parametric nonlinear Dirichlet problem driven by the p-Laplacian and with a Carathéodory reaction which is $(p-1)$-superlinear near $\pm\infty$ (but without satisfying the Ambrosetti-Rabinowitz condition) and $(p-1)$-sublinear near zero. We show that for all values of the parameter $\lambda > 0$, the problem has at least three nontrivial solutions (two of constant sign). If we alter the geometry near the origin by introducing a "concave" nonlinearity (problem with combined nonlinearities), we show the existence of at least five nontrivial solutions (four of constant sign and the fifth nodal), when the parameter $\lambda > 0$ is small. Also, we produce extremal constant sign solutions $u_\lambda^* \in \operatorname{int} C_+$ and $v_\lambda^* \in -\operatorname{int} C_+$. We investigate the monotonicity and continuity properties of the map $\lambda \longmapsto u_\lambda^*$.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following parametric nonlinear Dirichlet problem:

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u(z) = \lambda f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Here Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \forall u \in W_0^{1,p}(\Omega),$$

where $1 < p < +\infty$. Also $\lambda > 0$ is a parameter and $f(z, \zeta)$ is a Carathéodory reaction (i.e., for all $\zeta \in \mathbb{R}$, the function $z \longmapsto f(z, \zeta)$ is measurable and for almost all $z \in \Omega$, the function $\zeta \longmapsto f(z, \zeta)$ is continuous). We assume that $f(z, \cdot)$ is $(p-1)$-superlinear near $\pm\infty$ and $(p-1)$-sublinear near zero. However, to express the superlinearity near $\pm\infty$, we do not employ the usual in such cases Ambrosetti-Rabinowitz condition.

Problem (P_λ) with $p = 2$ (semilinear equation) was investigated by Miyagaki-Souto [26], who established the existence of at least one nontrivial weak solution

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for all $\lambda > 0$. Subsequently, their result was extended to p-Laplacian equations by Li-Yang [25]. The aim of this work is to improve the aforementioned papers. More precisely, under more general conditions on the reaction, we show that for every $\lambda > 0$ problem (P_λ) admits at least three nontrivial solutions, two of which have constant sign (one positive and the other negative). By changing the hypotheses on the reaction near zero (hence we have a new geometry for the problem), we can produce extremal constant sign solutions (i.e., the smallest positive and the biggest negative solutions) and using them, we can generate a nodal (sign-changing) solution. Finally, in the semilinear case (i.e., $p = 2$), if by u_λ^* we denote the smallest positive solution, then we provide conditions for the map $\lambda \longmapsto u_\lambda^*$ to be continuous and monotone. Finally for other boundary value problems with the so called resonance, we refer to Gasiński [11] and Gasiński-Papageorgiou [12, 13, 14].

Our approach is variational based on the critical point theory, combined with suitable truncations and comparison techniques and Morse theory (critical groups). In the next section, for easy reference, we recall the main mathematical tools which we will use in the sequel.

2. **Mathematical Background**

Let X be a Banach space and let X^* be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X)$, we say that φ satisfies the *Cerami condition*, if the following is true:

"Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$, such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$\big(1 + \|u_n\|_X\big)\varphi'(u_n) \ \longrightarrow\ 0 \quad \text{in } X^*,$$

admits a strongly convergent subsequence."

This compactness type condition on the functional φ, leads to a deformation theorem from which one derives the minimax theory for the critical values of φ. One of the main results in that theory is the so called mountain pass theorem due to Ambrosetti-Rabinowitz [3]. Here we state the result in a slightly more general form (see e.g., Gasiński-Papageorgiou [15]).

Theorem 2.1. *If X is a Banach space, $\varphi \in C^1(X)$ satisfies the Cerami condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > \varrho > 0$,*

$$\max\big\{\varphi(u_0), \varphi(u_1)\big\} < \inf\big\{\varphi(u) : \ \|u - u_0\| = \varrho\big\} = \eta_\varrho,$$

and

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi\big(\gamma(t)\big),$$

where

$$\Gamma = \big\{\gamma \in C\big([0,1]; X\big) : \gamma(0) = u_0,\ \gamma(1) = u_1\big\},$$

then $c \geqslant \eta_\varrho$ and c is a critical value of φ.

In the study of problem (P_λ), we will use the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space

$$C_0^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\right\}.$$

By $\|\cdot\|$ we denote the norm of $W_0^{1,p}(\Omega)$. By virtue of the Poincaré inequality, we have

$$\|u\| = \|\nabla u\|_p \quad \forall u \in W_0^{1,p}(\Omega).$$

Also, we will exploit the fact that $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive cone

$$C_+ = \left\{u \in C_0^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} < 0\right\},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

In what follows, by $\widehat{\lambda}_1$ we denote the first eigenvalue of $\big(-\Delta_p, W_0^{1,p}(\Omega)\big)$. We know that $\widehat{\lambda}_1$ is positive, isolated, simple and admits the following variational characterization

$$\widehat{\lambda}_1 = \inf\left\{\frac{\|\nabla u\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega),\ u \neq 0\right\}. \tag{2.1}$$

In this expression the infimum is realized on the corresponding one-dimensional eigenspace. Also, it is clear that the elements of this eigenspace do not change sign. Let $\widehat{u}_1$ be the L^p-normalized (that is, $\|\widehat{u}_1\|_p = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1 > 0$. The nonlinear regularity theory and the nonlinear maximum principle (see e.g., Gasiński-Papageorgiou [15, pp. 737–738]), imply that $\widehat{u}_1 \in \operatorname{int} C_+$.

Let $A\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega)$ (where $\frac{1}{p} + \frac{1}{p'} = 1$) be the nonlinear map defined by

$$\langle A(u), y\rangle = \int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla y)_{\mathbb{R}^N} dz \quad \forall u, y \in W_0^{1,p}(\Omega). \tag{2.2}$$

Proposition 2.2. *The map $A\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega)$ defined above is bounded (i.e., maps bounded sets into bounded ones), demicontinuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$ (i.e., if $u_n \longrightarrow u$ weakly in $W_0^{1,p}(\Omega)$ and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leqslant 0$, then $u_n \longrightarrow u$ in $W_0^{1,p}(\Omega)$).*

Let X be a Banach space and let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geqslant 0$ by $H_k(Y_1, Y_2)$ we denote the k-th singular homology group with coefficients in a field $\mathbb{F}$ of characteristic zero for the pair (Y_1, Y_2) (for example $\mathbb{F} = \mathbb{R}$). We know that each group $H_k(Y_1, Y_2)$ is an fact an $\mathbb{F}$-vector space. Recall that $H_k(Y_1, Y_2) = 0$ for all integers $k \geqslant 0$.

Given $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\begin{aligned} \varphi^c &= \{x \in X : \varphi(x) \leqslant c\}, \\ K_\varphi &= \{x \in X : \varphi'(x) = 0\}, \\ K_\varphi^c &= \{x \in K_\varphi : \varphi(x) = c\}. \end{aligned}$$

The critical groups of φ at an isolated element $u \in K_\varphi^c$ are defined by

$$C_k(\varphi, u) = H_k\big(\varphi^c \cap U,\ \varphi^c \cap U \setminus \{x\}\big) \quad \forall k \geqslant 0,$$

with U being a neighbourhood of $u \in X$, such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology, implies that the above definition of critical groups is independent of the particular choice of the neighbourhood U.

Suppose that φ satisfies the Cerami condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. Then the critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \forall k \geqslant 0.$$

The second deformation theorem (see e.g., Gasiński-Papageorgiou [15, Theorem 5.1.33, p. 628]), implies that the above definition is independent of the particular choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose that K_φ is finite. We introduce the following quantities:

$$M(t, u) = \sum_{k \geqslant 0} \operatorname{rank} C_k(\varphi, u) t^k \quad \forall t \in \mathbb{R},\ u \in K_\varphi$$

and

$$P(t, \infty) = \sum_{k \geqslant 0} \dim C_k(\varphi, \infty) t^k \quad \forall t \in \mathbb{R}.$$

The Morse relation says that

$$\sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1+t) Q(t), \tag{2.3}$$

where

$$Q(t) = \sum_{k \geqslant 0} \beta_k t^k$$

is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

Finally, let us fix some notation. For $\zeta \in \mathbb{R}$, we set $\zeta^\pm = \max\{\pm\zeta, 0\}$ and for $u \in W_0^{1,p}(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$. We have

$$u^\pm \in W_0^{1,p}(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^-.$$

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Also, if $h \colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function (for example a Carathéodory function), then we set

$$N_h(u)(\cdot) = h\big(\cdot, u(\cdot)\big) \quad \forall u \in W_0^{1,p}(\Omega)$$

(the Nemytski map corresponding to h). Note that $z \longmapsto N_h(u)(z)$ is measurable.

3. Multiple Solutions for $\lambda > 0$

In this section we prove a three nontrivial solutions theorem valid for all $\lambda > 0$. The hypotheses on the function $f(z,\zeta)$ are the following:

$\underline{H_1:}$ $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $f(z,0) = 0$ for almost all $z \in \Omega$ and

(i): there exist a function $a \in L^\infty(\Omega)_+$ and $r \in (p, p^*)$, where

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if} \quad p < N, \\ +\infty & \text{if} \quad N \leqslant p \end{cases}$$

such that

$$|f(z,\zeta)| \leqslant a(z)(1 + |\zeta|^{r-1}) \quad \text{for almost all } z \in \Omega, \ \text{all } \zeta \in \mathbb{R};$$

(ii): if

$$F(z,\zeta) = \int_0^\zeta f(z,s)\, ds,$$

then

$$\lim_{\zeta \to \pm\infty} \frac{F(z,\zeta)}{|\zeta|^p} = +\infty$$

uniformly for almost all $z \in \Omega$;

(iii): if

$$\xi(z,\zeta) = f(z,\zeta)\zeta - pF(z,\zeta),$$

then there exists $\beta \in L^1(\Omega)_+$ such that

$$\xi(z,\zeta) \leqslant \xi(z,y) + \beta(z) \quad \text{for almost all } z \in \Omega, \ \text{all } 0 \leqslant \zeta \leqslant y \text{ or } y \leqslant \zeta \leqslant 0;$$

(iv): we have

$$\lim_{\zeta \to 0} \frac{f(z,\zeta)}{|\zeta|^{p-2}\zeta} = 0$$

uniformly for almost all $z \in \Omega$.

Remark 3.1. Clearly hypotheses $H_1(ii)$ and (iii) imply that

$$\lim_{\zeta \to \pm\infty} \frac{f(z,\zeta)}{|\zeta|^{p-2}\zeta} = +\infty \quad \text{uniformly for almost all } z \in \Omega,$$

hence $f(z,\cdot)$ is $(p-1)$-superlinear near $\pm\infty$. Hypothesis $H_1(iii)$ is a quasimonotonicity condition on $\xi(z,\cdot)$. It is satisfied if, for example, we can find $M_1 > 0$ such that for almost all $z \in \Omega$

$$\zeta \longmapsto \frac{f(z,\zeta)}{\zeta^{p-1}} \text{ is nondecreasing on } [M_1, +\infty),$$

$$\zeta \longmapsto \frac{f(z,\zeta)}{|\zeta|^{p-2}\zeta} \text{ is nonincreasing on } (-\infty, -M_1]$$

(see Li-Yang [25]). Conditions $H_1(ii)$ and (iii) replace the Ambrosetti-Rabinowitz condition which says that there exist $q > p$ and $M_2 > 0$ such that

$$f(z,\zeta)\zeta \geqslant qF(z,\zeta) \ > \ 0 \quad \text{for almost all } z \in \Omega, \text{ all } |\zeta| \geqslant M_2$$

and

$$\operatorname*{ess\,inf}_{\Omega} F(\cdot, \pm M_2) \ > \ 0.$$

A direct integration, leads to the following unilateral growth estimate

$$c_1|\zeta|^q \leqslant F(z,\zeta) \quad \text{for almost all } z \in \Omega, \text{ all } |\zeta| \geqslant M_2,$$

for some $c_1 > 0$.

Hypotheses $H_1(ii)$ and (iii) incorporate in our framework superlinear reactions with "slower" growth near $\pm\infty$ which fail to satisfy the Ambrosetti-Rabinowitz condition (see Example 3.2 below)

Example 3.2. The following functionals satisfy hypotheses H_1. For the sake of simplicity, we drop the z-dependence:

$$\begin{aligned} f_1(\zeta) &= |\zeta|^{q-2}\zeta \quad \text{with } p < q < p^*, \\ f_2(\zeta) &= |\zeta|^{p-2}\zeta \ln(1+|\zeta|). \end{aligned}$$

Note that f_2 fails to satisfy the Ambrosetti-Rabinowitz condition.

