Dynamic Systems and Applications 25 (2016) 455-468

FINITE TIME BLOW UP FOR SEMILINEAR HEAT EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

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ABSTRACT. Let T and p be positive constants with $p \geq 1$, Ω be a bounded domain in $\mathbb{R}^n$, and Δ be the Laplace operator. This paper considers the semilinear integro-differential problem under nonlocal boundary conditions:

$$u_t(x,t) - \Delta u(x,t) = \left(\int_0^t [F(u(x,s))]^p ds\right) G(u(x,t)) \text{ in } \Omega \times (0,T),$$
$$\alpha_0 \frac{\partial u}{\partial \nu} + u(x,t) = \int_\Omega K(x,y) H(u(y,t)) dy \quad \text{in } \partial\Omega \times (0,T),$$
$$u(x,0) = u_0(x) \text{ on } \bar{\Omega}.$$

We determine some conditions on functions F, G and H for finding criterion for the solution to blow-up in a finite time.

AMS (MOS) Subject Classification. 35K38.

1. Introduction

Let T and p be positive constants with $p \geq 1$, Ω be a smooth bounded domain in $\mathbb{R}^n$, $\partial\Omega$ be the smooth boundary of Ω, $\bar{\Omega}$ be the closure of Ω, Δ be the Laplace operator, and $\partial/\partial t - \Delta = \partial/\partial t - \sum_{i=1}^n \partial^2/\partial x_i^2$ be the heat operator. This article considers the following semilinear integro-differential problem with nonlocal boundary conditions in the form:

$$(1.1) \quad \begin{cases} u_t(x,t) - \Delta u(x,t) = \left(\int_0^t [F(u(x,s))]^p ds\right) G(u(x,t)) & \text{in } \Omega \times (0,T), \\ \alpha_0 \frac{\partial u}{\partial \nu} + u(x,t) = \int_\Omega K(x,y) H(u(y,t)) dy & \text{in } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \text{ on } \bar{\Omega}, \end{cases}$$

where $K(x,y)$ and $u_0(x)$ are nonnegative continuous functions on $\bar{\Omega}$ such that $\int_\Omega K(x,y)dy \leq 1$ for $x \in \partial\Omega$, $\alpha_0 \geq 0$ and $\partial u/\partial \nu$ denotes the outward normal derivative of u on $\partial\Omega$.

Received June 20, 2016

Here are preliminary conditions on F, G and H that we require: (i) F, G and H are nonnegative continuous functions with $G(0) = 0 = H(0)$ such that $G(u) \geq F(u) \geq u$, $H(u) \leq u$ for $u \geq 0$, (ii) F', G' and H' exist, and $F'(u)$ and $G'(u)$ are nonnegative for $u \geq 0$, and (iii) $F''(u)$, $G''(u)$ and $H''(u)$ are nonnegative continuous functions for $u \geq 0$.

Notice that (1.1) with $F(u(x,s)) = |u(x,s)|$, $G(u(x,t)) = u(x,t)$ and $p = 1$ has been appeared in the theory of nuclear reactor kinetics [7]. We also found that the above nonlocal boundary condition where, $H(u(y,t)) = u(y,t)$ appears in quasi-static thermoelasticity. In there, the entropy is governed by a parabolic equation under a nonlocal boundary condition [10].

One of the basic approaches to our problem is the method of upper and lower solutions and the associated monotone iterations [10, 11, 12]. This technique allows us to show the existence and uniqueness of the solution of (1.1). Some qualitative properties of the solution can be extracted through suitable construction of upper and lower solutions.

One important property in the analysis of parabolic-type problems is the finite time blow-up of the solution.

Definition 1.1 ([8])**.** If $u(x,t)$ is unbounded in $\Omega \times (0,T)$, then the solution u is said to be blow-up in a finite time T.

A lot of researchers [2, 3, 5] have been studied the blow-up behavior for semilinear parabolic equations of the following type.

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= f(x,u) \text{ in } \Omega \times (0,T), \\ u(x,t) &= 0 \text{ on } \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x), x \in \Omega. \end{aligned}$$

Such a nonlocal parabolic equation including the time-integral nonlocal source term does not seem to be so much investigated. The nonlocal equations with space-integral terms, that is the nonlocal terms involved an integral over Ω have been studied extensively, for instant, Beberens and Ely [2] and Pao [9]. They considered the problem of the form

$$u_t - \Delta u = e^{u(x,t)} + \int_\Omega k e^{u(x,t)} dx, \quad t > 0, x \in \Omega, \tag{1.2}$$

where k is a positive constant. (1.2) is known as an ignition model for a compressible reactive gas. They could prove that the solution blows up in the whole domain. A similar problem of the form

$$u_t(x,t) - \Delta u(x,t) = \left(\int_\Omega u^p(y,t) dy \right) - k u^q(x,t) \text{ in } \Omega \times (0,T),$$

$$u(x,t) = 0 \text{ in } \partial\Omega \times (0,T),$$
$$u(x,0) = u_0(x), x \in \Omega,$$

was studied by Wang and Wang [13]. They explored the blow-up behavior of the positive solution u of the above problem.

