Dynamic Systems and Applications 25 (2016) 303-320

ON EXISTENCE THEOREMS FOR SOME NONLINEAR FUNCTIONAL-INTEGRAL EQUATIONS

LAKSHMI NARAYAN MISHRA[1,2] AND RAVI P. AGARWAL[3]

[1]Department of Mathematics, National Institute of Technology, Silchar 788 010, Cachar, Assam, India
[2]L. 1627 Awadh Puri Colony, Phase III, Beniganj, Opposite Industrial Training Institute (I.T.I.), Ayodhya Main Road, Faizabad 224 001, Uttar Pradesh, India
lakshminarayanmishra04@gmail.com, l_n_mishra@yahoo.co.in
[3]Department of Mathematics, Texas A&M University-Kingsville
700 University Blvd. Kingsville, TX 78363-8202, USA
Ravi.Agarwal@tamuk.edu

ABSTRACT. In this paper, we present sufficient conditions for the existence of solutions of two different types of nonlinear functional-integral equations in Banach space $C([0,a]\times[0,b],\mathbb{R})$ consisting of real functions, defined and continuous on the set $[0,a]\times[0,b]$. The main tools used in the proof are the concept of measures of noncompactness, Petryshyn fixed point theorem and Darbo's theorem in Banach space concerning the estimate on the solutions. Finally, we establish some examples of nonlinear functional-integral equations to show that our results are applicable.

AMS (MOS) Subject Classification. 45G10, 47H08, 47H10.

1. INTRODUCTION

Nonlinear integral equations have wide variety of applications in engineering, mechanics, physics, economics, vehicular traffic, optimization, biology, queuing theory and so on, for instance (cf. [2, 3, 5, 8, 9, 10, 14, 17, 22, 28, 37]). They yield an important tool for modeling various phenomena and processes occurring in heat conducting radiation, computer graphics, realistic illumination, particle transport problems of astrophysics, elasticity, electrostatics, radiative transfer, chemical kinetics, chemical reactor theory, theory of communication systems, quantum mechanics, magnetohydrodynamics and many other areas (for more applications of integral equations, see [37]). The theory of integral equations have been a significant growth with the help of tools in functional analysis, topology and fixed point theory (see [1, 3, 4, 5, 8, 9, 10, 12, 33]).

In this paper, we study the existence of solutions of nonlinear functional-integral equation

(1.1) $u(x,y) = f_1(x,y)$

Received June 1, 2016

$$+ f_2\left(x, y, u(x,y), \int_0^x p(x,y,\sigma,u(\sigma,y))d\sigma, \int_0^x\int_0^y q(x,y,s,t,u(s,t))dt\, ds\right),$$

for $(x,y) \in J = [0,a]\times[0,b]$, where $f_1 : J \to \mathbb{R}$, $f_2 : J_1\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $p : J\times[0,a]\times\mathbb{R} \to \mathbb{R}$, $q : J_2 \times \mathbb{R} \to \mathbb{R}$ are given continuous functions such that $J_1 = \{(x,y,u) : 0 \le x \le a; 0 \le y \le b; u \in \mathbb{R}\}$, and $J_2 = \{(x,y,s,t) \in J \times J : 0 \le s \le x \le a; 0 \le t \le y \le b\}$.

The main goal of this paper is to study the existence of solutions of certain nonlinear integral equations in two independent variables (see [1, 16, 23, 27, 29, 32, 34, 35, 38]). For the existence of solutions of integral equation (1.1), we use the Petryshyn fixed point theorem that has been analysed as a generalization of Darbo's fixed theorem.

We also present an existence theorem for the solutions of the following function-integral equation

$$u(x,y) = F\left(x, y, \int_0^x\int_0^y f(x,y,s,t,u(s,t))dt\, ds\right) \times G\left(x, y, \int_0^a\int_0^b g(x,y,s,t,u(s,t))dt\, ds\right), \tag{1.2}$$

for $(x,y) \in J = [0,a] \times [0,b]$, where $F, G : J \times \mathbb{R} \to \mathbb{R}$ and $f, g : J \times J \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

By using the fixed point theorem for the product of two operators which satisfy the Darbo condition with respect to a measure of noncompactness in the Banach algebra of continuous functions in the set $[0,a] \times [0,b]$, we obtain the existence of solutions for the functional-integral equation (1.2) and those solutions are continuous and stable.

The paper should be further motivated by somehow connecting the work with the works of several ones obtained earlier (cf. [3, 7, 9, 11, 13, 15, 17, 18, 19, 22, 25, 26, 30, 31, 32]). In Section 2, we recall some definitions and results and use them to obtain our main results in next Section. In Section 3, we prove our main results concerning the existence of solutions of the integral equations (1.1) and (1.2) and in Section 4, we provide some examples that verifies the applications of these kind of nonlinear functional-integral equations in nonlinear analysis and finally in Section 5, we give conclusion of this paper.

2. DEFINITIONS AND PRELIMINARIES

In this section, we gather some facts which will be needed in our further considerations.