First we will produce solutions of constant sign. To this end, we introduce the positive and negative truncations of $f(z,\cdot)$. So, we introduce the Carathéodory functions:

$$f_\pm(z,\zeta) = f(z, \pm\zeta^\pm) \quad \forall (z,\zeta) \in \Omega \times \mathbb{R}.$$

We set

$$F_\pm(z,\zeta) = \int_0^\zeta f_\pm(z,s)\,ds$$

and consider the C^1-functionals $\varphi_\lambda^\pm \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\varphi_\lambda^\pm(u) = \frac{1}{p}\|\nabla u\|_p^p - \lambda \int_\Omega F_\pm(z,u(z))\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Proposition 3.3. *If hypotheses H_1 hold and $\lambda > 0$, then the functionals $\varphi_\lambda^\pm$ satisfy the Cerami condition.*

Proof. We do the proof for the functional φ_λ^+, the proof for φ_λ^- being similar.

Let $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that $\{\varphi_\lambda^+(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$(1+\|u_n\|)(\varphi_\lambda^+)'(u_n) \ \longrightarrow \ 0 \quad \text{in } W^{-1,p'}(\Omega) \quad \text{as } n \to +\infty.$$

We have

$$\left| \langle A(u_n), h\rangle - \lambda \int_\Omega f_+(z,u_n)h\,dz \right| \leqslant \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \quad \forall h \in W_0^{1,p}(\Omega), \tag{3.1}$$

with $\varepsilon_n \searrow 0$.

In (3.1) first we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$. Then

$$\|\nabla u_n^-\|_p^p \leqslant \varepsilon_n \quad \forall n \geqslant 1,$$

so

$$u_n^- \longrightarrow 0 \quad \text{in } W_0^{1,p}(\Omega). \tag{3.2}$$

The boundedness of the sequence $\{\varphi_\lambda^+(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ and (3.2) imply that

$$\left| \|\nabla u_n^+\|_p^p - \lambda \int_\Omega pF(z, u_n^+)\, dz \right| \leqslant M_3 \quad \forall n \geqslant 1, \tag{3.3}$$

for some $M_3 > 0$. Also, if in (3.1) we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$, then

$$-\|\nabla u_n^+\|_p^p + \lambda \int_\Omega f(z, u_n^+) u_n^+\, dz \leqslant \varepsilon_n \quad \forall n \geqslant 1. \tag{3.4}$$

From (3.3) and (3.4), it follows that

$$\lambda \int_\Omega \xi(z, u_n^+)\, dz \leqslant M_4 \quad \forall n \geqslant 1, \tag{3.5}$$

for some $M_4 > 0$.

Claim. The sequence $\{u_n^+\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded.

We argue by contradiction. So, suppose that the Claim is not true. By passing to a subsequence if necessary, we may assume that

$$\|u_n^+\| = \|\nabla u_n^+\|_p \longrightarrow +\infty \quad \text{as } n \to +\infty. \tag{3.6}$$

Let $y_n = \frac{u_n^+}{\|u_n^+\|}$ for $n \geqslant 1$. Then $\|y_n\| = 1$, $y_n \geqslant 0$ for all $n \geqslant 1$ and so, passing to a subsequence if necessary, we may assume that

$$y_n \longrightarrow y \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{3.7}$$

$$y_n \longrightarrow y \quad \text{in } L^r(\Omega),\ y \geqslant 0. \tag{3.8}$$

If $y \neq 0$, then setting $\Omega_0 = \Omega \setminus y^{-1}(0)$, we have $|\Omega_0|_N > 0$ and

$$u_n^+(z) \longrightarrow +\infty \quad \text{as } n \to +\infty, \quad \forall z \in \Omega_0.$$

Then hypothesis $H_1(ii)$ implies that

$$\frac{F(z, u_n^+(z))}{\|u_n^+\|^p} = \frac{F(z, u_n^+(z))}{u_n^+(z)^p} y_n(z)^p \longrightarrow +\infty \quad \text{as } n \to +\infty \quad \forall z \in \Omega_0.$$

By virtue of Fatou's lemma (see $H_1(ii)$), we have

$$\int_\Omega \frac{F(z, u_n^+(z))}{\|u_n^+\|^p}\, dz \longrightarrow +\infty \quad \text{as } n \to +\infty. \tag{3.9}$$

From (3.3) we have

$$\lambda \int_\Omega p \frac{F(z, u_n^+)}{\|u_n^+\|^p}\, dz \leqslant M_5 \quad \forall n \geqslant 1, \tag{3.10}$$

for some $M_5 > 0$. Comparing (3.9) and (3.10), we reach a contradiction. This proves that $y \equiv 0$. We fix $k > 0$ and set $\widehat{y}_n = (2kp)^{\frac{1}{p}} y_n$. From (3.7) and hypothesis $H_1(i)$ it follows that

$$\int_\Omega F(z, \widehat{y}_n(z))\, dz \ \longrightarrow\ 0 \quad \text{as } n \to +\infty. \tag{3.11}$$

Also, from (3.6) we see that we can find an integer $n_0 \geqslant 1$ such that

$$0 < (2kp)^{\frac{1}{p}} \frac{1}{\|u_n^+\|} < 1 \quad \forall n \geqslant n_0. \tag{3.12}$$

Let $t_n \in [0, 1]$ be such that

$$\varphi_\lambda^+(t_n u_n^+) = \max_{t \in [0,1]} \varphi_\lambda^+(t u_n^+).$$

From (3.11) and (3.12) we see that we can find an integer $n_1 \geqslant n_0$ such that

$$\varphi_\lambda^+(t_n u_n^+) \geqslant \varphi_\lambda^+(\widehat{y}_n) = 2k - \lambda \int_\Omega F(z, \widehat{y}_n)\, dz \ >\ k \quad \forall n \geqslant n_1 \geqslant n_0.$$

Since $k > 0$ is arbitrary, it follows that

$$\varphi_\lambda^+(t_n u_n^+) \ \longrightarrow\ +\infty \quad \text{as } n \to +\infty. \tag{3.13}$$

Note that the sequence $\{\varphi_\lambda^+(u_n^+)\}_{n \geqslant 1}$ is bounded (see (3.2) and recall that the sequence $\{\varphi_\lambda^+(u_n)\}_{n \geqslant 1}$ is bounded). Also $\varphi_\lambda^+(0) = 0$. So, from (3.13) we infer that $t_n \in (0, 1)$ for all $n \geqslant 1$ and so we have

$$\frac{d}{dt} \varphi_\lambda^+(t u_n^+)\Big|_{t = t_n} = 0,$$

so

$$\langle A(t_n u_n^+),\ u_n^+ \rangle - \lambda \int_\Omega f(z, t_n u_n^+) u_n^+\, dz = 0,$$

thus

$$\|\nabla(t_n u_n^+)\|_p^p = \lambda \int_\Omega f(z, t_n u_n^+)(t_n u_n^+)\, dz \quad \forall n \geqslant 1. \tag{3.14}$$

We have $0 \leqslant t_n u_n^+ \leqslant u_n^+$ for all $n \geqslant 1$ and so hypothesis $H_1(iii)$ implies that

$$\lambda \int_\Omega \xi(z, t_n u_n^+)\, dz \leqslant \lambda \int_\Omega \xi(z, u_n^+)\, dz + \lambda \|\beta\|_1 \quad \forall n \geqslant 1,$$

so

$$\lambda \int_\Omega \xi(z, t_n u_n^+)\, dz \leqslant M_6 \quad \forall n \geqslant 1,$$

with $M_6 = M_4 + \lambda \|\beta\|_1$ (see (3.5)). Thus

$$\|\nabla(t_n u_n^+)\|_p^p - \lambda \int_\Omega pF(z, t_n u_n^+)\, dz \leqslant M_6 \quad \forall n \geqslant 1$$

(see (3.14)) and we get

$$\varphi_\lambda^+(t_n u_n^+) \leqslant \frac{M_6}{p} \quad \forall n \geqslant 1. \tag{3.15}$$

Comparing (3.13) and (3.15), we reach a contradiction. Therefore the sequence $\{u_n^+\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. This fact and (3.2) imply that the sequence $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. This proves the Claim.

By virtue of the Claim, passing to a subsequence if necessary, we may assume that

$$u_n \longrightarrow u \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{3.16}$$

$$u_n \longrightarrow u \quad \text{in } L^r(\Omega). \tag{3.17}$$

In (3.1) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (3.16). Then

$$\lim_{n\to+\infty} \langle A(u_n), u_n - u\rangle = 0,$$

so

$$u_n \longrightarrow u \quad \text{in } W_0^{1,p}(\Omega)$$

(see (3.16) and Proposition 2.2) and thus φ_λ^+ satisfies the Cerami condition.

Similarly we show that φ_λ^- also satisfies the Cerami condition. $\square$

Let $\varphi_\lambda \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the energy functional for problem (P_λ), namely

$$\varphi_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p - \lambda \int_\Omega F(z, u(z))\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Evidently $\varphi_\lambda \in C^1(W_0^{1,p}(\Omega))$.

Minor changes in the proof of Proposition 3.3 lead to the following result.

Proposition 3.4. *If hypotheses H_1 hold and $\lambda > 0$, then the energy functional φ_λ satisfies the Cerami condition.*

In the next two propositions, we will verify the mountain pass geometry for the functionals $\varphi_\lambda^\pm$, $\lambda > 0$.

Proposition 3.5. *If hypotheses H_1 hold and $\lambda > 0$, then $u = 0$ is a local minimizer for the functionals $\varphi_\lambda^\pm$ and φ_λ.*

Proof. We do the proof for φ_λ^+, the proofs for φ_λ^+ and φ_λ being similar.

By virtue of hypothesis $H_1(iv)$, we see that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon, \lambda) > 0$ such that

$$\lambda F(z, \zeta) \leqslant \frac{\varepsilon}{p}|\zeta|^p \quad \text{for almost all } z \in \Omega,\ \text{all } |\zeta| \leqslant \delta. \tag{3.18}$$

So, if $u \in C_0^1(\overline{\Omega})$ with $\|u\|_{C_0^1(\overline{\Omega})} \leqslant \delta$, then

$$\varphi_\lambda^+(u) \;=\; \frac{1}{p}\|\nabla u\|_p^p - \lambda \int_\Omega F(z, u^+)\,dz \geqslant \frac{1}{p}\|\nabla u\|_p^p - \frac{\varepsilon}{p}\|u\|_p^p$$

$$\geqslant \frac{1}{p}\left(1-\frac{\varepsilon}{\widehat{\lambda}_1(p)}\right)\|u\|^p$$

(see (3.18)), so $u = 0$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ_λ^+ and so it is also a local $W_0^{1,p}(\Omega)$-minimizer of φ_λ^+ (see García Azorero-Manfredi-Peral Alonso [10]).

Similarly for the functionals φ_λ^- and φ_λ. $\square$

The next result is a straightforward consequence of hypothesis $H_1(ii)$.

Proposition 3.6. *If hypotheses H_1 hold, $\lambda > 0$ and $u \in \operatorname{int} C_+$, then $\varphi_\lambda^\pm(tu) \longrightarrow -\infty$ as $t \to \pm\infty$.*

Now we are ready to produce two nontrivial solutions of constant sign.

Proposition 3.7. *If hypotheses H_1 hold and $\lambda > 0$, then problem (P_λ) admits at least two solutions of constant sign*

$$u_0 \in \operatorname{int} C_+ \quad \textit{and} \quad v_0 \in -\operatorname{int} C_+.$$

Proof. Proposition 3.5 implies that we can find $\varrho \in (0,1)$ small such that

$$\varphi_\lambda^+(0) = 0 < \inf\{\varphi_\lambda^+(u) : \|u\| = \varrho\} = \eta_{\lambda,\varrho}^+ \tag{3.19}$$

(see Aizicovici-Papageorgiou-Staicu [1, Proposition 29] or Gasiński-Papageorgiou [16, proof of Theorem 3.4]). Combining (3.19) with Propositions 3.3 and 3.6, we see that we can apply Theorem 2.1 (the mountain pass theorem). So, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\eta_{\lambda,\varrho}^+ \leqslant \varphi_\lambda^+(u_0) \quad \text{and} \quad (\varphi_\lambda^+)'(u_0) = 0. \tag{3.20}$$

From (3.19) and the inequality in (3.20), it follows that $u_0 \neq 0$. From the equality in (3.20), we have

$$A(u_0) = \lambda N_{f_+}(u_0). \tag{3.21}$$

On (3.21) we act with $-u_0^- \in W_0^{1,p}(\Omega)$ and obtain

$$\|\nabla u_0^-\|_p^p = 0,$$

so

$$u_0 \geqslant 0, \quad u_0 \neq 0.$$

Then (3.21) becomes

$$A(u_0) = \lambda N_f(u_0),$$

so

$$\begin{cases} -\Delta_p u_0(z) = \lambda f(z, u_0(z)) & \text{in } \Omega, \\ u_0|_{\partial\Omega} = 0. \end{cases} \tag{3.22}$$

The nonlinear regularity theory (see e.g., Gasiński-Papageorgiou [15, pp. 737–738]) implies that $u_0 \in C_+ \setminus \{0\}$.