In 2003, Chan and Tian [4] studied the following degenerate semilinear parabolic first initial-boundary value problem:

$$x^q u_t(x,t) - u_{xx}(x,t) = a^{mq+2}\delta(x-b)f(u(x,t))U^m(t) \text{ for } 0 < x < 1, 0 < t \le T,$$
$$u(x,0) = \psi(x) \text{ for } 0 \le x \le 1,$$
$$u(0,t) = 0 = u(1,t) \text{ for } 0 < t \le T,$$

where q, m and T are real numbers such that $q \ge 0$, $m > 1$, $T > 0$ and $U(t) = \int_0^1 x^q |u(x,t)| dx$, $\delta(x)$ is the Dirac delta function, f and ψ are given functions such that $f(0) \ge 0$, $f(u)$ and $f'(u)$ are positive for $u > 0$, ψ is nontrivial, nonnegative and continuous such that $\psi(0) = 0 = \psi(1)$, and $\psi'' + a^{mq+2}\delta(x-b)f(\psi(x))U^m(0) \ge 0$ for $0 < x < 1$. They showed that it has a unique solution before a blow-up occurs. A criterion for u to blow-up in a finite time is given. Finally, they also proved that if u blows up, then the blow-up set consists of the single-point b.

In 2008, a nonlocal parabolic problem involving the time-integral nonlocal source term has been studied by Liu and Chen [8]. They considered the semilinear integro-differential problem of the form

$$\text{(1.3)} \qquad \begin{cases} u_t(x,t) - \Delta u(x,t) = \left(\int_0^t |u(x,s)|^p ds\right) u(x,t) \text{ in } \Omega \times (0,T), \\ \alpha_0 \dfrac{\partial u}{\partial \nu} + u(x,t) = \displaystyle\int_\Omega K(x,y)u(y,t)dy \text{ in } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \text{ on } \bar{\Omega}. \end{cases}$$

They assumed that the function $K(x,y)$ satisfies the followings

$$K(x,y) \ge 0 \text{ and } \int_\Omega K(x,y)dy \le 1 \text{ for } x \in \partial\Omega, y \in \Omega,$$

and u_0 is a nonnegative continuous function on $\bar{\Omega}$. They showed the local existence and uniqueness of the solution of (1.3). The method of upper and lower solutions is used to obtain existence and comparison results for time dependent problem. The comparison results with the suitable lower solution are also used to show a finite time blow-up criterion for their problem.

In this study, we modify (1.3) by considering (1.1) with more general forcing term under some prescribed conditions. This extends the work of Liu and Chen [8] for the problem that have the time-integral nonlocal source.

Recently, Aiewcharoen and Boonklurb [1] studied the existence and uniqueness of the solution for (1.1). For ease of reference, we collect their results here in this paper.

First, let $D_T = \Omega \times (0,T)$ and $\bar{D}_T = \bar{\Omega} \times [0,T]$. By modifying the proof of Lemma 2.1 of Liu and Chen [8], we can obtain the following lemma which is proved by contradiction.

Lemma 1.2 ([1]). *Let $w(x,t) \in C(\bar{D}_T) \cap C^{2,1}(D_T)$ be such that*

$$\begin{aligned} w_t(x,t) - \Delta w(x,t) &\geq g_1(x,t)F(w(x,t)) + \left(\int_0^t g_2(x,s)F(w(x,s))ds\right) g_3(x,t) \ \textit{in } D_T, \\ \alpha_0 \frac{\partial w}{\partial \nu} + w(x,t) &\geq \int_\Omega K(x,y)g_4(y,t)H(w(y,t))dy \ \textit{in } \partial\Omega \times (0,T), \\ w(x,0) &\geq 0 \ \textit{on } \bar{\Omega}, \end{aligned}$$

where $g_1(x,t), g_2(x,s), g_3(x,t)$ and $g_4(y,t)$ are nonnegative continuous functions on D_T. Then, $w(x,t) \geq 0$ on $\bar{D}_T$.

By Lemma 1.2, if we let $g_1(x,t), g_2(x,s), g_3(x,t)$ and $g_4(y,t)$ as 0, $[F(u(x,s))]^{p-1}$, $G(u(x,t))$ and 1, respectively, then $u(x,t) \geq 0$ on $\bar{D}_T$, which is the following theorem.