Let E is a real Banach space with the norm $\|\cdot\|$ and zero element θ. Denote by $B(u,r)$ the closed ball centered at u and with radius r. The symbols $B_r = \{u \in E : \|u\| \le r\}$ and $\partial B_r = \{u \in E : \|u\| = r\}$ for the sphere in E around θ with radius r. If X is a nonempty subset of E we write $\bar{X}, ConvX$ in order to denote the closure and convex closure of X, respectively. We denote the standard algebraic operations on sets by the symbols λX and $X + Y$. Moreover, we denote by $\mathcal{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact sets.

We use the following definitions on the concept of a measure of noncompactness [6, 20, 24].

Definition 2.1. The Kuratowski measure of nocompactness (or set measure of noncompactness)

(2.1)
$$\nu(X) = \inf\{\epsilon > 0 : X \text{ may be covered by finitely many sets of diameter} \le \epsilon\}.$$

Goldenštein introduced the following measure of noncompactness.

Definition 2.2. The Hausdorff (or ball) measure of noncompactness

$$\mu(X) = \inf\{\epsilon > 0 : \text{there exists a finite } \epsilon\text{-net for } X \text{ in } E\}, \tag{2.2}$$

where by a finite ϵ-net for X in E it means, as usual, a set $\{p_1, p_2, \ldots, p_m\} \subset E$ such that the balls $B_\epsilon(E;p_1), B_\epsilon(E;p_2), \ldots, B_\epsilon(E;p_m)$ over X. These measures of noncompactness are mutually equivalent in the sense given by

$$\mu(X) \le \nu(x) \le 2\mu(X),$$

for any bounded set $X \subset E$.

Theorem 2.3. *Let $X, Y \in \mathcal{M}_E$ and $\lambda \in \mathbb{R}$. Then*

(i) $\mu(X) = 0$ *if and only if* $X \in \mathcal{N}_E$*;*
(ii) $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$*;*
(iii) $\mu(\bar{X}) = \mu(ConvX) = \mu(X)$*;*
(iv) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$*;*
(v) $\mu(\lambda X) = |\lambda|\mu(X)$*, where* $\lambda X = \{\lambda u : u \in X\}$*;*
(vi) $\mu(X + Y) \le \mu(X) + \mu(Y)$*, where* $X + Y = \{u + v : u \in X, v \in Y\}$*;*
(vii) $|\mu(X) - \mu(Y)| \le 2d_h(X,Y)$*, where* $d_h(X,Y)$ *denotes the Hausdorff metric of* X *and* Y*, i.e.*

$$d_h(X,Y) = \max\left\{\sup_{v\in Y} d(v,X), \sup_{u\in X} d(u,Y)\right\},$$

where $d(\cdot,\cdot)$ *is the distance from an element of* E *to a set of* E*.*

Further on, every function $\mu : \mathcal{M}_E \to [0, \infty)$, satisfying conditions (i)–(vi) of Theorem 2.3, will be called a regular measure of noncompactness in the Banach space E (cf. [9]).

Now let us assume that Ω is a nonempty subset of a Banach space E and $S : \Omega \to E$ is a continuous operator transforming bounded subsets of Ω to bounded ones. Moreover, let μ be a regular measure of noncompactness in E.

Definition 2.4 (see [6])**.** We say that S satisfies the Darbo condition with a constant k with respect to measure μ provided

$$\mu(SX) \leq k\mu(X)$$

for each $X \in \mathcal{M}_E$ such that $X \subset \Omega$. If $k < 1$, then S is called a contraction with respect to μ and if $\mu(SX) < \mu(X)$, for all $\mu(X) > 0$, then S is called densifying or condensing map. A k-set contraction with $0 < k < 1$ is densifying, but converse may not be true.

In the sequel, we will work in the space $C(J, \mathbb{R})$ consisting of all real functions defined and continuous on the set J. The space $C(J, \mathbb{R})$ is equipped with standard norm

$$\|u\| = \sup\{|u(x, y)| : (x, y) \in J\}.$$

Obviously, the space $C(J, \mathbb{R})$ has also the structure of Banach algebra.

In our considerations, we will use a regular measure of noncompactness defined in [7] (cf. also [6]). In order to recall the definitions of that measure let us fix a set $X \in \mathcal{M}_{C(J,\mathbb{R})}$. For $u \in X$ and for a given $\epsilon > 0$ denote by $w(u, \epsilon)$ the modulus of continuity of u, i.e.,

$$w(u, \epsilon) = \sup\{|u(x, y) - u(s, t)| : x, s \in [0, a];\ y, t \in [0, b]; |x - s| \leq \epsilon, |y - t| \leq \epsilon\}.$$

Further, put

$$w(X, \epsilon) = \sup\{w(u, \epsilon) : u \in X\},$$

$$w_0(X) = \lim_{\epsilon \to 0} w(X, \epsilon).$$

It can be shown in [7] that the function $w_0(X)$ is a regular measure of noncompactness in the space $C(J, \mathbb{R})$.

For our purposes we will need the following theorems and lemma [7, 21].

Theorem 2.5 (see [36])**.** *Let $M : B_r \to E$ is a densifying mapping which satisfies the boundary condition*

$$M(u) = ku, \ \textit{for some } u \textit{ in } \partial B_r \textit{ with } k \leq 1,$$

then the set of fixed points of M in B_r is nonempty which is known by Petryshyn fixed point theorem.