Note that hypotheses $H_1(i)$ and (iv) imply that given $\varrho > 0$, we can find $\xi_\varrho = \xi_\varrho(\lambda) > 0$ such that

$$\lambda f(z,\zeta)\zeta + \xi_p|\zeta|^p \geqslant 0 \quad \text{for almost all } z \in \Omega, \text{ all } |\zeta| \leqslant \varrho.$$

Let $\varrho = \|u_0\|_\infty$ and let $\xi_\varrho = \xi_\varrho(\lambda) > 0$ be as above. From (3.22) we have

$$-\Delta_p u_0(z) + \xi_\varrho u_0(z)^{p-1} = \lambda f(z, u_0(z)) + \xi_\varrho u_0(z)^{p-1} \geqslant 0,$$

almost everywhere in Ω, so

$$\Delta_p u_0(z) \leqslant \xi_p u_0(z)^{p-1} \quad \text{almost everywhere in } \Omega.$$

Invoking the nonlinear maximum principle (see e.g., Gasiński-Papageorgiou [15, p. 738]), we have that $u_0 \in \operatorname{int} C_+$.

Similarly, working with the functional φ_λ^-, we produce a second constant sign solution $v_0 \in -\operatorname{int} C_+$. □

In fact, we can produce extremal constant sign solutions for problem (P_λ), that is the smallest positive solution and the biggest negative solution of (P_λ).

We introduce the following solution sets:

$$\begin{aligned} S_\lambda^+ &= \big\{u \in W_0^{1,p}(\Omega) : u \text{ is a positive solution of } (P_\lambda)\big\}, \\ S_\lambda^- &= \big\{u \in W_0^{1,p}(\Omega) : u \text{ is a negative solution of } (P_\lambda)\big\}. \end{aligned}$$

From Proposition 3.7 and its proof, we have

$$\emptyset \neq S_\lambda^+ \subseteq \operatorname{int} C_+ \quad \text{and} \quad \emptyset \neq S_\lambda^- \subseteq -\operatorname{int} C_+.$$

Moreover, from Filippakis-Kristaly-Papageorgiou [9], we know that S_λ^+ is downward directed (that is, if $u_1, u_2 \subseteq S_\lambda^+$, then there exists $u \in S_\lambda^+$ such that $u \leqslant u_1$, $u \leqslant u_2$) and S_λ^- is upward directed (that is, if $v_1, v_2 \subseteq S_\lambda^-$, then there exists $v \in S_\lambda^-$ such that $v_1 \leqslant v$, $v_2 \leqslant v$; see also Gasiński-Papageorgiou [16]).

Proposition 3.8. *If hypotheses H_1 hold and $\lambda > 0$, then problem (P_λ) admits the smallest positive solution $u_\lambda \in \operatorname{int} C_+$ and a biggest negative solution $v_\lambda \in -\operatorname{int} C_+$.*

Proof. Since S_λ^+ is downward directed and we are looking for the minimal positive solution, without any loss of generality, we may assume that

$$\|u\|_\infty \leqslant M_7 \quad \forall u \in S_\lambda^+, \tag{3.23}$$

for some $M_7 > 0$. From Dunford-Schwartz [8, p. 336], we know that we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq S_\lambda^+$ such that

$$\inf S_\lambda^+ = \inf_{n\geqslant 1} u_n.$$

We have

$$A(u_n) = \lambda N_f(u_n), \quad u_n \geqslant 0 \quad \forall n \geqslant 1, \tag{3.24}$$

so the sequence $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded (see (3.23)).

So, we may assume that

$$u_n \longrightarrow u_\lambda \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{3.25}$$

$$u_n \longrightarrow u_\lambda \quad \text{in } L^r(\Omega). \tag{3.26}$$

On (3.24) we act with $u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (3.25). Then

$$\lim_{n\to+\infty} \langle A(u_n), u_n - u_\lambda \rangle = 0,$$

so

$$u_n \longrightarrow u_\lambda \quad \text{in } W_0^{1,p}(\Omega) \tag{3.27}$$

(see Proposition 2.2).

In (3.24) we pass to the limit as $n \to +\infty$ and use (3.27). Then

$$A(u_\lambda) = \lambda N_f(u_\lambda),$$

so

$$u_\lambda \text{ is a solution of } (P_\lambda),\ u_\lambda \geqslant 0. \tag{3.28}$$

We show that $u_\lambda \neq 0$. We argue by contradiction. So, suppose that $u_\lambda = 0$. Then

$$u_n \longrightarrow 0 \quad \text{in } W_0^{1,p}(\Omega)$$

(see (3.27)).

Let $y_n = \frac{u_n}{\|u_n\|}$ for $n \geqslant 1$. Then $\|y_n\| = 1$ for all $n \geqslant 1$ and so, passing to a subsequence if necessary, we may assume that

$$y_n \longrightarrow y \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{3.29}$$

$$y_n \longrightarrow y \quad \text{in } L^r(\Omega). \tag{3.30}$$

From (3.24), we have

$$A(y_n) = \frac{\lambda N_f(u_n)}{\|u_n\|^{p-1}} \quad \forall n \geqslant 1. \tag{3.31}$$

Hypotheses $H_1(i)$ and (iv) imply that

$$|f(z,\zeta)| \leqslant c_1(|\zeta|^{p-1} + |\zeta|^{r-1}) \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R},$$

for some $c_1 > 0$, so

$$\text{the sequence } \left\{ \frac{N_f(u_n)}{\|u_n\|^{p-1}} \right\}_{n\geqslant 1} \subseteq L^{p'}(\Omega) \text{ is bounded}$$

(see (3.23)).

So, by passing to a subsequence if necessary and using hypothesis $H_1(iv)$, we have

$$\frac{N_f(u_n)}{\|u_n\|^{p-1}} \longrightarrow 0 \quad \text{weakly in } L^{p'}(\Omega). \tag{3.32}$$

On (3.31) we act with $y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (3.29) and (3.32). Then

$$\lim_{n\to+\infty} \langle A(y_n), y_n - y\rangle = 0,$$

so

$$y_n \longrightarrow y \quad \text{in } W_0^{1,p}(\Omega), \quad \|y\| = 1. \tag{3.33}$$

So, if in (3.31) we pass to the limit as $n \to +\infty$ and use (3.32), (3.33), we obtain

$$A(y) = 0,$$

so

$$y = 0,$$

which contradicts (3.33).

Therefore $u_\lambda \neq 0$ and so

$$u_\lambda \in S_\lambda^+$$

(see (3.28)) and $u_\lambda = \inf S_\lambda^+$.

Similarly, working this time with S_λ^-, we produce the biggest negative solution $v_\lambda \in -\operatorname{int} C_+$ of (P_λ). $\square$

Next using tools from Morse theory (critical groups), we will produce a third nontrivial solution of S_λ^+ but we are not able to provide sign information for this third solution.

First we compute the critical groups of φ_λ at infinity. Our proposition extends a similar result proved by Wang [28] (for $p = 2$) and Liu [24] (for $1 < p < +\infty$), for reactions satisfying the Ambrosetti-Rabinowitz condition.

Proposition 3.9. *If hypotheses H_1 hold and $\lambda > 0$, then*

$$C_k(\varphi_\lambda, \infty) = 0 \quad \forall k \geqslant 0.$$

Proof. By virtue of hypotheses $H_1(i), (ii)$ given any $\eta > 0$, we can find $c_2 = c_2(\eta) > 0$ such that

$$F(z,\zeta) \geqslant \frac{\eta}{p}|\zeta|^p - c_2 \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R}. \tag{3.34}$$

Let $u \in \partial B_1 = \{y \in W_0^{1,p}(\Omega) : \|y\| = 1\}$ and $t > 0$. Then

$$\varphi_\lambda(tu) \quad = \quad \frac{t^p}{p}\|\nabla u\|_p^p - \lambda \int_\Omega F(z,tu)\,dz$$

$$\begin{aligned} &\leqslant \frac{t^p}{p}\|\nabla u\|_p^p - \frac{t^p}{p}\eta\lambda\|u\|_p^p + c_2\lambda|\Omega|_N \\ &= \frac{t^p}{p}\big(1-\eta\lambda\|u\|_p^p\big) + c_2\lambda|\Omega|_N \end{aligned} \tag{3.35}$$

(see (3.34) and use the fact that $u \in \partial B_1$). So, if $\eta > \frac{1}{\lambda\|u\|_p^p}$, then from (3.35) we see that

$$\varphi_\lambda(tu) \longrightarrow -\infty \quad \text{as } t \to +\infty. \tag{3.36}$$

By virtue of hypothesis $H_1(iii)$, we have

$$0 = \xi(z,0) \leqslant \xi(z,u^+(z)) + \beta(z) \quad \text{almost everywhere in } \Omega$$

and

$$0 = \xi(z,0) \leqslant \xi(z,-u^-(z)) + \beta(z) \quad \text{almost everywhere in } \Omega,$$

so

$$0 = \xi(z,0) \leqslant \xi(z,u(z)) + \beta(z) \quad \text{almost everywhere in } \Omega$$

and thus

$$-\xi(z,u(z)) = pF(z,u(z)) - f(z,u(z))u(z) \leqslant \beta(z) \tag{3.37}$$

almost everywhere in Ω. Let $u \in W_0^{1,p}(\Omega)$ and let $t > 0$. then

$$\begin{aligned} \frac{d}{dt}\varphi_\lambda(tu) &= \langle \varphi_\lambda'(tu), u\rangle = \frac{1}{t}\langle \varphi_\lambda'(tu), tu\rangle \\ &= \frac{1}{t}\bigg(\|\nabla(tu)\|_p^p - \lambda\int_\Omega f(z,tu)tu\,dz\bigg) \\ &\leqslant \frac{1}{t}\bigg(\|\nabla(tu)\|_p^p - \lambda\int_\Omega pF(z,tu)\,dz + \lambda\|\beta\|_1\bigg) \\ &= \frac{1}{t}\big(p\varphi_\lambda(tu) + \lambda\|\beta\|_1\big) \end{aligned} \tag{3.38}$$

(see (3.37)). From (3.36) it is clear that, if $u \neq 0$ and $t > 0$ is big, then

$$\varphi_\lambda(tu) \leqslant \vartheta_0 < -\frac{\lambda\|\beta\|_1}{p},$$

so

$$\frac{d}{dt}\varphi_\lambda(tu) < 0 \quad \text{for } t > 0 \text{ big}. \tag{3.39}$$

So, for $u \in \partial B_1$ we can find a unique $\gamma(u) > 0$ such that $\varphi_\lambda(\gamma(u)u) = \vartheta_0$ and the implicit function theorem (see (3.39)) implies that $\gamma \in C(\partial B_1)$. We extend γ to $W_0^{1,p}(\Omega) \setminus \{0\}$ by setting

$$\gamma_0(u) = \frac{1}{\|u\|}\gamma\bigg(\frac{u}{\|u\|}\bigg) \quad \forall u \in W_0^{1,p}(\Omega) \setminus \{0\}.$$

Evidently $\gamma_0 \in C(W_0^{1,p}(\Omega) \setminus \{0\})$ and $\varphi_\lambda(\gamma_0(u)u) = \vartheta_0$. Note that, if $\varphi(u) = \vartheta_0$, then $\gamma_0(u) = 1$. So, if we define

$$\widehat{\gamma}_0(u) = \begin{cases} 1 & \text{if } \varphi_\lambda(u) < \vartheta_0, \\ \gamma_0(u) & \text{if } \varphi_\lambda(u) \geqslant \vartheta_0, \end{cases}$$

then clearly $\widehat{\gamma}_0 \in C(W_0^{1,p}(\Omega) \setminus \{0\})$.

We introduce the deformation

$$h(t,u) \ = \ (1-t)u + t\widehat{\gamma}_0(u)u \quad \forall (t,u) \in [0,1] \times (W_0^{1,p}(\Omega) \setminus \{0\}).$$

We have

$$h(0,u) = u, \quad h(1,u) = \widehat{\gamma}_0(u)u \in \varphi_\lambda^{\vartheta_0}$$

and

$$h(t,\cdot)|_{\varphi_\lambda^{\vartheta_0}} = id|_{\varphi_\lambda^{\vartheta_0}} \quad \forall t \in [0,1],$$

so $\varphi_\lambda^{\vartheta_0}$ is a strong deformation retract of $W_0^{1,p}(\Omega) \setminus \{0\}$.