Theorem 1.3 ([1]). *If $u(x,t) \in C(\bar{D}_T) \cap C^{2,1}(D_T)$ is a solution of the problem*

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \left(\int_0^t [F(u(x,s))]^p ds\right) G(u(x,t)) \quad \textit{in } \Omega \times (0,T), \\ \alpha_0 \frac{\partial u(x,t)}{\partial \nu} + u(x,t) &= \int_\Omega K(x,y)H(u(y,t))dy \quad \textit{in } \partial\Omega \times (0,T) \textit{ and} \\ u(x,0) &\geq 0 \quad \textit{on } \bar{\Omega}, \end{aligned}$$

then $u(x,t) \geq 0$ for $(x,t) \in \bar{D}_T$.

Aiewcharoen and Boonklurb [1] used the monotone iterative method to show the local existence of the solution u of (1.1). First, let us give the following definition for upper and lower solutions.

Definition 1.4. A nonnegative function $\tilde{u}(x,t) \in C(\bar{D}_T) \cap C^{2,1}(D_T)$ is called an upper solution of (1.1) if it satisfies

$$\left\{ \begin{aligned} \tilde{u}_t(x,t) - \Delta \tilde{u}(x,t) &\geq \left(\int_0^t [F(\tilde{u}(x,s))]^p ds\right) G(\tilde{u}(x,t)) \text{ in } \Omega \times (0,T), \\ \alpha_0 \frac{\partial \tilde{u}}{\partial \nu} + \tilde{u}(x,t) &\geq \int_\Omega K(x,y)H(\tilde{u}(y,t))dy \text{ in } \partial\Omega \times (0,T), \\ \tilde{u}(x,0) &\geq u_0(x) \text{ on } \bar{\Omega}. \end{aligned} \right. \tag{1.4}$$

Similarly, a nonnegative function $\hat{u}(x,t)$ is called a lower solution of (1.1) if it satisfies

$$\begin{cases} \hat{u}_t(x,t) - \Delta \hat{u}(x,t) \leq \left(\int_0^t [F(\hat{u}(x,s))]^p ds \right) G(\hat{u}(x,t)) \text{ in } \Omega \times (0,T), \\ \alpha_0 \frac{\partial \hat{u}}{\partial \nu} + \hat{u}(x,t) \leq \int_\Omega K(x,y) H(\hat{u}(y,t)) dy \text{ in } \partial\Omega \times (0,T), \\ \hat{u}(x,0) \leq u_0(x) \text{ on } \bar{\Omega}. \end{cases} \tag{1.5}$$

We can see from the definition that $\hat{u}(x,t) \equiv 0$ is a lower solution of (1.1). Next, Aiewcharoen and Boonklurb [1] used the idea of Liu and Chen [8] to construct an upper solution for (1.1). For $a > 0$, consider the following initial value problem

$$g'(t) = \left(\int_0^t [F(g(s))]^p ds \right) G(g(t)), \quad t \in [0,a], \tag{1.6}$$

with $g(0) > 0$. The solution of (1.6) has the integral representation form as follow

$$g(t) = g(0) + \int_0^t \left(\int_0^s [F(g(\sigma))]^p G(g(s)) d\sigma \right) ds.$$

By modifying the lemma 3.2 from Liu and Chen [8], Aiewcharoen and Boonklurb [1] can have the following lemma that gives an existence of $g(t)$ which satisfies (1.6).

Lemma 1.5 ([1])**.** *Let a and b be positive real numbers, and $Z = \{\phi : \phi$ is nonnegative continuous on $[0,a]$, $\phi(0) = g(0)$ and $\|\phi(t) - g(0)\|_\infty \leq b\}$. Then, the integro-differential equation* (1.6) *possesses at least one solution $g(t)$ on $[0,\alpha]$, for some $0 < \alpha \leq a$.*

Next, Aiewcharoen and Boonklurb [1] consider $\tilde{u}(x,t) = g(t)$, where $g(t)$ satisfies (1.6) with $g(0) \geq \|u_0(x)\|_\infty$. Then, $\tilde{u}(x,t)$ is an upper solution for (1.1). They showed that the upper solution, (1.4), is greater than or equal to the lower solution, (1.5), whenever both exist. In particular, any solution of (1.1) should lie between the upper and lower solutions, which is the following lemma.