Lemma 2.6. *Let F be a bounded, closed and convex subset of E. If operator $S : F \to F$ is a strict set contraction, then S has a fixed point in F.*

Theorem 2.7. *Assume that Ω is a nonempty, bounded, convex and closed subset of $C(J,\mathbb{R})$ and the operators P and T transform continuously the set Ω into $C(J,\mathbb{R})$ in such a way that $P(\Omega)$ and $T(\Omega)$ are bounded. Moreover, assume that the operator $S = P \cdot T$ transform Ω into itself. If the operators P and T satisfy on the set Ω the Darbo condition with the constant k_1 and k_2, respectively, then the operator S satisfies the Darbo condition on Ω with the constant*

$$\|P(\Omega)\|k_2 + \|T(\Omega)\|k_1.$$

Remark 2.8. In Theorem 2.7, if $\|P(\Omega)\|k_2 + \|T(\Omega)\|k_1 < 1$, then S is a contraction with respect to the measure w_0 and has at least one fixed point in the set Ω.

These properties will permit us to identify solutions of the integral equations (1.1) and (1.2).

3. MAIN RESULTS

In this section, we prove the main results of this paper under the following special assumptions for integral equation (1.1).

(A_1) $u, f_1 \in C(J,\mathbb{R})$, $f_2 \in C(J_1 \times \mathbb{R} \times \mathbb{R},\mathbb{R})$, $p \in C(J \times J_a \times \mathbb{R},\mathbb{R})$, $q \in C(J_2 \times \mathbb{R},\mathbb{R})$, and the function f_1 is bounded, where, $J = J_a \times J_b$, $J_1 = \{(x,y,u) : 0 \le x \le a; 0 \le y \le b;\ u \in \mathbb{R}\}$, $J_2 = \{(x,y,s,t) \in J \times J : 0 \le s \le x \le a; 0 \le t \le y \le b\}$.

(A_2) There exist nonnegative constants $l_1, l_2, l_3 \in (0,1)$ such that $|f_2(x,y,u,w,v) - f_2(x,y,u^*,w^*,v^*) \le l_1|u-u^*| + l_2|w-w^*| + l_3|v-v^*|$.

(A_3) There exists the number $r \ge 0$ such that the following bounded condition is satisfied $\sup\{|f_1(x,y)| : (x,y) \in J\} + A \le r$, where $A = \sup\{|f_2(x,y,u,w,v)| : (x,y) \in J,\ -r \le u \le r,\ -aX_1 \le w \le aX_1;\ -abX_2 \le v \le abX_2\}$, $X_1 = \sup\{|p(x,y,\sigma,u)| : \text{for all } (x,y) \in J \text{ and } \sigma \in J_a, u \in [-r,r]\}$,
$X_2 = \sup\{|q(x,y,s,t,u)| : \text{ for all } (x,y,s,t) \in J_2 \text{ and } u \in [-r,r]\}$.

Theorem 3.1. *Under assumptions (A_1)–(A_3), equation (1.1) has at least one solution in the Banach space $E = C(J,\mathbb{R})$.*

Proof. To achieve this result, we will use Theorem 2.5 as our main tool, we require to define the operator $M : B_r \to E$ as follows:

$$(Mu)(x,y) = f_1(x,y) + f_2\left(x, y, u(x,y), \int_0^x p(x,y,\sigma,u(\sigma,y))d\sigma, \int_0^x \int_0^y q(x,y,s,t,u(s,t))dt\, ds\right).$$

Now, we have to prove that the operator M is continuous on the ball B_r. For that, we consider $\epsilon > 0$ and take arbitrary $u, v \in B_r$ with $\|u - v\| \leq \epsilon$, for $(x, y) \in J$, we obtain

$$\begin{aligned}
&|(Mu)(x,y) - (Mv)(x,y)| \\
&= \Bigg| f_2\left(x, y, u(x,y), \int_0^x p(x,y,\sigma,u(\sigma,y))d\sigma, \int_0^x\int_0^y q(x,y,s,t,u(s,t))dt\, ds\right) \\
&\quad - f_2\left(x, y, v(x,y), \int_0^x p(x,y,\sigma,v(\sigma,y))d\sigma, \int_0^x\int_0^y q(x,y,s,t,v(s,t))dt\, ds\right)\Bigg| \\
&\leq l_1|u(x,y) - v(x,y)| + l_3\int_0^x\int_0^y |q(x,y,s,t,u(s,t)) - q(x,y,s,t,v(s,t))|dt\, ds \\
&\quad + l_2\int_0^x |p(x,y,\sigma,u(\sigma,y)) - p(x,y,\sigma,v(\sigma,y))|d\sigma \\
&\leq l_1\|u - v\| + l_3 ab w_q(\epsilon) + l_2 a w_p(\epsilon),
\end{aligned} \tag{3.1}$$

where for $\epsilon > 0$, we denote $w_q(\epsilon) = \sup\{|q(x,y,s,t,u) - q(x,y,s,t,v)| : (x,y,s,t) \in J_2; u, v \in [-r, r]; |u - v| \leq \epsilon\}$, $w_p(\epsilon) = \sup\{|p(x,y,\sigma,u) - p(x,y,\sigma,v)| : (x,y) \in J; \sigma \in J_a; u, v \in [-r, r]; |u - v| \leq \epsilon\}$. Now, using the uniform continuity of the functions $p(x,y,\sigma,u)$ and $q(x,y,s,t,u)$ on the set $J \times J_a \times [-r, r]$ and $J_2 \times [-r, r]$, respectively, we derive that $w_p(\epsilon) \to 0$ and $w_q(\epsilon) \to 0$ as $\epsilon \to 0$. Hence, in view of our assumptions and from the above estimate (3.1) we conclude that the operator M is continuous on B_r.