The radial retraction implies that ∂B_1 is a retract of $W_0^{1,p}(\Omega) \setminus \{0\}$ and the latter is deformable onto ∂B_1. So, from Dugundji [7, p. 325], we have that ∂B_1 is a deformation retract of $W_0^{1,p}(\Omega) \setminus \{0\}$. Therefore, it follows that

$$\varphi_\lambda^{\vartheta_0} \text{ and } \partial B_1 \text{ are homotopy equivalent,}$$

so

$$H_k(W_0^{1,p}(\Omega),\ \varphi_\lambda^{\vartheta_0}) = H_k(W_0^{1,p}(\Omega),\ \partial B_1) \quad \forall k \geqslant 0. \tag{3.40}$$

Since $W_0^{1,p}(\Omega)$ is infinite dimensional, we have that ∂B_1 is contractible in itself. Therefore we have

$$H_k(W_0^{1,p}(\Omega),\ \partial B_1) = 0 \quad \forall k \geqslant 0$$

(see Granas-Dugundji [21, p. 389]), so

$$H_k(W_0^{1,p}(\Omega),\ \varphi_\lambda^{\vartheta_0}) = 0 \quad \forall k \geqslant 0$$

(see (3.40)) and finally

$$C_k(\varphi_\lambda, \infty) = 0 \quad \forall k \geqslant 0$$

(choosing $\varphi_0 < 0$ even smaller if necessary). $\square$

We can have an analogous result for the truncated functionals $\varphi_\lambda^\pm$.

Proposition 3.10. *If hypotheses H_1 hold and $\lambda > 0$, then*

$$C_k(\varphi_\lambda^\pm, \infty) = 0 \quad \forall k \geqslant 0.$$

Proof. Let $\partial B_1^+ = \{u \in \partial B_1 : u^+ \neq 0\}$. We consider the deformation $h_+\colon [0,1] \times \partial B_1^+ \longrightarrow \partial B_1^+$ defined by

$$h_+(t,u) = \frac{(1-t)u + t\widehat{u}_1}{\|(1-t)u + t\widehat{u}_1\|} \quad \forall (t,u) \in [0,1] \times \partial B_1^+.$$

Note that

$$h_+(1,u) = \frac{\widehat{u}_1}{\|\widehat{u}_1\|} \in \partial B_1^+,$$

so ∂B_1^+ is contractible in itself.

By virtue of hypothesis $H_1(ii)$, for every $u \in \partial B_1^+$ we have

$$\varphi_\lambda^+(tu) \longrightarrow -\infty \quad \text{as } t \to +\infty. \tag{3.41}$$

From hypothesis $H_1(iii)$, we have

$$0 = \xi(z,0) \leqslant \xi(z,u^+(z)) + \beta(z) \quad \text{almost everywhere in } \Omega,$$

so

$$pF(z,u^+(z)) - f(z,u^+(z))u^+(z) \leqslant \beta(z) \quad \text{almost everywhere in } \Omega,$$

thus

$$-f_+(z,u(z))u(z) \leqslant \beta(z) - pF_+(z,u(z)) \quad \text{almost everywhere in } \Omega. \tag{3.42}$$

We fix $u \in \partial B_1^+$. Then for $t > 0$, we have

$$\begin{aligned} \frac{d}{dt}\varphi_\lambda^+(tu) &= \langle (\varphi_\lambda^+)'(tu), u\rangle = \frac{1}{t}\langle (\varphi_\lambda^+)'(tu), tu\rangle \\ &= \frac{1}{t}\Big(\|\nabla(tu)\|_p^p - \lambda\int_\Omega f_+(z,tu)(tu)\,dz\Big) \\ &\leqslant \frac{1}{t}\Big(\|\nabla(tu)\|_p^p - \lambda\int_\Omega pF_+(z,tu)\,dz + \lambda\|\beta\|_1\Big) \\ &= \frac{1}{t}\big(p\varphi_\lambda^+(tu) + \lambda\|\beta\|_1\big). \end{aligned} \tag{3.43}$$

From (3.41) it follows that

$$\frac{d}{dt}\varphi_\lambda^+(tu) < 0 \quad \text{for } t > 0 \text{ big} \tag{3.44}$$

(such that $\varphi_\lambda^+(tu) < -\frac{\lambda\|\beta\|_1}{p}$). Let $\eta \in \mathbb{R}$ be such that

$$\eta < \min\Big\{-\frac{\lambda\|\beta\|_1}{p},\ \inf_{\overline{B}_1}\varphi_\lambda^+\Big\}.$$

From (3.44) we infer that there exists unique $\mu(u) \geqslant 1$ such that

$$\varphi_\lambda^+(tu) \begin{cases} > \eta & \text{if } \ t \in [0,\mu(u)), \\ = \eta & \text{if } \ t = \mu(u), \\ < \eta & \text{if } \ t > \mu(u). \end{cases} \tag{3.45}$$

Moreover, as before the implicit function theorem implies that $\mu \in C(\partial B_1^+)$. Also, we have

$$(\varphi_\lambda^+)^\eta = \big\{tu : u \in \partial B_1^+,\ t \geqslant \mu(u)\big\}$$

(see (3.45)). Let $E_+ = \{tu : u \in \partial B_1^+,\ t \geqslant 1\}$. Then $(\varphi_\lambda^+)^\eta \subseteq E_+$. We consider the deformation $\widehat{h}_+ \colon [0,1] \times E_+ \longrightarrow E_+$ defined by

$$\widehat{h}_+(s,tu) = \begin{cases} (1-s)tu + s\mu(u)u & \text{if} \quad t \in [1,\mu(u)], \\ tu & \text{if} \quad \mu(u) < t, \end{cases}$$

for all $s \in [0,1]$, all $t \geqslant 1$ and all $u \in \partial B_1^+$.

We have

$$\widehat{h}_+(0,tu) = tu, \quad \widehat{h}_+(1,tu) \in (\varphi_\lambda^+)^\eta$$

(see (3.45)) and

$$\widehat{h}_+(s,\cdot)\big|_{(\varphi_\lambda^+)^\eta} = id\big|_{(\varphi_\lambda^+)^\eta} \quad \forall s \in [0,1],$$

so $(\varphi_\lambda^+)^\eta$ is a strong deformation retract of E_+ and thus

$$H_k(W_0^{1,p}(\Omega),(\varphi_\lambda^+)^\eta) = H_k(W_0^{1,p}(\Omega),E_+) \quad \forall k \geqslant 0. \tag{3.46}$$

We consider the deformation $\widetilde{h}_+ \colon [0,1] \times E_+ \longrightarrow E_+$ defined by

$$\widetilde{h}_+(s,tu) = (1-s)tu + s\frac{tu}{\|tu\|} \quad \forall s \in [0,1],\ t \geqslant 1,\ u \in \partial B_1.$$

This deformation shows that E_+ is deformable onto ∂B_1^+, which is a retract of E_+. Therefore from Dugundji [7, p. 325], we have that ∂B_1 is a deformation retract of E_+. Hence

$$H_k(W_0^{1,p}(\Omega),E_+) = H_k(W_0^{1,p}(\Omega),\partial B_1^+) \quad \forall k \geqslant 0. \tag{3.47}$$

From (3.46) and (3.47) it follows that

$$H_k(W_0^{1,p}(\Omega),(\varphi_\lambda^+)^\eta) = H_k(W_0^{1,p}(\Omega),\partial B_1^+) \quad \forall k \geqslant 0. \tag{3.48}$$

Recall that ∂B_1^+ is contractible in itself. Hence

$$H_k(W_0^{1,p}(\Omega),\partial B_1^+) = 0 \quad \forall k \geqslant 0$$

(see Granas-Dugundji [21, p. 389]) and so

$$H_k(W_0^{1,p}(\Omega),(\varphi_\lambda^+)^\eta) = 0 \quad \forall k \geqslant 0$$

(see (3.48)), thus finally

$$C_k(\varphi_\lambda^+,\infty) = 0 \quad \forall k \geqslant 0$$

(choosing $\eta < 0$ even smaller if necessary). $\square$

The two constant sign solutions $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$ produced in Proposition 3.7 are critical points of mountain pass type for the functionals φ_λ^+ and φ_λ^- respectively. So we have

$$C_1(\varphi_\lambda^+,u_0) \neq 0 \quad \text{and} \quad C_1(\varphi_\lambda^-,v_0) \neq 0. \tag{3.49}$$

Using Proposition 3.10 we can improve (3.49).

Proposition 3.11. *If hypotheses H_1 hold, $\lambda > 0$ and $K_{\varphi_\lambda} = \{0, u_0, v_0\}$, then*

$$C_k(\varphi_\lambda, u_0) = C_k(\varphi_\lambda, v_0) = \delta_{k,1}\mathbb{F} \quad \forall k \geqslant 0.$$

Proof. We can easily verify that $K_{\varphi_\lambda^+} \subseteq C_+$. Since $\varphi_\lambda|_{C_+} = \varphi_\lambda^+|_{C_+}$ it follows that

$$K_{\varphi_\lambda^+} = \{0, u_0\}.$$

Let $\eta < 0 < \gamma < \eta_{\lambda,\varrho}^+$ (see (3.19)) and consider the following triple of sets

$$(\varphi_\lambda^+)^\eta \ \subseteq \ (\varphi_\lambda^+)^\gamma \ \subseteq \ W_0^{1,p}(\Omega).$$

We consider the corresponding long exact sequence of homology groups

$$\ldots H_k(W_0^{1,p}(\Omega), (\varphi_\lambda^+)^\eta) \xrightarrow{i_*} H_k(W_0^{1,p}(\Omega), (\varphi_\lambda^+)^\gamma) \xrightarrow{\partial_*} H_{k-1}((\varphi_\lambda^+)^\gamma, (\varphi_\lambda^+)^\eta) \ldots \tag{3.50}$$

with i_* being the homomorphism induced by the inclusion $(\varphi_\lambda^+)^\eta \xrightarrow{i} (\varphi_\lambda^+)^\gamma$ and ∂_* is the boundary homomorphism. Since $K_{\varphi_\lambda^+} = \{0, u_0\}$ and $0 = \varphi_\lambda^+(0) < \eta_{\lambda,\varrho}^+ \leqslant \varphi_\lambda^+(u_0)$ (see (3.19)), from the choice of $\eta > 0$, we have

$$H_k(W_0^{1,p}(\Omega), (\varphi_\lambda^+)^\eta) \ = \ C_k(\varphi_\lambda^+, \infty) = 0 \quad \forall k \geqslant 0 \tag{3.51}$$

(see Proposition 3.10). Also, since $\gamma \in (0, \eta_{\lambda,\varrho}^+)$, we have

$$H_k(W_0^{1,p}(\Omega), (\varphi_\lambda^+)^\gamma) \ = \ C_k(\varphi_\lambda^+, u_0) \quad \forall k \geqslant 0. \tag{3.52}$$

Finally, since $\eta < 0 < \gamma < \eta_{\lambda,\eta}^+$, we have

$$H_{k-1}((\varphi_\lambda^+)^\gamma, (\varphi_\lambda^+)^\eta) = \delta_{k-1,0}\mathbb{F} = \delta_{k,1}\mathbb{F} \quad \forall k \geqslant 0 \tag{3.53}$$

(see Proposition 3.5). From (3.51), (3.52), (3.53), it follows that in (3.50) only the tail (that is $k = 1$) is nontrivial. The exactness of (3.50) and the rank theorem, imply that

$$\begin{aligned} \dim C_1(\varphi_\lambda^+, u_0) \ &= \ \dim(\ker \partial_*) + \dim(\operatorname{im} \partial_*) \\ &\leqslant \ \dim(\operatorname{im} i_*) + 1 = 1 \end{aligned}$$

(see (3.53) and (3.51)), so

$$C_k(\varphi_\lambda^+, u_0) = \delta_{k,1}\mathbb{F} \quad \forall k \geqslant 0$$

(see (3.49)).

Similarly, we show that

$$C_k(\varphi_\lambda^-, v_0) = \delta_{k,1}\mathbb{F} \quad \forall k \geqslant 0.$$

Since $\varphi_\lambda\big|_{C_+} = \varphi_\lambda^+\big|_{C_+}$, $\varphi_\lambda\big|_{-C_+} = \varphi_\lambda^-\big|_{-C_+}$ and $u_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$, via the homotopy invariance of critical groups, we obtain

$$C_k(\varphi_\lambda^-, u_0) = C_k(\varphi_\lambda^-, v_0) = \delta_{k,1}\mathbb{F} \quad \forall k \geqslant 0.$$

$\square$

Now we are ready for the first multiplicity theorem.