Lemma 1.6 ([1])**.** *Let $\hat{u}(x,t)$ and $\tilde{u}(x,t)$ be nonnegative lower and upper solutions for* (1.1)*, respectively. Then, $\hat{u}(x,t) \leq \tilde{u}(x,t)$ on $\bar{D}_T$. Moreover, if $u^*(x,t)$ is a solution of* (1.1)*, then $\hat{u}(x,t) \leq u^*(x,t) \leq \tilde{u}(x,t)$.*

If we let u_1 and u_2 be two solutions of (1.1), then by the definition, we can regard u_1 as an upper solution and u_2 as a lower solutions of (1.1). Thus, by Lemma 1.6, $u_1 \leq u_2$. By interchanging the role between upper and lower solution of u_1 and u_2, we can use Lemma 1.6 again to conclude the uniqueness result of (1.1).

Theorem 1.7 ([1])**.** (1.1) *has at most one solution.*

Aiewcharoen and Boonklurb [1] constructed sequences of upper and lower solutions of (1.1) in order to prove existence of the (1.1). Next, using the iterative process to construct $\{u^{(k)}\}$ successively in the sector $<\hat{u},\tilde{u}>=\{u\in C(\bar{D}_T):0\leq\hat{u}\leq u\leq\tilde{u}\}$. Let $u^{(0)}$ be either the upper solution $\tilde{u}$ or the lower solution $\hat{u}$, and for $k\geq 1$, define $u^{(k)}$ to be the solution of the problem:

(1.7)

$$\begin{cases} u_t^{(k)}(x,t)-\Delta u^{(k)}(x,t)=\left(\int_0^t [F(u^{(k-1)}(x,s))]^p ds\right)G(u^{(k-1)}(x,t)) \text{ in } \Omega\times(0,T),\\ \alpha_0\dfrac{\partial u^{(k)}(x,t)}{\partial\nu}+u^{(k)}(x,t)=\displaystyle\int_\Omega K(x,y)H(u^{(k-1)}(y,t))dy \text{ in } \partial\Omega\times(0,T),\\ u^{(k)}(x,0)=u_0(x) \text{ on } \bar{\Omega}.\end{cases}$$

If $\bar{u}^{(0)}=\tilde{u}$, the constructed sequence will be denoted by $\{\bar{u}^{(k)}\}$. Otherwise, it will be $\{\underline{u}^{(k)}\}$. By Theorem 7 of Friedman [6] (pp. 65–68), (1.7) has a solution $u^{(k)}(x,t)\in C(\bar{D_T})\cap C^{2+\alpha,1+\alpha/2}(D_T)$ for some $\alpha\in(0,1)$.

Since $\underline{u}^{(0)}=\hat{u}$ and $\bar{u}^{(0)}=\tilde{u}$, from Lemma 1.6, we have $\underline{u}^{(0)}\leq\bar{u}^{(0)}$. Now, we can use Lemma 1.2 to show that $\{\bar{u}^{(k)}\}$ is a monotone nonincreasing sequence, and $\{\underline{u}^{(k)}\}$ is a monotone nondecreasing sequence.

Lemma 1.8 ([1])**.** *The sequences $\{\bar{u}^{(k)}\}$ and $\{\underline{u}^{(k)}\}$ possess the monotone property, that is,*

$$0\leq\hat{u}\leq\underline{u}^{(k)}\leq\underline{u}^{(k+1)}\leq\bar{u}^{(k+1)}\leq\bar{u}^{(k)}\leq\tilde{u} \text{ on } \bar{D_T},$$

for every $k\in\mathbb{N}$.

Aiewcharoen and Boonklurb [1] used the sequences $\{\bar{u}^{(k)}\}$ and $\{\underline{u}^{(k)}\}$ to prove the existence of the solution of (1.1).

Theorem 1.9. (1.1) *has a unique solution in $\bar{D}_T$.*

2. Comparison Results and Finite Time Blow-Up

After Aiewcharoen and Boonklurb [1] established the existence and uniqueness of (1.1), we can now study the behavior of the solution of (1.1). Comparison theorem is one of the key concepts in studying parabolic differential equations. Here, we explore some comparison results which will be need in main results. Let $D_T=\Omega\times(0,T)$ and $\bar{D_T}=\bar{\Omega}\times[0,T]$, where $T>0$ is a finite constant. Let $\widehat{K}(x)=\int_\Omega K(x,y)dy$ and assume that $\widehat{K}(x)\leq 1$ for $x\in\partial\Omega$.