Next, we have to show that the operator M satisfies densifying condition. For that, we choose a fixed arbitrary $\epsilon > 0$ and take $u \in X$, where X is bounded subset of E, $(x_1, y_1), (x_2, y_2) \in J$, with $x_1 \leq x_2, y_1 \leq y_2$ such that $x_2 - x_1 \leq \epsilon, y_2 - y_1 \leq \epsilon$, we have

(3.2)

$$\begin{aligned}
&|(Mu)(x_2,y_2) - (Mu)(x_1,y_1)| \leq |f_1(x_2,y_2) - f_1(x_1,y_1)| \\
&\quad + \Bigg| f_2\left(x_2, y_2, u(x_2,y_2), \int_0^{x_2} p(x_2,y_2,\sigma,u(\sigma,y_2))d\sigma, \int_0^{x_2}\int_0^{y_2} q(x_2,y_2,s,t,u(s,t))dt\, ds\right) \\
&\quad - f_2\left(x_1, y_1, u(x_1,y_1), \int_0^{x_1} p(x_1,y_1,\sigma,u(\sigma,y_1))d\sigma, \int_0^{x_1}\int_0^{y_1} q(x_1,y_1,s,t,u(s,t))dt\, ds\right)\Bigg| \\
&\leq w(f_1, \epsilon)
\end{aligned}$$

$$+\left|f_2\left(x_2,y_2,u(x_2,y_2),\int_0^{x_2}p(x_2,y_2,\sigma,u(\sigma,y_2))d\sigma,\int_0^{x_2}\int_0^{y_2}q(x_2,y_2,s,t,u(s,t))dt\,ds\right)\right.$$

$$\left.-f_2\left(x_2,y_2,u(x_2,y_2),\int_0^{x_2}p(x_2,y_2,\sigma,u(\sigma,y_2))d\sigma,\int_0^{x_1}\int_0^{y_1}q(x_1,y_1,s,t,u(s,t))dt\,ds\right)\right|$$

$$+\left|f_2\left(x_2,y_2,u(x_2,y_2),\int_0^{x_2}p(x_2,y_2,\sigma,u(\sigma,y_2))d\sigma,\int_0^{x_1}\int_0^{y_1}q(x_1,y_1,s,t,u(s,t))dt\,ds\right)\right.$$

$$\left.-f_2\left(x_2,y_2,u(x_2,y_2),\int_0^{x_1}p(x_1,y_1,\sigma,u(\sigma,y_1))d\sigma,\int_0^{x_1}\int_0^{y_1}q(x_1,y_1,s,t,u(s,t))dt\,ds\right)\right|$$

$$+\left|f_2\left(x_2,y_2,u(x_2,y_2),\int_0^{x_1}p(x_1,y_1,\sigma,u(\sigma,y_1))d\sigma,\int_0^{x_1}\int_0^{y_1}q(x_1,y_1,s,t,u(s,t))dt\,ds\right)\right.$$

$$\left.-f_2\left(x_2,y_2,u(x_1,y_1),\int_0^{x_1}p(x_1,y_1,\sigma,u(\sigma,y_1))d\sigma,\int_0^{x_1}\int_0^{y_1}q(x_1,y_1,s,t,u(s,t))dt\,ds\right)\right|$$

$$+\left|f_2\left(x_2,y_2,u(x_1,y_1),\int_0^{x_1}p(x_1,y_1,\sigma,u(\sigma,y_1))d\sigma,\int_0^{x_1}\int_0^{y_1}q(x_1,y_1,s,t,u(s,t))dt\,ds\right)\right.$$

$$\left.-f_2\left(x_1,y_1,u(x_1,y_1),\int_0^{x_1}p(x_1,y_1,\sigma,u(\sigma,y_1))d\sigma,\int_0^{x_1}\int_0^{y_1}q(x_1,y_1,s,t,u(s,t))dt\,ds\right)\right|$$

$$\leq w(f_1,\epsilon)+l_3\left|\int_0^{x_2}\int_0^{y_2}q(x_2,y_2,s,t,u(s,t))dt\,ds-\int_0^{x_1}\int_0^{y_1}q(x_1,y_1,s,t,u(s,t))dt\,ds\right|$$

$$+l_2\left|\int_0^{x_2}p(x_2,y_2,\sigma,u(\sigma,y_2))d\sigma-\int_0^{x_1}p(x_1,y_1,\sigma,u(\sigma,y_1))d\sigma\right|$$

$$+l_1|u(x_2,y_2)-u(x_1,y_1)|+w_1(f_2,\epsilon)$$

$$\leq w(f_1,\epsilon)+l_3\int_0^{x_1}\int_0^{y_1}|q(x_2,y_2,s,t,u(s,t))-q(x_1,y_1,s,t,u(s,t))|dt\,ds$$