Theorem 3.12. *If hypotheses H_1 hold, then for every $\lambda > 0$, problem (P_λ) admits at least three solutions*

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+ \quad \textit{and} \quad y_0 \in C_0^1(\overline{\Omega}) \setminus \{0\}.$$

Proof. From Proposition 3.7, we already have two nontrivial constant sign solutions

$$u_0 \in \operatorname{int} C_+, \quad \text{and} \quad v_0 \in -\operatorname{int} C_+.$$

Suppose that $K_{\varphi_\lambda} = \{0, u_0, v_0\}$. Then Proposition 3.11 implies that

$$C_k(\varphi_\lambda, u_0) = C_k(\varphi_\lambda, v_0) = \delta_{k,1}\mathbb{F} \quad \forall k \geqslant 0. \tag{3.54}$$

Also, from Propositions 3.5 and 3.9, we have

$$C_k(\varphi_\lambda, 0) = \delta_{k,0}\mathbb{F} \quad \text{and} \quad C_k(\varphi_\lambda, \infty) = 0 \quad \forall k \geqslant 0. \tag{3.55}$$

From (3.54), (3.55) and the Morse relation with $t = -1$ (see 2.3), we have

$$2(-1)^1 + (-1)^0 = 0,$$

a contradiction. This means that there exists $y_0 \in K_{\varphi_\lambda}$, $y_0 \notin \{0, u_0, v_0\}$. Evidently this is the third nontrivial solution of (P_λ) and $y_0 \in C_0^1(\overline{\Omega})$ (nonlinear regularity). $\square$

Remark 3.13. Theorem 3.12 extends Theorem 1.1 of Miyagaki-Souto [26] (where $p = 2$) an Theorem 1.1 of Li-Yang [25] (where $1 < p < +\infty$). Both works produce only one solution for all $\lambda > 0$, under stronger conditions on the reaction $f(z, \zeta)$.

4. **Problems with Combined Nonlinearities**

In Section 3, although we were able to produce extremal constant sign solutions (see Proposition 3.8), the geometry of the problem, with $u = 0$ being a local minimizer of the energy functional φ_λ (see Proposition 3.5), does not allow us to obtain nodal (sign-changing) solutions. We can do this for the following alternative parametric Dirichlet problem:

$$\left\{ \begin{array}{l} -\Delta_p u(z) = \lambda\big(|u(z)|^{q-2}u(z) + f(z, u(z))\big) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{array} \right. \tag{C_λ}$$

where $1 < q < p$. Again $f(z, \zeta)$ is a Carathéodory function, which exhibits $(p-1)$-superlinear growth in the ζ-variable, without satisfying the Ambrosetti-Rabinowitz condition. Since $1 < q < p$ we see that in (C_λ), we have the combined effect of both concave and convex nonlinearities. Such equations were studied by Ambrosetti-Brézis-Cerami [2], García Azorero-Manfredi-Peral Alonso [10], Papageorgiou-Smyrlis [27]. In all these works, the emphasis is on the existence, nonexistence and multiplicity of positive solutions (bifurcation-type results). Nodal solutions were produced by Hu-Papageorgiou [22], but under stronger conditions on the superlinear nonlinearity. We point out that in all the aforementioned works the parameter $\lambda > 0$ multiplies only the concave term $|u|^{q-2}u$, that is the reaction of the problem, has the form

$$\lambda|\zeta|^{q-2}\zeta + f(z,\zeta).$$

In fact, in Ambrosetti-Brézis-Cerami [2] and in García Azorero-Manfredi-Peral Alonso [10], we have

$$f(z,\zeta) = f(\zeta) = |\zeta|^{r-2}\zeta,$$

with $p < r < p^*$. For other problems with combined nonlinearities we refer to Gasiński-Papageorgiou [17, 18, 19, 20].

For every $\lambda > 0$, let $\widehat{\varphi}_\lambda \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the energy functional for the problem (C_λ), namely

$$\widehat{\varphi}_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p - \frac{\lambda}{p}\|u\|_q^q - \int_\Omega \lambda F(z,u(z))\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Also, we introduce the following Carathéodory functions

$$g_\lambda^+(z,\zeta) = \begin{cases} 0 & \text{if } \zeta \leqslant 0, \\ \lambda(\zeta^{q-1} + f(z,\zeta)) & \text{if } 0 < \zeta, \end{cases} \tag{4.1}$$

$$g_\lambda^-(z,\zeta) = \begin{cases} \lambda(|\zeta|^{q-2}\zeta + f(z,\zeta)) & \text{if } \zeta \leqslant 0, \\ 0 & \text{if } 0 < \zeta. \end{cases} \tag{4.2}$$

We set

$$G_\lambda^\pm(\zeta) = \int_0^\zeta g_\lambda^\pm(z,s)\,ds$$

and consider the C^1-functionals $\widehat{\varphi}_\lambda^\pm \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$, namely

$$\varphi_\lambda^\pm(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega G_\lambda^\pm(z,u(z))\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

We strengthen the conditions of f as follows:

$\underline{H_2:}$ $f \colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $f(z,0) = 0$ for almost all $z \in \Omega$, hypotheses $H_2(i) - (iv)$ are the same as corresponding hypotheses $H_1(i) - (iv)$ and

- **(v):** $f(z,\zeta)\zeta \geqslant 0$ for almost all $z \in \Omega$ and all $\zeta \geqslant 0$ and for every $\varrho > 0$, there exists $\xi_\varrho > 0$ such that for almost all $z \in \Omega$ the function $\zeta \longmapsto f(z,\zeta) + \xi_\varrho|\zeta|^{p-2}\zeta$ is nondecreasing on $[-\varrho, \varrho]$.

The presence of the concave term $\lambda|\eta|^{q-2}\zeta$ alters drastically the geometry of the problem near the origin and leads to a nodal solution but only when $\lambda > 0$ is small enough.

A careful reading of the proof of Proposition 3.3, reveals that the result remains valid for the functionals $\widehat{\varphi}_\lambda$ and $\widehat{\varphi}_\lambda^\pm$.

Proposition 4.1. *If hypotheses H_2 hold and $\lambda > 0$, then the functionals $\widehat{\varphi}_\lambda$ and $\widehat{\varphi}_\lambda^\pm$ satisfy the Cerami condition.*

In contrast to the previous case in Section 3, now in order to proceed we need to restrict the range of the parameter. This is necessary in order to satisfy the mountain pass geometry (see Theorem 2.1).

Proposition 4.2. *If hypotheses H_2 hold, then there exist $\lambda^* > 0$ and $\varrho_\pm > 0$ such that for all $\lambda \in (0, \lambda^*)$, we have*

$$\inf\left\{\widehat{\varphi}_\lambda^\pm : \|u\| = \varrho_\pm\right\} = \eta_\lambda^\pm \;>\; 0.$$

Proof. We do the proof for the functional $\widehat{\varphi}_\lambda^+$, the proof for $\widehat{\varphi}_\lambda^-$ being similar.

Hypotheses $H_1(i)$ and (iv) imply that given $\varepsilon > 0$, we can find $c_3 = c_3(\varepsilon) > 0$ such that

$$f(z,\zeta) \leqslant \varepsilon\zeta^{p-1} + c_3\zeta^{r-1} \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \geqslant 0,$$

so

$$F(z,\zeta) \leqslant \frac{\varepsilon}{p}\zeta^p + \frac{c_3}{r}\zeta^r \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \geqslant 0. \tag{4.3}$$

Then for every $u \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}
\widehat{\varphi}_\lambda^+(u) &= \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega G_\lambda^+(z,u)\,dz \\
&= \frac{1}{p}\|\nabla u\|_p^p - \frac{\lambda}{q}\|u^+\|_q^q - \int_\Omega F(z,u^+)\,dz \\
&\geqslant \frac{1}{p}\left(1 - \frac{\varepsilon}{\widehat{\lambda}_1}\right)\|\nabla u\|_p^p - \frac{\lambda}{q}\|u\|_q^q - \frac{\lambda c_3}{r}\|u\|_r^r
\end{aligned}$$

(see (4.3)).

Choosing $\varepsilon \in (0, \widehat{\lambda}_1)$ and since $1 < q < r < p^*$, we have

$$\begin{aligned}
\widehat{\varphi}_\lambda^+(u) &\geqslant c_4\|u\|^p - \lambda c_5(\|u\|^q + \|u\|^r) \\
&= \big(c_4 - \lambda c_5(\|u\|^{q-p} + \|u\|^{r-p})\big)\|u\|^p,
\end{aligned} \tag{4.4}$$

with $c_4 = \frac{1}{p}(1 - \frac{\varepsilon}{\widehat{\lambda}_1}) > 0$ and $c_5 > 0$.

We consider the function

$$\gamma(t) = t^{q-p} + t^{r-p} \quad \forall t \geqslant 0.$$

Evidently $\gamma \in C^1(0,+\infty)$ and since $q < p < r$, we see that

$$\gamma(t) \;\longrightarrow\; +\infty \quad \text{as } t \searrow 0 \text{ and as } t \to +\infty.$$

So, we can find $t_0 \in (0,+\infty)$ such that

$$\gamma(t_0) = \inf_{t>0}\gamma \;>\; 0,$$

so

$$\gamma'(t_0) = 0,$$

thus

$$(p-q)t_0^{q-p-1} = (r-p)t_0^{r-p-1}$$

and hence

$$t_0 = \left(\frac{p-q}{r-p}\right)^{\frac{1}{r-q}}.$$

We return to (4.4) and we see that, if $\|u\| = t_0$, then

$$\widehat{\varphi}_\lambda^+(u) \geqslant \big(c_4 - \lambda c_5 \gamma(t_0)\big) t_0^p.$$

So, if $\lambda^* = \frac{c_4}{c_5\gamma(t_0)}$ and $\varrho_+ = t_0$, then for every $\lambda \in (0, \lambda^*)$, we have

$$\inf\left\{\widehat{\varphi}_\lambda^+(u) : \|u\| = \varrho_+\right\} = \eta_\lambda^+ \ > \ 0$$

(see (4.4)). In a similar fashion, we show the corresponding result for the functional $\widehat{\varphi}_\lambda^-$. $\square$

As before (see Proposition 3.6), using hypothesis $H_2(ii)$, we infer that:

Proposition 4.3. *If hypotheses H_2 hold, $\lambda > 0$ and $u \in \operatorname{int} C_+$, then $\widehat{\varphi}_\lambda^\pm(tu) \longrightarrow -\infty$ as $t \to \pm\infty$.*

Now we are ready to produce nontrivial constant sign solutions for problem (C_λ). More precisely, we show that for all $\lambda \in (0, \lambda^*)$ (here $\lambda^* > 0$ is as in Proposition 4.2) problem (C_λ) admits at least four nontrivial constant sign solutions which are ordered.

Proposition 4.4. *If hypotheses H_2 hold and $\lambda \in (0, \lambda^*)$, then problem (C_λ) has at least four nontrivial constant sign solutions*

$$u_0, \widehat{u} \in \operatorname{int} C_+, \quad u_0 \leqslant \widehat{u}, \quad u_0 \neq \widehat{u},$$
$$v_0, \widehat{v} \in -\operatorname{int} C_+, \quad \widehat{v} \leqslant v_0, \quad v_0 \neq \widehat{v}.$$

Proof. First we produce the two positive solutions.

Propositions 4.1, 4.2 and 4.3 permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_\lambda^+(0) = 0 < \eta_\lambda^+ \leqslant \widehat{\varphi}_\lambda^+(u_0) \quad \text{and} \quad (\widehat{\varphi}_\lambda^+)'(u_0) = 0. \tag{4.5}$$

From the inequality in (4.5) it is clear that $u_0 \neq 0$. From the equality in (4.5), we have

$$A(u_0) = N_{g_\lambda^+}(u_0). \tag{4.6}$$

On (4.6) we act with $-u_0^- \in W_0^{1,p}(\Omega)$. From (4.1) we have

$$\|\nabla u_0^-\|_p^p = 0,$$

hence $u_0 \geqslant 0$, $u_0 \neq 0$. So, (4.6) becomes

$$A(u_0) = \lambda(u_0^{q-1} + N_f(u_0))$$

(see (4.1)), thus u_0 is a nontrivial nonnegative solution of (C_λ) and hence $u_0 \in C_+ \setminus \{0\}$ (by the nonlinear regularity theory).

Let $\varrho = \|u_0\|_\infty$ and let $\xi_p > 0$ be as postulated by hypothesis $H_2(v)$. Then

$$-\Delta_p u_0(z) + \lambda \xi_p u_0(z)^{p-1} = \lambda\big(u_0(z)^{q-1} + f(z, u_0(z))\big) + \lambda \xi_p u_0(z) \geqslant 0$$

almost everywhere on Ω, so

$$\Delta_p u_0(z) \leqslant \lambda \xi_\varrho u_0(z)^{p-1}$$

almost everywhere on Ω, thus $u_0 \in \operatorname{int} C_+$ (see, e.g., Gasiński-Papageorgiou [15, p. 738]).