Theorem 2.1. *If $u(x,t),v(x,t)\in C(\bar{D_T})\cap C^{2,1}(D_T)$ and satisfy the relations*

$$u_t(x,t)-\Delta u(x,t)\geq g_1(x,t)F(u(x,t)) + \left(\int_0^t g_2(x,s)F(u(x,s))ds\right)g_3(x,t) \text{ in } \Omega\times(0,T),$$

$$\alpha_0\frac{\partial u}{\partial\nu}+u(x,t)\geq\int_\Omega K(x,y)u(y,t)dy \text{ in } \partial\Omega\times(0,T),$$

and

$$
\begin{aligned}
v_t(x,t) - \Delta v(x,t) &\le g_1(x,t)F(v(x,t)) \\
&\qquad + \left(\int_0^t g_2(x,s)F(v(x,s))ds\right) g_3(x,t) \ \textit{in}\ \Omega \times (0,T), \\
\alpha_0 \frac{\partial v}{\partial \nu} + v(x,t) &\le \int_\Omega K(x,y)v(y,t)dy \ \textit{in}\ \partial\Omega \times (0,T),
\end{aligned}
$$

respectively, and $u(x,0) \ge v(x,0)$ *for* $x \in \bar{\Omega}$, *where* $g_1(x,t), g_2(x,t)$ *and* $g_3(x,t)$ *are nonnegative continuous functions on* D_T, *then* $u(x,t) \ge v(x,t)$ *on* $\bar{D_T}$.

Proof. From the above assumptions, in $\Omega \times (0,T)$, by the Mean-Value theorem, we have

$$
\begin{aligned}
&\frac{\partial}{\partial t}(u(x,t) - v(x,t)) - \Delta\left(u(x,t) - v(x,t)\right) \\
&\quad \ge g_1(x,t)[F(u(x,t)) - F(v(x,t))] \\
&\qquad + \left(\int_0^t g_2(x,s)[F(u(x,s)) - F(v(x,s))]ds\right) g_3(x,t) \\
&\quad = (u(x,t) - v(x,t))F'(\xi_1(x,t))g_1(x,t) \\
&\qquad + \left(\int_0^t g_2(x,s)[u(x,s) - v(x,s)]F'(\xi_2(x,s))ds\right) g_3(x,t),
\end{aligned}
$$

where $\xi_1(x,t)$ and $\xi_2(x,s)$ are between $u(x,t), v(x,t)$ and $u(x,s), v(x,s)$, respectively.

For the boundary conditions, we have

$$
\alpha_0 \frac{\partial}{\partial \nu}(u - v) + u(x,t) - v(x,t) \ge \int_\Omega K(x,y)[u(y,t) - v(y,t)]dy
$$

Next, the initial condition gives

$$
u(x,0) - v(x,0) \ge 0, \qquad \text{for } x \in \bar{\Omega}.
$$

By Lemma 2.1 of [8], we have $u(x,t) - v(x,t) \ge 0$ on $\bar{D_T}$. That is, $u(x,t) \ge v(x,t)$ on $\bar{D_T}$. □

Comparison results obtained above will be used later. Theorem 1.9 shows that the solution u of (1.1) exists in D_T for some $T > 0$. Moreover, as in Definition 1.1, if the solution becomes unbounded in D_T, then we say that u blows up in a finite time T. In this section, we show that for sufficiently large initial data, the solution u always blows up in a finite time.

Let λ_1 be the first eigenvalue of the eigenvalue problem

$$
\begin{cases} -\Delta\varphi(x) = \lambda\varphi(x) \text{ in } \Omega, \\ \quad \varphi(x) = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}
$$

and $\varphi_1(x)$ be the eigenfunction corresponding to λ_1. Then, $\varphi_1(x)$ is a nonnegative smooth function on $\bar{\Omega}$, and $\varphi_1(x)$ is positive in Ω. For convenience, we normalize $\varphi_1(x)$ in sup-norm, that is, we let

$$\sup_{x\in\bar{\Omega}} \varphi_1(x) = 1.$$

In order to prove the blow-up result, we first prove the following useful inequality.

Lemma 2.2. *For $1 \le p < \infty$, let F be nonnegative function. Then,*

$$\int_0^t [F(u(x,s))\varphi_1(x)]^p ds \ge t^{-(p-1)} \left(\int_0^t F(u(x,s))\varphi_1(x) ds\right)^p.$$

Proof. Let $1 \le p < \infty$.
Case $p = 1$. We have

$$\int_0^t F(u(x,s))\varphi_1(x) ds = t^{-(1-1)} \left(\int_0^t F(u(x,s))\varphi_1(x) ds\right)^1.$$

Case $1 < p < \infty$. Since $\frac{1}{p} + \frac{p-1}{p} = 1$, Hölder's inequality implies

$$\begin{aligned} & t^{-(p-1)} \left(\int_0^t F(u(x,s))\varphi_1(x) ds\right)^p \\ & \le t^{-(p-1)} \left(\int_0^t [F(u(x,s))\varphi_1(x)]^p ds\right) \left(\int_0^t 1^{\frac{p}{p-1}} ds\right)^{p-1} \\ & = \int_0^t [F(u(x,s))\varphi_1(x)]^p ds. \end{aligned}$$