$$+l_3\int_{x_1}^{x_2}\int_{y_1}^{y_2}|q(x_2,y_2,s,t,u(s,t))|dt\,ds+l_3\int_0^{x_1}\int_{y_1}^{y_2}|q(x_2,y_2,s,t,u(s,t))|dt\,ds$$

$$+l_3\int_{x_1}^{x_2}\int_0^{y_1}|q(x_2,y_2,s,t,u(s,t))|dt\,ds+l_2\int_0^{x_1}|p(x_2,y_2,\sigma,u(\sigma,y_2))$$

$$-p(x_1,y_1,\sigma,u(\sigma,y_1))|d\sigma+l_2\int_{x_1}^{x_2}|p(x_2,y_2,\sigma,u(\sigma,y_2))|d\sigma+l_1|u(x_2,y_2)-u(x_1,y_1)|$$

$+ w_1(f_2, \epsilon)$.

For our convenience, we use the following quantities:

$$\begin{aligned} w_1(q,\epsilon) &= \sup\{|q(x,y,s,t,u) - q(x^*,y^*,s,t,u)| : |x-x^*| \le \epsilon, |y-y^*| \le \epsilon; \\ &\qquad (x,y,s,t) \in J_2; u \in [-r,r]\}, \\ w_1(p,\epsilon) &= \sup\{|p(x,y,\sigma,u) - p(x^*,y^*,\sigma,u)| : |x-x^*| \le \epsilon, |y-y^*| \le \epsilon; \\ &\qquad (x,y) \in J; \sigma \in J_a; u \in [-r,r]\}, \\ w_1(f_2,\epsilon) &= \sup\{|f_2(x,y,u,w,v) - f_2(x^*,y^*,u,w,v)| : |x-x^*| \le \epsilon, |y-y^*| \le \epsilon; \\ &\qquad u \in [-r,r]; w \in [-aX_1, aX_1]; v \in [-abX_2, abX_2]\}. \end{aligned}$$

Thus from the estimate (3.2), we obtain

$$\begin{aligned} |(Mu)(x_2,y_2) - (Mu)(x_1,y_1)| &\le w(f_1,\epsilon) + l_3 ab w_1(q,\epsilon) + \epsilon^2 l_3 X_2 + \epsilon a l_3 X_2 + \epsilon b l_3 X_2 \\ &\quad + l_2 a w_1(p,\epsilon) + \epsilon l_2 X_1 + l_1 w(u,\epsilon) + w_1(f_2,\epsilon). \end{aligned}$$

Thus taking the limit as $\epsilon \to 0$, the above estimate yields as follows

$$w_0(MX) \le l_1 w_0(X)$$

which shows that M is densifying map. Now let $u \in \partial B_r$ and if $Mu = ku$ then we obtain $\|Mu\| = k\|u\| = kr$ and by using assumption (A_3), we conclude

$$\begin{aligned} &|(Mu)(x,y)| \\ &= \left| f_1(x,y) + f_2\left(x, y, u(x,y), \int_0^x p(x,y,\sigma,u(\sigma,y))d\sigma, \int_0^x\int_0^y q(x,y,s,t,u(s,t))dt\,ds\right)\right| \\ &\le r, \end{aligned}$$

for all $(x,y) \in J$, and hence $\|Mu\| \le r$, this means $k \le 1$. □

Now, we will study the solvability of the nonlinear functional-integral equation (1.2) for $u \in C(J,\mathbb{R})$. Then we have the following result.

Theorem 3.2. *Under the following assumptions (H_1)–(H_6):*

- (H_1) *The functions $F, G : J \times \mathbb{R} \to \mathbb{R}$ are continuous and there exists a nonnegative constant m such that $|F(x,y,0)| \le m$, $|G(x,y,0)| \le m$, for $(x,y) \in J$.*
- (H_2) *There exists the continuous functions $a_1, b_1 : J \to \mathbb{R}_+$ such that $|F(x,y,z_1) - F(x,y,z_2)| \le a_1(x,y)|z_1 - z_2|$, $|G(x,y,z_1) - G(x,y,z_2)| \le b_1(x,y)|z_1 - z_2|$, for all $z_i \in \mathbb{R}, i = 1,2$ and $(x,y) \in J$.*
- (H_3) *The functions $f(x,y,s,t,u)$ and $g(x,y,s,t,u)$ act continuously from the set $J \times J \times \mathbb{R} \to \mathbb{R}$. Moreover, there exist the functions $n, n^* : J \to \mathbb{R}_+$ being continuous on J and the functions $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$, continuous and nondecreasing on $\mathbb{R}_+$ with $\phi(0) = 0$ and $\psi(0) = 0$ such that $|f(x,y,s,t,u) - f(x,y,s,t,v)| \le$*

$n(x,y)\phi(|u-v|)$, $|g(x,y,s,t,u)-g(x,y,s,t,v)| \le n^*(x,y)\psi(|u-v|)$, for all $(s,t)\in J$ such that $s\le x; t\le y$ and for all $u,v\in\mathbb{R}$. Let us define the function $f_1 : J\to\mathbb{R}_+$, $f_1(x,y)=\max\{|f(x,y,s,t,0)| : 0\le s\le x, 0\le t\le y\}$. The function f_1 is continuous on J.