Using $u_0 \in \operatorname{int} C_+$, we introduce the following truncation of the reaction in the problem (C_λ):

$$k_\lambda^+(z,\zeta) = \begin{cases} \lambda\big(u_0(z)^{q-1} + f(z,u_0(z))\big) & \text{if} \quad \zeta \leqslant u_0(z), \\ \lambda(\zeta^{q-1} + f(z,\zeta)) & \text{if} \quad u_0(z) < \zeta. \end{cases} \tag{4.7}$$

This is a Carathéodory function. We set

$$K_\lambda^+(z,\zeta) = \int_0^\zeta k_\lambda^+(z,s)\,ds$$

and consider the C^1-functional $\sigma_\lambda^+ \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\sigma_\lambda^+(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega K_\lambda^+(z,u(z))\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Claim 1. $K_{\sigma_\lambda^+} \subseteq [u_0)$, where

$$[u_0) = \big\{u \in W_0^{1,p}(\Omega) : u_0(z) \leqslant u(z) \text{ almost everywhere in } \Omega\big\}.$$

Let $u \in K_{\sigma_\lambda^+}$. Then we have

$$A(u) = N_{k_\lambda^+}(u). \tag{4.8}$$

On (4.8) we act with $(u_0 - u)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} \langle A(u), (u_0-u)^+\rangle &= \int_\Omega k_\lambda^+(z,u)(u_0-u)^+\,dz \\ &= \int_\Omega \lambda(u_0^{q-1} + f(z,u_0))(u_0-u)^+\,dz \\ &= \langle A(u_0), (u_0-u)^+\rangle \end{aligned}$$

(see (4.7) and use the fact that u_0 is a solution of (C_λ)), so

$$\int_{\{u_0>u\}} \big(|\nabla u_0|^{p-2}\nabla u_0 - |\nabla u|^{p-2}\nabla u, \nabla u_0 - \nabla u\big)_{\mathbb{R}^N} dz = 0$$

and thus

$$|\{u_0 > u\}|_N = 0,$$

hence $u_0 \leqslant u$.

Therefore $u \in [u_0)$ and this proves Claim 1.

Claim 2. $u_0 \in \operatorname{int} C_+$ is a local minimizer of the functional σ_λ^+.

Let $\mu \in (\lambda, \lambda^*)$. From the first part of the proof, via the mountain pass theorem (see Theorem 2.1), we show that problem (C_μ) admits a solution $\overline{u}_\mu \in \operatorname{int} C_+$. We will show that without any loss of generality, we may assume that

$$u_0 \leqslant \overline{u}_\mu.$$

Indeed, note that by virtue of hypothesis $H_2(v)$, we have

$$\begin{aligned} -\Delta_p \overline{u}_\mu(z) &= \mu\big(\overline{u}_\mu(z)^{q-1} + f(z, \overline{u}_\mu(z))\big) \\ &> \lambda\big(\overline{u}_\mu(z) + f(z, \overline{u}_\mu(z))\big) \quad \text{for almost all } z \in \Omega. \end{aligned} \tag{4.9}$$

So, we truncate the reaction of problem (C_λ) at $\{0, \overline{u}_\mu(z)\}$ and define the Carathéodory function

$$h_\lambda^+(z,\zeta) = \begin{cases} 0 & \text{if } \ \zeta < 0, \\ \lambda(\zeta^{q-1} + f(z,\zeta)) & \text{if } \ 0 \leqslant \zeta \leqslant \overline{u}_\mu(z). \\ \lambda(\overline{u}_\mu(z) + f(z, \overline{u}_\mu(z))) & \text{if } \ \overline{u}_\mu(z) < \zeta. \end{cases} \tag{4.10}$$

We set

$$H_\lambda^+(z,\zeta) = \int_0^\zeta h_\lambda^+(z,s)\, ds$$

and consider the C^1-functional $\eta_\lambda^+ \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\eta_\lambda^+(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega H_\lambda^+(z, u(z))\, dz \quad \forall u \in W_0^{1,p}(\Omega).$$

From (4.10) it is clear that η_λ^+ is coercive. Also, using the Sobolev embedding theorem, we see that η_λ^+ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $\overline{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\eta_\lambda^+(\overline{u}_0) = \inf_{u \in W_0^{1,p}(\Omega)} \eta_\lambda^+(u). \tag{4.11}$$

Since $q < p$, we can easily see that for $u \in \operatorname{int} C_+$ and for $t \in (0,1)$ small (at least such that $tu \leqslant \overline{u}_\mu$; see Filippakis-Kristaly-Papageorgiou [9, Lemma 3.3]), we have

$$\eta_\lambda^+(tu) < 0,$$

so

$$\eta_\lambda^+(\overline{u}_0) < 0 = \eta_\lambda^+(0)$$

(see (4.11)), hence $\overline{u}_0 \neq 0$.

From (4.11) we have

$$(\eta_\lambda^+)'(\overline{u}_0) = 0,$$

so

$$A(\overline{u}_0) = N_{h_\lambda^+}(\overline{u}_0). \tag{4.12}$$

On (4.12) we act with $-\overline{u}_0^- \in W_0^{1,p}(\Omega)$ and with $(\overline{u}_0 - \overline{u}_\mu)^+ \in W_0^{1,p}(\Omega)$ and using (4.10), we obtain

$$\overline{u}_0 \in [0, \overline{u}_\mu],$$

where $[0, \overline{u}_\mu] = \big\{u \in W_0^{1,p}(\Omega) : 0 \leqslant u(z) \leqslant \overline{u}_\mu(z) \text{ almost everywhere in } \Omega\big\}$, so $\overline{u}_0$ is a positive solution of (C_λ) (see (4.10)).

Therefore, without any loss of generality, we may assume that $u_0 \leqslant \overline{u}_\mu$. Moreover, we can assume that $K_{\sigma_\lambda^+} \cap [0, \overline{u}_\mu] = \{u_0\}$ or otherwise we already have the second positive solution $\widehat{u} \in \operatorname{int} C_+$ with $\widehat{u} \in [u_0, \overline{u}_\mu]$ (see Claim 1).

Using $\{u_0, \overline{u}_\mu\}$, we introduce the following Carathéodory function

$$\gamma_\lambda^+(z,\zeta) = \begin{cases} \lambda\big(u_0(z)^{q-1} + f(z, u_0(z))\big) & \text{if} \quad \zeta < u_0(z), \\ \lambda(\zeta^{q-1} + f(z,\zeta)) & \text{if} \quad u_0(z) \leqslant \zeta \leqslant \overline{u}_\mu(z). \\ \lambda(\overline{u}_\mu(z) + f(z, \overline{u}_\mu(z))) & \text{if} \quad \overline{u}_\mu(z) < \zeta. \end{cases} \tag{4.13}$$

We set

$$\Gamma_\lambda^+(z,\zeta) = \int_0^\zeta \gamma_\lambda^+(z,s)\,ds$$

and consider the C^1-functional $\vartheta_\lambda^+ \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\vartheta_\lambda^+(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega \Gamma_\lambda^+(z, u(z))\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

As before, via the direct method, we obtain $\widetilde{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\vartheta_\lambda^+(\widetilde{u}_0) = \inf_{u \in W_0^{1,p}(\Omega)} \vartheta_\lambda^+(u), \tag{4.14}$$

so

$$(\vartheta_\lambda^+)'(\widetilde{u}_0) = 0,$$

hence

$$A(\widetilde{u}_0) = N_{\gamma_\lambda^+}(\widetilde{u}_0).$$

Acting with $(u_0 - \widetilde{u}_0)^+ \in W_0^{1,p}(\Omega)$ and with $(\widetilde{u}_0 - \overline{u}_\mu)^+ \in W_0^{1,p}(\Omega)$ and using (4.9) and (4.13), we obtain that

$$\widetilde{u}_0 \in [u_0, \overline{u}_\mu],$$

where

$$[u_0, \overline{u}_\mu] = \big\{u \in W_0^{1,p}(\Omega) : u_0(z) \leqslant u(z) \leqslant \overline{u}_\mu(z) \text{ almost everywhere in } \Omega\big\},$$

so

$$\widetilde{u}_0 \in K_{\sigma_\lambda^+}$$

(see (4.13)) and so

$$\widetilde{u}_0 = u_0$$

(since $K_{\sigma_\lambda^+} \cap [0, \overline{u}_\mu] = \{u_0\}$).

Let $\varrho = \|\overline{u}_\mu\|_\infty$ and $\xi_\varrho > 0$ be as postulated by hypothesis $H_2(v)$. Then

$$\begin{aligned} & -\Delta_p u_0(z) + \lambda\xi_\varrho u_0(z)^{p-1} \\ = \;& \lambda\big(u_0(z)^{q-1} + f(z, u_0(z)) + \xi_\varrho u_0(z)^{p-1}\big) \\ < \;& \mu u_0(z)^{q-1} + \lambda\big(f(z, u_0(z)) + \xi_\varrho u_0(z)^{p-1}\big) \\ \leqslant \;& \mu\big(\overline{u}_\mu(z)^{q-1} + f(z, \overline{u}_\mu(z))\big) + \lambda\xi_\varrho\overline{u}_\nu(z)^{p-1} \\ = \;& -\Delta_p\overline{u}_\mu(z) + \lambda\xi_\varrho\overline{u}_\mu(z)^{p-1} \quad \text{almost everywhere in } \Omega \end{aligned}$$

(since $\lambda < \mu$, $u_0 \in \operatorname{int} C_+$, $u_0 \leqslant \overline{u}_\mu$; see hypothesis $H_2(v)$), so

$$\overline{u}_\mu - u_0 \in \operatorname{int} C_+$$

(see Arcoya-Ruiz [4, Proposition 2.6]).

From (4.7) and (4.13) it is clear that $\sigma_\lambda^+\big|_{[0,\overline{u}_\mu]} = \vartheta_\lambda^+\big|_{[0,\overline{u}_\mu]}$. So, it follows that $u_0 \in \operatorname{int} C_+$ is a local $C_0^1(\overline{\Omega})$-minimizer of σ_λ^+ (see (4.14)). From García Azorero-Manfredi-Peral Alonso [10, Theorem 1.2] it follows that u_0 is a local $W_0^{1,p}(\Omega)$-minimizer of σ_λ^+. This proves Claim 2.

We claim that $K_{\sigma_\lambda^+}$ is finite (otherwise we already have an infinity of positive solutions $\widehat{u} \geqslant u_0$ and so we are done). Then by virtue of Claim 1, we can find $\varrho \in (0, 1)$ small, such that

$$\sigma_\lambda^+(u_0) < \inf\big\{\sigma_\lambda^+(u) : \|u - u_0\| = \varrho\big\} = \widehat{m}_\lambda^+, \quad \|u_0\| > \varrho. \tag{4.15}$$

Moreover, (4.7) and hypothesis $H_2(ii)$ imply that for all $u \in \operatorname{int} C_+$, we have

$$\sigma_\lambda^+(tu) \longrightarrow -\infty \quad \text{as } t \to +\infty. \tag{4.16}$$

Finally, a slight modification of the proof of Proposition 3.3, reveals that

$$\sigma_\lambda^+ \text{ satisfies the Cerami condition.} \tag{4.17}$$

Note that (4.15), (4.16) and (4.17) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $\widehat{u} \in W_0^{1,p}(\Omega)$ such that

$$\widehat{u} \in K_{\sigma_\lambda^+} \quad \text{and} \quad \widehat{m}_\lambda^+ \leqslant \sigma_\lambda^+(\widehat{u}),$$

thus

$$\widehat{u} \neq u_0 \quad \text{and} \quad u_0 \leqslant \widehat{u}$$

(see (4.15) and Claim 1), and hence

$$\widehat{u} \in \operatorname{int} C_+ \text{ is a positive solution of } (C_\lambda)$$

(see (4.7)) and $u_0 \leqslant \widehat{u}$, $u_0 \neq \widehat{u}$.

Similarly, starting with the functional $\widehat{\varphi}_\lambda^-$, we produce two ordered negative solutions $v_0, \widehat{v} \in -\operatorname{int} C_+$ with $\widehat{v} \leqslant v_0$, $\widehat{v} \neq v_0$. $\square$

Next, as in Section 3, we generate extremal nontrivial constant sign solutions. To this end, we consider the following auxiliary Dirichlet problem:

$$(Au)_\lambda \qquad \begin{cases} -\Delta_p u(z) = \lambda |u(z)|^{q-2}u(z) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Using the direct method (recall that $q < p$) and Diaz-Saa [6, Theorem 1] we can state the following existence and uniqueness result for problem $(Au)_\lambda$.

Proposition 4.5. *If $1 < q < p$ and $\lambda > 0$, then problem $(Au)_\lambda$ admits a unique positive solution $\widetilde{u}_\lambda \in \operatorname{int} C_+$ and since the problem is odd, $\widetilde{v}_\lambda = -\widetilde{u}_\lambda \in -\operatorname{int} C_+$ is the unique negative solution.*

We introduce the following solution sets.

$$S_\pm(\lambda) = \left\{u \in W_0^{1,p}(\Omega) : u \text{ is a positive (resp. negative) solution of } (C_\lambda).\right\}$$

From Proposition 4.4 and its proof, we know that for all $\lambda \in (0, \lambda^*)$, we have

$$\emptyset \neq S_+(\lambda) \subseteq \operatorname{int} C_+ \quad \text{and} \quad \emptyset \neq S_-(\lambda) \subseteq -\operatorname{int} C_+.$$

Moreover, as before, we have that $S_+(\lambda)$ is downward directed and $S_-(\lambda)$ is upward directed (see Filippakis-Kristaly-Papageorgiou [9]).