The lemma is proved. □

Theorem 2.3. *For sufficiently large initial data u_0, the solution u of the problem* (1.1) *blows up in a finite time.*

Proof. Let $(x,t) \in \Omega \times (0,T)$. Then, we multiply the equation (1.1) by $\varphi_1(x)$. Using the fact that $(\varphi_1(x))^p \le \varphi_1(x) \le 1$ on $\bar{\Omega}$, and $G(u) \ge F(u)$, we get

$$\begin{aligned} u_t(x,t)\varphi_1(x) - \Delta u(x,t)\varphi_1(x) &= \left(\int_0^t [F(u(x,s))]^p ds\right) G(u(x,t))\varphi_1(x) \\ &\ge \left(\int_0^t [F(u(x,s))]^p [\varphi_1(x)]^p ds\right) G(u(x,t))\varphi_1(x) \\ &\ge \left(\int_0^t [F(u(x,s))\varphi_1(x)]^p ds\right) F(u(x,t))\varphi_1(x). \end{aligned}$$

By Lemma 2.2, the right-hand side of the above inequality becomes

$$\begin{aligned} & \left(\int_0^t [F(u(x,s))\varphi_1(x)]^p ds\right) F(u(x,t))\varphi_1(x) \\ & \ge t^{-(p-1)} \left(\int_0^t F(u(x,s))\varphi_1(x) ds\right)^p F(u(x,t))\varphi_1(x) \end{aligned}$$

$$= \frac{1}{p+1} t^{-(p-1)} \frac{\partial}{\partial t} \left(\int_0^t F(u(x,s))\varphi_1(x) ds \right)^{p+1}.$$

Thus, we have

$$u_t(x,t)\varphi_1(x) - \Delta u(x,t)\varphi_1(x) \geq \frac{1}{p+1} t^{-(p-1)} \frac{\partial}{\partial t} \left(\int_0^t F(u(x,s))\varphi_1(x) ds \right)^{p+1}.$$

Integrating the above inequality over $\Omega \times (0,t)$ to obtain

$$\begin{aligned} &\int_\Omega \int_0^t u_t(x,s)\varphi_1(x) ds dx - \int_\Omega \int_0^t \Delta u(x,s)\varphi_1(x) ds dx \\ &\geq \int_\Omega \int_0^t \left[\frac{1}{p+1} s^{-(p-1)} \frac{\partial}{\partial s} \left(\int_0^s F(u(x,\sigma))\varphi_1(x) d\sigma \right)^{p+1} \right] ds dx. \end{aligned} \tag{2.2}$$

For ease of reference, we recall here some of the main conditions on F, G and p which are $G(u) \geq F(u) \geq u$ and $p \geq 1$. Now, we are ready to prove the result about the blow-up of our solution. Consider

$$\begin{aligned} \int_\Omega \int_0^t u_t(x,s)\varphi_1(x) ds dx &= \int_\Omega u(x,t)\varphi_1(x) - u(x,0)\varphi_1(x) dx \\ &= \int_\Omega u(x,t)\varphi_1(x) dx - \int_\Omega u_0(x)\varphi_1(x) dx. \end{aligned}$$

Next, by the Green's second identity, and (2.1), we have

$$\begin{aligned} \int_0^t \int_\Omega \Delta u(x,s)\varphi_1(x) dx ds &= \int_0^t \int_\Omega u(x,s)\Delta\varphi_1(x) dx ds \\ &\quad + \int_0^t \int_{\partial\Omega} \left(\varphi_1(x) \frac{\partial u}{\partial \nu} - u(x,s) \frac{\partial \varphi_1(x)}{\partial \nu} \right) dx ds \\ &\geq -\lambda_1 \int_0^t \int_\Omega u(x,s)\varphi_1(x) dx ds. \end{aligned}$$

Finally, we use integration by parts, and Jensen's inequality to get

$$\begin{aligned} &\int_\Omega \int_0^t \left[\frac{1}{p+1} s^{-(p-1)} \frac{\partial}{\partial s} \left(\int_0^s F(u(x,\sigma))\varphi_1(x) d\sigma \right)^{p+1} \right] ds dx \\ &= \frac{1}{p+1} t^{-(p-1)} \int_\Omega \left(\int_0^t F(u(x,s))\varphi_1(x) ds \right)^{p+1} dx \\ &\quad + \int_\Omega \int_0^t \left(\int_0^s F(u(x,\sigma))\varphi_1(x) d\sigma \right)^{p+1} \left(\frac{p-1}{p+1} s^{-p} \right) ds dx \\ &\geq \frac{1}{p+1} t^{-(p-1)} \int_\Omega \left(\int_0^t F(u(x,s))\varphi_1(x) ds \right)^{p+1} dx \\ &\geq \frac{1}{p+1} t^{-(p-1)} \left(\int_\Omega \int_0^t F(u(x,s))\varphi_1(x) ds dx \right)^{p+1} \frac{1}{|\Omega|^{p+1}}, \end{aligned}$$