(H_4) *(Sublinearity condition) There exists the constants ξ and η such that $|f(x,y,s,t,u)| \le \xi+\eta|u|$, $|g(x,y,s,t,u)|\le \xi+\eta|u|$, also $ab\eta > 1$ for all $(x,y),(s,t)\in J$ and $u\in\mathbb{R}$.*

(H_5) *There exists a nonnegative constant k such that $\max\{a_1(x,y), b_1(x,y)\}\le k$, for all $(x,y)\in J$.*

(H_6) *$4\xi'\eta' < 1$ for $\xi' = kab\xi + m$ and $\eta' = kab\eta + k$.*

Then equation (1.2) has at least one solution in the Banach algebra $E = C(J,\mathbb{R})$.

Proof. Let us consider the operators P and T defined on the Banach algebra E by the formula

$$(Pu)(x,y) = F\left(x,y,\int_0^x\int_0^y f(x,y,s,t,u(s,t))dt\,ds\right),$$

$$(Tu)(x,y) = G\left(x,y,\int_0^a\int_0^b g(x,y,s,t,u(s,t))dt\,ds\right),$$

for $(x,y)\in J$.

From assumptions (H_1) and (H_3), it follows that P and T transform the algebra E into itself. Further, let us define the operator S on the algebra E by putting

$$Su = (Pu)\cdot(Tu).$$

Obviously, S transform E into itself. Now, let us fix $u\in E$. Then using our assumptions for $(x,y)\in J$, we get

$$\begin{aligned}
&|(Su)(x,y)| = |(Pu)(x,y)|\times|(Tu)(x,y)|\\
&\le\left\{\left|F\left(x,y,\int_0^x\int_0^y f(x,y,s,t,u(s,t))dt\,ds,\right)-F(x,y,0)\right|+|F(x,y,0)|\right\}\\
&\quad\times\left\{\left|G\left(x,y,\int_0^a\int_0^b g(x,y,s,t,u(s,t))dt\,ds\right)-G(x,y,0)\right|+|G(x,y,0)|\right\}\\
&\le\left\{a_1(x,y)\int_0^x\int_0^y |f(x,y,s,t,u(s,t))|dt\,ds+|F(x,y,0)|\right\}\\
&\quad\times\left\{b_1(x,y)\int_0^a\int_0^b |g(x,y,s,t,u(s,t))|dt\,ds+|G(x,y,0)|\right\}
\end{aligned}$$

$$\le \{k(\xi+\eta|u(x,y)|).ab+m\}\cdot\{k(\xi+\eta|u(x,y)|).ab+m\}$$
$$\le \{k(\xi+\eta\|u\|)ab+m\}\cdot\{k(\xi+\eta\|u\|)ab+m\}$$
$$\le \{(kab\eta)\|u\|+kab\xi+m\}^2.$$

Let $\eta^{'} = kab\eta$ and $\xi^{'} = kab\xi + m$ then from the above estimate, we obtain

$$\|Pu\| \le \eta^{'}\|u\| + \xi^{'}, \tag{3.3}$$

$$\|Tu\| \le \eta^{'}\|u\| + \xi^{'}, \tag{3.4}$$

$$\|Su\| \le (\eta^{'}\|u\| + \xi^{'})^2, \tag{3.5}$$

for $u \in C(J,\mathbb{R})$.

From estimate (3.5), we deduce that the operator S maps the ball $B_r \subset C(J,\mathbb{R})$ into itself for $r_1 \le r \le r_2$, where

$$r_1 = \frac{1-2\eta^{'}\xi^{'} - \sqrt{1-4\eta'\xi'}}{2\eta'^2},$$

$$r_2 = \frac{1-2\eta^{'}\xi^{'} + \sqrt{1-4\eta'\xi'}}{2\eta'^2}.$$

Also, from estimate (3.3) and (3.4), it follows easily that

$$\|PB_r\| \le \eta^{'} r + \xi^{'}, \tag{3.6}$$

$$\|TB_r\| \le \eta^{'} r + \xi^{'}. \tag{3.7}$$

Now we prove that the operator P is continuous on the ball B_r. For that we fix $\epsilon > 0$ and take arbitrary $u, v \in B_r$ such that $\|u - v\| \le \epsilon$. Then for $(x,y) \in J$, we have

$$|(Pu)(x,y) - (Pv)(x,y)| = \left|F\left(x,y,\int_0^x\int_0^y f(x,y,s,t,u(s,t))dt\,ds\right)\right.$$
$$\left. -F\left(x,y,\int_0^x\int_0^y f(x,y,s,t,v(s,t))dt\,ds\right)\right|$$
$$\le a_1(x,y)\int_0^x\int_0^y |f(x,y,s,t,u(s,t)) - f(x,y,s,t,v(s,t))|dt\,ds$$
$$\le k\int_0^x\int_0^y |f(x,y,s,t,u(s,t)) - f(x,y,s,t,v(s,t))|dt\,ds$$
$$\le kab\; w(f,\epsilon)$$

where $w(f,\epsilon) = \sup\{|f(x,y,s,t,u) - f(x,y,s,t,v)| : s \le x, t \le y; (x,y) \in J;\ u, v \in [-r,r]; |u-v| \le \epsilon\}$.