Proposition 4.6. *If hypotheses H_2 hold and $\lambda \in (0, \lambda^*)$, then $\widetilde{u}_\lambda \leqslant u$ for all $u \in S_+(\lambda)$ and $v \leqslant \widetilde{v}_\lambda$ for all $v \in S_-(\lambda)$.*

Proof. Let $u \in S_+(\lambda) \subseteq C_+$ and consider the following Carathéodory function

$$v_\lambda^+(z, \zeta) = \begin{cases} 0 & \text{if } \zeta < 0, \\ \lambda \zeta^{q-1} & \text{if } 0 \leqslant \zeta \leqslant u(z), \\ \lambda u(z)^{q-1} & \text{if } u(z) < \zeta. \end{cases} \tag{4.18}$$

Let

$$V_\lambda^+(z, \zeta) = \int_0^\zeta v_\lambda^+(z, s)\, ds$$

and introduce the C^1-functional $\beta_\lambda^+ \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\beta_\lambda^+(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega V_\lambda^+(z, u(z))\, dz \quad \forall u \in W_0^{1,p}(\Omega).$$

From (4.18) it is clear that β_λ^+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u} \in W_0^{1,p}(\Omega)$ such that

$$\beta_\lambda^+(\widetilde{u}) = \inf\left\{\beta_\lambda^+(u) : u \in W_0^{1,p}(\Omega).\right\} \tag{4.19}$$

Since $q < p$, as before (see the proof of Proposition 4.4), we have

$$\beta_\lambda^+(\widetilde{u}) < 0 = \beta_\lambda^+(0),$$

hence $\widetilde{u} \neq 0$. From (4.19), we have

$$(\beta_\lambda^+)'(\widetilde{u}) = 0,$$

so

$$A(\widetilde{u}) = N_{V_\lambda^+}(\widetilde{u}).$$

As before, acting with $-\widetilde{u}^- \in W_0^{1,p}(\Omega)$ and with $(\widetilde{u}-u)^+ \in W_0^{1,p}(\Omega)$ and using (4.18), we obtain

$$\widetilde{u} \in [0,u],$$

so

$$\widetilde{u} = \widetilde{u}_\lambda \in \operatorname{int} C_+$$

(see (4.18) and Proposition 4.5), thus

$$\widetilde{u}_\lambda \leqslant u \quad \forall u \in S_+(\lambda).$$

In a similar fashion, we show that $v \leqslant \widetilde{v}_\lambda$ for all $v \in S_-(\lambda) \subseteq -\operatorname{int} C_+$. $\square$

As we did in Section 3, we can establish the existence of extremal nontrivial constant sign solutions for problem (C_λ).

Proposition 4.7. *If hypotheses H_2 hold and $\lambda \in (0,\lambda^*)$, then problem (C_λ) admits a smallest positive solution $u_\lambda^* \in \operatorname{int} C_+$ and a biggest negative solution $v_\lambda^* \in -\operatorname{int} C_+$.*

Proof. As in proof of Proposition 3.8, without any loss of generality, we may assume that

$$\|u\|_\infty \leqslant M_8 \quad \forall u \in S_+(\lambda), \tag{4.20}$$

for some $M_8 > 0$. Then we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq S_+(\lambda)$ such that

$$\inf S_+(\lambda) = \inf_{n\geqslant 1} u_n.$$

We have

$$A(u_n) = \lambda(u_n^{q-1} + N_f(u_n)) \quad \forall n \geqslant 1, \tag{4.21}$$

so the sequence $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded (see (4.20)). So, we may assume that

$$u_n \longrightarrow u_\lambda^* \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{4.22}$$

$$u_n \longrightarrow u_\lambda^* \quad \text{in } L^r(\Omega). \tag{4.23}$$

Acting on (4.21) with $u_n - u_\lambda^* \in W_0^{1,p}(\Omega)$, passing to the limit as $n \to +\infty$ and using (4.22), we obtain

$$\lim_{n\to+\infty} \langle A(u_n), u_n - u_\lambda^* \rangle = 0,$$

so

$$u_n \longrightarrow u_\lambda^* \quad \text{in } W_0^{1,p}(\Omega)$$

(see Proposition 2.2), thus

$$A(u_\lambda^*) = \lambda\big((u_\lambda^*)^{q-1} + N_f(u_\lambda^*)\big)$$

(see (4.21). Since $\widetilde{u}_\lambda \leqslant u_\lambda^*$ (see Proposition 4.6), it follows that

$$u_\lambda^* \in S_+(\lambda) \ \subseteq\ \operatorname{int} C_+ \quad \text{and} \quad u_\lambda^* = \inf S_+(\lambda),$$

so $u_\lambda^* \in \operatorname{int} C_+$ is the desired smallest positive solution of problem (C_λ).

Similarly, we produce the biggest negative solution $v_\lambda^* \in -\operatorname{int} C_+$ of (C_λ). □

Now we are ready to generate nodal solutions.

Proposition 4.8. *If hypotheses H_2 hold and $\lambda \in (0, \lambda^*)$, then problem (C_λ) admits a nodal solution $u_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C_0^1(\overline{\Omega})$, where $u_\lambda^* \in \operatorname{int} C_+$ and $v_\lambda^* \in -\operatorname{int} C_+$ are the two extremal nontrivial constant sign solutions of problem (C_λ) produced in Proposition 4.7.*

Proof. We introduce the following Carathéodory function

$$\mu_\lambda(z,\zeta) = \begin{cases} \lambda\big(|v_\lambda^*(z)|^{q-2} v_\lambda^*(z) + f(z, v_\lambda^*(z))\big) & \text{if} \quad \zeta < v_\lambda^*(z), \\ \lambda\big(|\zeta|^{q-2}\zeta + f(z,\zeta)\big) & \text{if} \quad v_\lambda^*(z) \leqslant \zeta \leqslant u_\lambda^*(z), \\ \lambda\big((u_\lambda^*)^{q-1} + f(z, u_\lambda^*(z))\big) & \text{if} \quad u_\lambda^*(z) \leqslant \zeta. \end{cases} \tag{4.24}$$

We set

$$M_\lambda(z,\zeta) = \int_0^\zeta \mu_\lambda(z,s)\, ds$$

and consider the C^1-functional $\widehat{\xi}_\lambda \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\xi}_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega M_\lambda(z, u(z))\, dz \quad \forall u \in W_0^{1,p}(\Omega).$$

We also consider the positive and negative truncations of $\mu_\lambda(z,\cdot)$. So, we introduce the Carathéodory functions

$$\mu_\lambda^\pm(z,\zeta) = \mu_\lambda(z, \pm\zeta^\pm).$$

We set

$$M_\lambda^\pm(z,\zeta) = \int_0^\zeta \mu_\lambda^\pm(z,s)\, ds$$

an consider the C^1-functionals $\widehat{\xi}_\lambda^\pm \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\xi}_\lambda^\pm = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega M_\lambda^\pm(z, u(z))\, dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Using (4.24) as before (see the proof of Proposition 4.4), we can check that

$$K_{\widehat{\xi}_\lambda} \ \subseteq\ [v_\lambda^*, u_\lambda^*], \quad K_{\widehat{\xi}_\lambda^+} \ \subseteq\ [0, u_\lambda^*], \quad K_{\widehat{\xi}_\lambda^-} \ \subseteq\ [v_\lambda^*, 0].$$

The extremality of $v_\lambda^* \in -\operatorname{int} C_+$ and $u_\lambda^* \in \operatorname{int} C_+$, implies that

$$K_{\widehat{\xi}_\lambda} \ \subseteq\ [v_\lambda^*, u_\lambda^*], \quad K_{\widehat{\xi}_\lambda^+} = \{0, u_\lambda^*\}, \quad K_{\widehat{\xi}_\lambda^-} = \{v_\lambda^*, 0\}. \tag{4.25}$$

Claim. $u_\lambda^* \in \operatorname{int} C_+$ and $v_\lambda^* \in -\operatorname{int} C_+$ are local minimizers of the functional $\widehat{\xi}_\lambda$.

From (4.24) it is clear that $\widehat{\xi}_\lambda^+$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_\lambda^* \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\xi}_\lambda^+(\widetilde{u}_\lambda^*) = \inf\big\{\xi_\lambda^+(u) : u \in W_0^{1,p}(\Omega)\big\}.$$

As before, since $q < p$, we have

$$\widehat{\xi}_\lambda^+(\widetilde{u}_\lambda^*) < 0 = \widehat{\xi}_\lambda^*(0),$$

hence $\widetilde{u}_\lambda^* \neq 0$, so

$$\widetilde{u}_\lambda^* = u_\lambda^* \in \operatorname{int} C_+$$

(see (4.25)). Evidently $\widehat{\xi}_\lambda\big|_{C_+} = \widehat{\xi}_\lambda^+\big|_{C_+}$, so $u_\lambda^* \in \operatorname{int} C_+$ is a local $C_0^1(\overline{\Omega})$-minimizer of $\widehat{\xi}_\lambda$. Hence, from García Azorero-Manfredi-Peral Alonso [10, Theorem 1.1], we have that u_λ^* is also a local $W_0^{1,p}(\Omega)$-minimizer of $\widehat{\xi}_\lambda$.

Similarly for $v_\lambda^* \in -\operatorname{int} C_+$ using this time the functional ξ_λ^-. This proves the Claim.

Without any loss of generality, we may assume that $\widehat{\xi}_\lambda(v_\lambda^*) \leqslant \widehat{\xi}_\lambda(u_\lambda^*)$ (the analysis is similar if the opposite inequality holds). By virtue of the Claim, we can find $\varrho \in (0,1)$ small such that

$$\widehat{\xi}_\lambda(v_\lambda^*) \leqslant \widehat{\xi}_\lambda(u_\lambda^*) < \inf\big\{\widehat{\xi}_\lambda(u) : \|u - u_\lambda^*\| = \varrho\big\} = \widehat{\eta}_\lambda, \quad \|v_\lambda^* - u_\lambda^*\| \ > \ \varrho. \tag{4.26}$$

Since $\widehat{\xi}_\lambda$ is coercive (see (4.24)), it satisfies the Cerami condition. This fact and (4.26) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $y_\lambda \in W_0^{1,p}(\Omega)$ such that

$$y_\lambda \in K_{\xi_\lambda} \quad \text{and} \quad \eta_\lambda \leqslant \widehat{\xi}_\lambda(y_\lambda). \tag{4.27}$$

From (4.25), (4.26) and (4.27), it follows that

$$y_\lambda \in [v_\lambda^*, u_\lambda^*] \setminus \{v_\lambda^*, u_\lambda^*\}.$$

If we can show that $y_\lambda \neq 0$, then by virtue of the extremality of v_λ^* and u_λ^*, we will have that y_λ is nodal and $y_\lambda \in C_0^1(\overline{\Omega}) \setminus \{0\}$ (nonlinear regularity theory).

Since $y_\lambda \in K_{\widehat{\xi}_\lambda}$ is of mountain pass type, we have

$$C_1(\widehat{\xi}_\lambda, y_\lambda) \neq 0. \tag{4.28}$$

On the other hand, the presence of the concave term and hypothesis $H_2(v)$ imply that

$$C_k(\widehat{\varphi}_\lambda, 0) = 0 \quad \forall k \geqslant 0 \tag{4.29}$$

(see Jiu-Su [23, proof of Proposition 2.1]). Then since $u_\lambda^* \in \operatorname{int} C_+$, $v_\lambda^* \in -\operatorname{int} C_+$ and $\widehat{\varphi}_\lambda\big|_{[v_\lambda^*, u_\lambda^*]} = \widehat{\xi}_\lambda\big|_{[v_\lambda^*, u_\lambda^*]}$ (see (4.24)), from the homotopy invariance of critical groups and (4.29), we have

$$C_k(\widehat{\xi}_\lambda, 0) = 0 \quad \forall k \geqslant 0. \tag{4.30}$$

Comparing (4.28) and (4.30), we infer that $y_\lambda \neq 0$. So, $y_\lambda \in C_0^1(\overline{\Omega})$ is a nodal solution for problem (C_λ) $(\lambda \in (0, \lambda^*))$. □

We can state the following multiplicity theorem for problem (C_λ).

Theorem 4.9. *If hypotheses H_2 hold, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem (C_λ) admits at least five nontrivial solutions*

$$u_0, \widehat{u} \in \operatorname{int} C_+, \quad u_0 \leqslant \widehat{u}, \quad u_0 \neq \widehat{u},$$
$$v_0, \widehat{v} \in -\operatorname{int} C_+, \quad \widehat{v} \leqslant v_0, \quad v_0 \neq \widehat{v},$$
$$y_0 \in [v_0, u_0] \cap \operatorname{int} C_0^1(\overline{\Omega}) \text{ nodal.}$$

Moreover problem (C_λ) admits extremal nontrivial constant sign solutions

$$u_\lambda^* \in \operatorname{int} C_+ \quad \text{and} \quad v_\lambda^* \in -\operatorname{int} C_+.$$

Next we examine the monotonicity properties of the map $(0, \lambda^*) \ni \lambda \longmapsto u_\lambda^* \in C_0^1(\overline{\Omega})$.