where $|\Omega| = \int_\Omega 1dx$ is the volume of Ω. Since $F(u) \geq u$, (2.2) becomes

$$\begin{aligned}&\int_\Omega u(x,t)\varphi_1(x)dx - \int_\Omega u_0(x)\varphi_1(x)dx + \lambda_1 \int_0^t \int_\Omega u(x,s)\varphi_1(x)dxds \\ &\geq \frac{1}{|\Omega|^{p+1}} \frac{1}{p+1} t^{-(p-1)} \left(\int_0^t \int_\Omega F(u(x,s))\varphi_1(x)dxds \right)^{p+1} \\ &\geq \frac{1}{|\Omega|^{p+1}} \frac{1}{p+1} t^{-(p-1)} \left(\int_0^t \int_\Omega u(x,s)\varphi_1(x)dxds \right)^{p+1}.\end{aligned}$$

Let us define

$$A(t) \equiv \int_0^t \int_\Omega u(x,s)\varphi_1(x)dxds.$$

Then, $A(t) \in C^1((0,T))$ with $A(0) = 0$, $A'(t) = \int_\Omega u(x,t)\varphi_1(x)dx \geq 0$, and $A(t)$ satisfies the integro-differential inequality

$$A'(t) + \lambda_1 A(t) - \frac{1}{|\Omega|^{p+1}} \frac{1}{p+1} t^{-(p-1)} (A(t))^{p+1} - \int_\Omega u_0(x)\varphi_1(x)dx \geq 0.$$

Next, we modify the idea of Liu and Chen [8] to construct a function $B(t)$ such that $B(t) \leq A(t)$ and $B(t)$ tends to infinity in a finite time. Let $a > 1$. Then, we define $[t]_a$ and $B(t)$ as follows

$$[t]_a \equiv \begin{cases} t - a, & \text{if } t > a, \\ 0, & \text{otherwise} \end{cases}$$

and

$$B(t) = \frac{\mu t}{(a - [t]_a)^{\frac{1}{p}}} \text{ for } 0 \leq t < 2a, \tag{2.3}$$

where $\mu > 0$ is a finite number to be chosen. Then, $B(t) \in C([0,2a))$ and $B(t)$ is differentiable in $[0,a) \cup (a,2a)$.

Case 1: For $0 \leq t < a$. We obtain $B'(t) = \frac{\mu}{a^{1/p}}$, and by direct calculation

$$\begin{aligned}&B'(t) + \lambda_1 B(t) - \frac{1}{|\Omega|^{p+1}} \frac{1}{p+1} t^{-(p-1)} (B(t))^{p+1} - \int_\Omega u_0(x)\varphi_1(x)dx \\ &= \frac{\mu}{a^{1/p}} + \lambda_1 \frac{\mu t}{a^{1/p}} - \frac{1}{p+1} \frac{1}{|\Omega|^{p+1}} t^2 \frac{\mu^{p+1}}{a^{(p+1)/p}} - \int_\Omega u_0(x)\varphi_1(x)dx.\end{aligned}$$

Note that $(\lambda_1 \mu t / a^{1/p}) - (\mu^{p+1} t^2 / (p+1)|\Omega|^{p+1} a^{(p+1)/p})$ attains its maximum value at

$$t^* = \frac{\lambda_1 (p+1)|\Omega|^{p+1} a}{2\mu^p},$$

and the maximum value is $(p+1)|\Omega|^{p+1}\lambda_1^2 \mu^{1-p} a^{1-\frac{1}{p}}/4$. By choosing

$$\mu \geq \left(\frac{\lambda_1 (p+1)}{2} \right)^{\frac{1}{p}} |\Omega|^{\frac{p+1}{p}}, \tag{2.4}$$

we have $0 < t^* < a$. Furthermore, if $u_0(x)$ satisfies

$$\int_\Omega u_0(x)\varphi_1(x)dx \geq \frac{\mu}{a^{1/p}} + \frac{1}{4}(p+1)|\Omega|^{p+1}\lambda_1^2\mu^{1-p}a^{1-\frac{1}{p}}, \tag{2.5}$$

we have

$$B'(t) + \lambda_1 B(t) - \frac{1}{|\Omega|^{p+1}}\frac{1}{p+1}t^{-(p-1)}(B(t))^{p+1} - \int_\Omega u_0(x)\varphi_1(x)dx \leq 0.$$

By regarding $F(u(x,t)) = u(x,t) = B(t)$, $F(v(x,t)) = v(x,t) = A(t)$, $g_1(x,t) = \lambda_1$, $g_2(x,s) = A(s)^{p-1}, B(s)^{p-1}$ and $g_3(x,t) = t^{1-p}/|\Omega|^{p+1}$ in Theorem 2.1, we obtain $B(t) \leq A(t)$ for $0 \leq t < a$. Since $B(t)$ and $A(t)$ are continuous on $[0,a]$, we get $B(a) \leq A(a)$. Hence, $B(t) \leq A(t)$ on $[0,a]$.