Since, we know that the function $f = f(x,y,s,t,u(s,t))$ is uniformly continuous on the bounded subset $J \times J \times [-r,r]$, we conclude that $w(f,\epsilon) \to 0$ as $\epsilon \to 0$.

Thus the operator P is continuous on B_r. Similarly, one can easily show that T is continuous on B_r and consequently we deduce that S is continuous on B_r.

Now, we show that the operators P and T satisfy the Darbo condition with respect to measure w_0 defined in Section 2, in the ball B_r. To do this, we take a nonempty subset X of B_r and $u \in X$, Let $\epsilon > 0$ be fixed and $x_1, x_2 \in [0,a];\ y_1, y_2 \in [0,b]$ such that $x_2 - x_1 \leq \epsilon, y_2 - y_1 \leq \epsilon$. Without loss of generality we can assume that $x_1 \leq x_2$ and $y_1 \leq y_2$. Then, in view of imposed assumptions, we have

$$\begin{aligned}(3.8)\quad |(Pu)(x_2,y_2)-(Pu)(x_1,y_1)| &= \Bigg| F\left(x_2,y_2,\int_0^{x_2}\int_0^{y_2} f(x_2,y_2,s,t,u(s,t))dt\,ds\right) \\ &\quad -F\left(x_1,y_1,\int_0^{x_1}\int_0^{y_1} f(x_1,y_1,s,t,u(s,t))dt\,ds\right)\Bigg| \\ &\leq a_1(x,y)\left|\int_0^{x_2}\int_0^{y_2} f(x_2,y_2,s,t,u(s,t))dt\,ds - \int_0^{x_1}\int_0^{y_1} f(x_1,y_1,s,t,u(s,t))dt\,ds\right| \\ &\quad + \Bigg| F\left(x_2,y_2,\int_0^{x_1}\int_0^{y_1} f(x_1,y_1,s,t,u(s,t))dt\,ds\right) \\ &\quad -F\left(x_1,y_1,\int_0^{x_1}\int_0^{y_1} f(x_1,y_1,s,t,u(s,t))dt\,ds\right)\Bigg|\end{aligned}$$

For our convenience, we define the following quantities

$$\begin{aligned} w_f(\epsilon,\cdot,\cdot) &= \sup\{|f(x_2,y_2,s,t,u)-f(x_1,y_1,s,t,u)| : x_1,x_2\in[0,a];\ y_1,y_2\in[0,b]; \\ &\qquad s\leq x_1,x_2;\ t\leq y_1,y_2;\ |x_1-x_2|\leq\epsilon, |y_1-y_2|\leq\epsilon \text{ and } u\in[-r,r]\}, \\ w_F(\epsilon,\cdot,\cdot) &= \sup\{|F(x_2,y_2,z)-F(x_1,y_1,z)| : x_1,x_2\in[0,a];\ y_1,y_2\in[0,b]; \\ &\qquad |x_1-x_2|\leq\epsilon, |y_1-y_2|\leq\epsilon, z\in[-k'ab,k'ab] \text{ and } u\in[-r,r]\}, \\ k' &= \sup\{|f(x,y,s,t,u)| : (x,y),(s,t)\in J;\ u\in[-r,r]\}. \end{aligned}$$

Then using relation (3.8) we obtain the following

(3.9)

$$\begin{aligned} &|(Pu)(x_2,y_2)-(Pu)(x_1,y_1)| \\ &\leq k\left|\int_0^{x_2}\int_0^{y_2} f(x_2,y_2,s,t,u(s,t))dt\,ds - \int_0^{x_1}\int_0^{y_1} f(x_1,y_1,s,t,u(s,t))dt\,ds\right| + w_F(\epsilon,\cdot,\cdot) \\ &\leq k\Bigg\{\int_0^{x_1}\int_0^{y_1} |f(x_2,y_2,s,t,u(s,t)) - f(x_1,y_1,s,t,u(s,t))|dt\,ds \end{aligned}$$