Proposition 4.10. *If hypotheses H_2 hold, then the map $(0, \lambda^*) \ni \lambda \longmapsto u_\lambda^* \in C_0^1(\overline{\Omega})$ is strictly monotone, that is if $\lambda < \mu$, then $u_\mu^* - u_\lambda^* \in \operatorname{int} C_+$.*

Proof. We consider the following truncation of the reaction of problem (C_λ):

$$\kappa_\lambda(z, \zeta) = \begin{cases} 0 & \text{if } \zeta < 0, \\ \lambda(\zeta^{q-1} + f(z, \zeta)) & \text{if } 0 \leqslant \zeta \leqslant u_\mu^*(z), \\ \lambda\big(u_\mu^*(z)^{q-1} + f(z, u_\mu^*(z))\big) & \text{if } u_\mu^*(z) < \zeta. \end{cases} \tag{4.31}$$

This is a Carathéodory function. We set

$$K_\lambda(z, \zeta) = \int_0^\zeta \kappa_\lambda(x, s)\, ds$$

and consider the C^1-functional $\tau_\lambda \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\tau_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega K_\lambda(z, u(z))\, dz \quad \forall u \in W_0^{1,p}(\Omega).$$

As before (see the proof of Proposition 4.4), using the direct method, we can find $u_\lambda \in \operatorname{int} C_+$ such that

$$\tau_\lambda(u_\lambda) = \inf_{u \in W_0^{1,p}(\Omega)} \tau_\lambda(u),$$

so

$$u_\lambda \in K_{\tau_\lambda} \ \subseteq \ [0, u_\mu^*] \cap \operatorname{int} C_+$$

(see the proof of Proposition 4.4). So, $u_\lambda \in \operatorname{int} C_+$ is a solution of problem (C_λ). Let $\varrho = \|u_\mu^*\|_\infty$ and let $\xi_\varrho > 0$ be as postulated by hypothesis $H_2(v)$. We have

$$\begin{aligned}
& -\Delta_p u_\lambda(z) + \lambda\xi_\varrho u_\lambda(z)^{p-1} \\
= \; & \lambda\big(u_\lambda(z)^{q-1} + f(z, u_\lambda(z))\big) + \lambda\xi_\varrho u_\lambda(z)^{p-1} \\
< \; & \mu u_\mu^*(z)^{q-1} + \lambda f(z, u_\lambda^*(z)) + \lambda\xi_\varrho u_\mu^*(z)^{p-1} \\
\leqslant \; & \mu u_\mu^*(z)^{q-1} + \mu f(z, u_\mu^*(z)) + \lambda\xi_\varrho u_\mu^*(z)^{p-1} \\
= \; & -\Delta_p u_\mu^*(z) + \lambda\xi_\varrho u_\mu^*(z)^{p-1} \quad \text{almost everywhere on } \Omega
\end{aligned}$$

(since $\lambda < \mu$, $u_\mu^* \in \operatorname{int} C_+$ and $u_\lambda \leqslant u_\mu^*$ and see hypothesis $H_2(v)$), so

$$u_\mu^* - u_\lambda \in \operatorname{int} C_+$$

(see Arcoya-Ruiz [4, Proposition 2.6]) and thus

$$u_\mu^* - u_\lambda^* \in \operatorname{int} C_+$$

□

For the continuity of the map $\lambda \longmapsto u_\lambda^*$, we restrict ourselves to the semilinear problem (that is $p = 2$). This is done in the next section.

5. **Semilinear Problem**

In this section, we establish a continuity property of the map $(0, \lambda) \ni \lambda \longmapsto u_\lambda^* \in C(\overline{\Omega})$.

The problem under consideration is the following:

$$(SC_\lambda) \qquad \begin{cases} -\Delta u(z) = \lambda\big(u(z)^{q-1} + f(z, u(z))\big) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \ u > 0. \end{cases}$$

We impose the following stronger conditions on the perturbation f:

$\underline{H_3:}$ $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function, such that $f(z, 0) = 0$ and $f(z, \cdot) \in C^1(\mathbb{R})$ for almost all $z \in \Omega$ and

(i): there exist a function $a \in L^\infty(\Omega)_+$ and $r \in (2, 2^*)$, such that

$$|f'_\zeta(z, \zeta)| \leqslant a(z)(1 + \zeta^{r-2}) \quad \text{for almost all } z \in \Omega, \ \text{all } \zeta \geqslant 0;$$

(ii): if

$$F(z, \zeta) = \int_0^\zeta f(z, s)\, ds,$$

then

$$\lim_{\zeta \to +\infty} \frac{F(z, \zeta)}{\zeta^2} = +\infty$$

uniformly for almost all $z \in \Omega$;

(iii): if

$$\xi(z,\zeta) = f(z,\zeta)\zeta - 2F(z,\zeta),$$

then there exists $\beta \in L^1(\Omega)_+$ such that

$$\xi(z,\zeta) \leqslant \xi(z,y) + \beta(z) \quad \text{for almost all } z \in \Omega, \text{ all } 0 \leqslant \zeta \leqslant y;$$

(iv): we have

$$f'_\zeta(z,0) = \lim_{\zeta \to 0^+} \frac{f(z,\zeta)}{\zeta} = 0$$

uniformly for almost all $z \in \Omega$;

(v): we have

$$f(z,\zeta) \geqslant 0 \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geqslant 0.$$

Remark 5.1. Since we are interested in positive solutions and all the above hypotheses concern the positive semiaxis $(0,+\infty)$, without any loss of generality, we may assume that $f(z,\zeta) = 0$ for almost all $\zeta \leqslant 0$. Note that in this case, the extra regularity of $f(z,\cdot)$ together with hypothesis $H_3(i)$ imply that for every $\varrho > 0$, we can find $\xi_\varrho > 0$ such that for almost all $z \in \Omega$, the map $\zeta \longmapsto f(z,\zeta) + \xi_\varrho \zeta$ is nondecreasing on $[0,\varrho]$.

Then Theorem 4.9 can be applied and we can guarantee the existence of a smallest positive solution $u^*_\lambda \in \operatorname{int} C_+$ for problem (SC_λ) for all $\lambda \in (0,\lambda^*)$. Inspired by the work of Cazenave-Escobedo-Pozio [5], we can establish the continuity of the map $(0,\lambda) \ni \lambda \longmapsto u^*_\lambda \in C(\overline{\Omega})$. Note that in what follows for $m \in L^\infty(\Omega)$, by $\widehat{\lambda}_1(-\Delta - mI)$ we denote the first (principal) eigenvalue of

$$\begin{cases} -\Delta u(z) - m(z)u(z) = \widehat{\lambda} u(z) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

(see Gasiński-Papageorgiou [15, Section 6.1]).

Proposition 5.2. *If hypotheses H_1 hold and for some $\lambda_0 \in (0,\lambda^*)$, we have*

$$\widehat{\lambda}_1\big(-\Delta - \lambda\big((q-1)(u^*_\lambda)^{q-2} + f'_\zeta(\cdot, u^*_\lambda(\cdot))\big)\big) \;>\; 0,$$

then the map $(0,\lambda) \ni \lambda \longmapsto u^*_\lambda \in C(\overline{\Omega})$ is continuous at $\lambda = \lambda_0$.

Proof. Let $\{\lambda_n\}_{n\geqslant 1} \subseteq (0,\lambda^*)$ be a sequence such that $\lambda_n \longrightarrow \lambda_0 \in (0,\lambda^*)$. We set $u^*_n = u^*_{\lambda_n} \in \operatorname{int} C_+$ for $n \geqslant 1$ (see Theorem 4.9).

If $\lambda_n \longrightarrow \lambda_0^-$, then from Proposition 4.10, we know that the sequence $\{u^*_n\}_{n\geqslant 1} \subseteq \operatorname{int} C_+$ is strictly increasing and so we have

$$\widetilde{u}^* = \lim_{n\to+\infty} u^*_n, \quad \widetilde{u}^* \leqslant u^*_{\lambda_0}. \tag{5.1}$$

We have

$$A(u^*_n) = \lambda_n\big((u^*_n)^{q-1} + N_f(u^*_n)\big) \quad \forall n \geqslant 1$$

so the sequence $\{u_n^*\}_{n\geqslant 1} \subseteq H_0^1(\Omega)$ is bounded (see (5.1)) and thus, passing to a subsequence if necessary, we have

$$\begin{aligned} u_n^* &\longrightarrow \widetilde{u}^* \quad \text{weakly in } H_0^1(\Omega), \\ u_n^* &\longrightarrow \widetilde{u}^* \quad \text{in } L^r(\Omega) \end{aligned}$$

(see (5.1)). So, in the limit as $n \to +\infty$, we obtain

$$A(\widetilde{u}^*) = \lambda\big((\widetilde{u}^*)^{q-1} + N_f(\widetilde{u}^*)\big),$$

thus

$$u_{\lambda_0}^* \leqslant \widetilde{u}^*$$

and hence

$$u_{\lambda_0}^* = \widetilde{u}^*$$

(see (5.1)).

If $\lambda_n \longrightarrow \lambda_0^+$, then once again Proposition 4.10 implies that the sequence $\{u_n^*\}_{n\geqslant 1} \subseteq \operatorname{int} C_+$ is strictly decreasing and so we have

$$\overline{u}^* = \lim_{n\to+\infty} u_n^*, \quad u_{\lambda_n}^* \leqslant \overline{u}^*. \tag{5.2}$$

We consider the following auxiliary Dirichlet problem

$$\begin{cases} -\Delta y(z) - \lambda\big((q-1)u_{\lambda_0}^*(z)^{q-2} + f'_\zeta(z, u_{\lambda_0}^*(z))\big)y(z) = 1 \quad \text{in } \Omega, \\ y|_{\partial\Omega} = 0, \ y > 0. \end{cases}$$

From our hypothesis this problem has a solution $y \in \operatorname{int} C_+$. We set

$$v = u_{\lambda_0}^* + \delta y,$$

with $\delta > 0$. Then for $\mu > 0$, we have

$$\begin{aligned} & -\Delta v - (\lambda_0 + \mu)\big(v^{q-1} + f(z,v)\big) \\ = \ & \delta - \mu\big(v^{q-1} + f(z,v)\big) - \lambda_0\big(v^{q-1} + f(z,v) - (u_\lambda^*)^{q-1} - f(z, u_{\lambda_0}^*) \\ & -(v - u_\lambda^*)\big((q-1)(u_{\lambda_0}^*)^{q-2} + f'_\zeta(z, u_{\lambda_0}^*)\big)\big) \\ = \ & \delta - \mu\big(v^{q-1} + f(z,v)\big) - \lambda_0 o(\delta). \end{aligned}$$

Since $v^{q-1} \leqslant \|u_{\lambda_0}^*\|_\infty^{q-1} + \delta^{q-1}\|y\|_\infty^{q-1}$ and using hypothesis $H_3(i)$, we see that for $\delta \in (0,1)$ small, we can find $\mu(\delta) > 0$ such that

$$-\Delta v(z) - (\lambda_0 + \mu)\big(v(z)^{q-1} + f(z, v(z))\big) \geqslant 0 \quad \text{almost everywhere in } \Omega. \tag{5.3}$$

Then truncating the reaction of problem $(SC_{\lambda_0+\mu})$ at $v(z)$ and using the direct method and (5.3), we obtain a solution $u_{\lambda_0+\mu} \in S_+(\lambda_0+\mu)$ such that $u_{\lambda_0+\mu} \leqslant v$, hence $u_{\lambda_0+\mu}^* \leqslant v$ and so $\overline{u}^* \leqslant v$. Since $\delta > 0$ is arbitrary, we let $\delta \searrow 0$ and obtain

$$\overline{u}^* \leqslant u_{\lambda_0}^*,$$

so

$$\overline{u}^* = u^*_{\lambda_0} \in \operatorname{int} C_+$$

(see (5.2)) and thus

$$\overline{u}^* = \widetilde{u}^* = u^*_{\lambda_0}.$$

Therefore, we have

$$u^*_n(z) \longrightarrow u^*_{\lambda_0}(z) \quad \forall z \in \overline{\Omega}.$$

Moreover, by Dini's theorem, it follows that

$$u^*_n \longrightarrow u^*_{\lambda_0} \quad \text{in } C(\overline{\Omega}),$$

which proves the continuity of the map $\lambda \longmapsto u^*_\lambda$ at $\lambda = \lambda_0$. □

Remark 5.3. Note that the map $\zeta \longmapsto \zeta^{q-1} + f(z,\zeta)$ in general is neither convex nor concave and so, we cannot have general criteria for the condition on the principal eigenvalue to hold. For the scalar case (that is $N = 1$; ordinary differential equations), some such conditions were produced by Cazenave-Escobedo-Pozio [5, Proposition 4.1].

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