Case 2: For $a < t < 2a$. We have $B(t) = \mu t/(2a-t)^{1/p}$, and hence,

$$\begin{aligned}
&B'(t) + \lambda_1 B(t) - \frac{1}{|\Omega|^{p+1}}\frac{1}{p+1}t^{-(p-1)}(B(t))^{p+1} - \int_\Omega u_0(x)\varphi_1(x)dx \\
&= \frac{1}{p}\mu t(2a-t)^{-\frac{1}{p}-1} + \mu(2a-t)^{-\frac{1}{p}} + \lambda_1\mu t(2a-t)^{-\frac{1}{p}} \\
&\quad - \frac{1}{|\Omega|^{p+1}}\frac{1}{p+1}t^2\mu^{p+1}(2a-t)^{-\frac{1}{p}-1} - \int_\Omega u_0(x)\varphi_1(x)dx \\
&= \mu(2a-t)^{-\frac{1}{p}-1}\left\{2a + \left(\frac{1}{p} + 2a\lambda_1 - 1\right)t - \left(\lambda_1 + \frac{1}{|\Omega|^{p+1}}\frac{\mu^p}{p+1}\right)t^2\right\} \\
&\quad - \int_\Omega u_0(x)\varphi_1(x)dx.
\end{aligned} \tag{2.6}$$

If we take

$$\mu \geq \left\{\left(\frac{p(2a-1)+1}{p}\right)(p+1) + (2a-1)\lambda_1(p+1)\right\}^{\frac{1}{p}}|\Omega|^{\frac{1}{p}+1}, \tag{2.7}$$

then

$$(2a-1)\lambda_1 + \frac{p(2a-1)+1}{p} \leq \frac{1}{|\Omega|^{p+1}}\frac{\mu^p}{p+1}.$$

This gives

$$2a + \left(\frac{1}{p} + 2a\lambda_1 - 1\right) - \left(\lambda_1 + \frac{1}{|\Omega|^{p+1}}\frac{\mu^p}{p+1}\right) \leq 0,$$

Since

$$\begin{aligned}
&\frac{2a}{t} + \left(\frac{1}{p} + 2a\lambda_1 - 1\right) - \left(\lambda_1 + \frac{1}{|\Omega|^{p+1}}\frac{\mu^p}{p+1}\right)t \\
&< 2a + \left(\frac{1}{p} + 2a\lambda_1 - 1\right) - \left(\lambda_1 + \frac{1}{|\Omega|^{p+1}}\frac{\mu^p}{p+1}\right) \\
&\leq 0,
\end{aligned}$$

we have

$$2a + \left(\frac{1}{p} + 2a\lambda_1 - 1\right)t - \left(\lambda_1 + \frac{1}{|\Omega|^{p+1}}\frac{\mu^p}{p+1}\right)t^2 \leq 0$$

for any $t \in (a, 2a)$. Since $\mu(2a-t)^{-\frac{1}{p}-1} \geq 0$ in the open interval $(a, 2a)$, we have the right-hand side of (2.6) is nonpositive in $(a, 2a)$. This implies

$$B'(t) + \lambda_1 B(t) - \frac{1}{|\Omega|^{p+1}} \frac{1}{p+1} t^{-(p-1)} (B(t))^{p+1} - \int_\Omega u_0(x)\varphi_1(x)dx \leq 0$$

in the open interval $(a, 2a)$. By Theorem 2.1 again, we have $B(t) \leq A(t)$ for $t \in (a, 2a)$.

From both cases, we can see that if μ satisfies (2.7), then it must satisfy (2.4). Thus, if μ satisfies (2.7) and u_0 is large enough such that (2.5) holds, we can conclude that $B(t) \leq A(t)$ for $t \in [0, 2a)$.

Now, from (2.3) we can see that as t tends to $2a$, $B(t)$ tends to infinity, this implies $A(t)$ is unbounded in $[0, 2a)$. Since

$$A(t) = \int_0^t \int_\Omega u(x,s)\varphi_1(x)dxds,$$

we have $u(x,t)$ blows up in $\Omega \times [0, 2a)$. □

3. **Discussion**

One can investigate this type of problem further by approximating the blow-up time using numerical technique or considering the asymptotic behavior of a solution.

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