$$
\begin{aligned}
&+\int_{x_1}^{x_2}\int_{y_1}^{y_2} |f(x_2,y_2,s,t,u(s,t))-f(x_2,y_2,s,t,0)+f(x_2,y_2,s,t,0)|dt\ ds \\
&+\int_{0}^{x_1}\int_{y_1}^{y_2} |f(x_2,y_2,s,t,u(s,t))-f(x_2,y_2,s,t,0)+f(x_2,y_2,s,t,0)|dt\ ds \\
&\left.+\int_{x_1}^{x_2}\int_{0}^{y_1} |f(x_2,y_2,s,t,u(s,t))-f(x_2,y_2,s,t,0)+f(x_2,y_2,s,t,0)|dt\ ds\right\} \\
&+w_F(\epsilon,\cdot,\cdot) \\
\le\ & k\left[w_f(\epsilon,.,.).ab+\int_{x_1}^{x_2}\int_{y_1}^{y_2}\{|f(x_2,y_2,s,t,u(s,t))-f(x_2,y_2,s,t,0)|\right. \\
&+|f(x_2,y_2,s,t,0)|\}dt\ ds \\
&+\int_{0}^{x_1}\int_{y_1}^{y_2}\{|f(x_2,y_2,s,t,u(s,t))-f(x_2,y_2,s,t,0)|+|f(x_2,y_2,s,t,0)|\}dt\ ds \\
&\left.+\int_{x_1}^{x_2}\int_{0}^{y_1}\{|f(x_2,y_2,s,t,u(s,t))-f(x_2,y_2,s,t,0)|+|f(x_2,y_2,s,t,0)|\}dt\ ds\right] \\
&+w_F(\epsilon,\cdot,\cdot) \\
\le\ & k\left[w_f(\epsilon,.,.).ab+\int_{x_1}^{x_2}\int_{y_1}^{y_2}\{n(x_2,y_2)\phi(|u(s,t)|)+f_1(x_2,y_2)\}dt\ ds\right. \\
&+\int_{0}^{x_1}\int_{y_1}^{y_2}\{n(x_2,y_2)\phi(|u(s,t)|)+f_1(x_2,y_2)\}dt\ ds \\
&\left.+\int_{x_1}^{x_2}\int_{0}^{y_1}\{n(x_2,y_2)\phi(|u(s,t)|)+f_1(x_2,y_2)\}dt\ ds\right]+w_F(\epsilon,\cdot,\cdot) \\
\le\ & k\left[w_f(\epsilon,\cdot,\cdot)\cdot ab+\{n(x_2,y_2)\phi(\|u\|)+f_1(x_2,y_2)\}\{(x_2-x_1)(y_2-y_1)\right. \\
&\left.+x_1(y_2-y_1)+(x_2-x_1)y_1\}\right]+w_F(\epsilon,\cdot,\cdot) \\
\le\ & k\left[w_f(\epsilon,\cdot,\cdot)\cdot ab+\{n(x_2,y_2)\phi(\|u\|)+f_1(x_2,y_2)\}\{\epsilon(\epsilon+x_1+y_1)\}\right] \\
&+w_F(\epsilon,\cdot,\cdot)
\end{aligned}
$$

Let us denote

$$
\bar{n}(a,b)=\max\{n(x,y):(x,y)\in J\},
$$
$$
\bar{f}_1(a,b)=\max\{f_1(x,y):(x,y)\in J\}.
$$

Further, keeping in mind the estimate (3.9), we get

$$w(Pu,\epsilon) \le k\left[w_f(\epsilon,\cdot,\cdot)\cdot ab + \{\bar{n}(a,b)\phi(\|u\|) + \bar{f}_1(a,b)\}\{\epsilon(\epsilon + x_1 + y_1)\}\right] + w_F(\epsilon,\cdot,\cdot).$$

Observe that invoking the uniform continuity of the function $F(x,y,z)$ on the set $J\times\mathbb{R}$ and the function $f(x,y,s,t,u)$ on $J\times J\times\mathbb{R}$, we deduce that $w_F(\epsilon,\cdot,\cdot)\to 0$ and $w_f(\epsilon,\cdot,\cdot)\to 0$ as $\epsilon\to 0$. Consequently, from the above estimate we conclude

$$w_0(PX) \le k w_0(X). \tag{3.10}$$

Similarly, we can show that

$$w_0(TX) \le k w_0(X). \tag{3.11}$$

Finally, in view of the estimates (3.6), (3.7), (3.10) and (3.11) and Theorem 2.7, we infer that the operator S satisfies the Darbo condition on B_r with respect to measure w_0 with constant $(\eta' r + \xi')k + (\eta' r + \xi')k$. Thus, we have

$$\begin{aligned}
(\eta' r + \xi')k + (\eta' r + \xi')k &= 2k(\eta' r + \xi')\\
&\le 2k(\eta' r_2 + \xi')\\
&= 2k\left\{\eta'\left(\frac{(1-2\eta'\xi') + \sqrt{1-4\eta'\xi'}}{2\eta'^2}\right) + \xi'\right\}\\
&= k\left(\frac{1+\sqrt{1-4\eta'\xi'}}{\eta'}\right)\\
&< 1.
\end{aligned}$$

Hence, the operator S is a contraction on B_r with respect to w_0. Thus, by applying Theorem 2.7 and Remark 2.8 we get that S has at least one fixed point in B_r. Consequently, the nonlinear functional-integral equation (1.2) has at least one solution in B_r. □

4. **EXAMPLES**

As applications and to illustrate our results, we present some examples.

Example 4.1. If $f_2(x,y,u,w,v) = l_2 w + l_3 v$ and $l_2, l_3 \in (0,1)$, then equation (1.1) appears in the following form of the integral equation

$$u(x,y) = f_1(x,y) + \int_0^x l_2 p(x,y,\sigma,u(\sigma,y))d\sigma + \int_0^x\int_0^y l_3 q(x,y,s,t,u(s,t))dt\,ds, \tag{4.1}$$

where $(x,y)\in[0,a]\times[0,b]$. The above equation (4.1) is studied by various authors in the literature [14, 35]. Notice that the above equation contains as a particular case of the following integral equation

$$u(x,y) = f_1(x,y) + \int_0^a \int_0^b R(x,y,s,t)S(s,t,u(s,t))dt\,ds,$$

which may be conceived as a two independent variables generalization of the famous Hammerstein type integral equation [35].

Example 4.2. In the above example, if we put $l_3 = \frac{1}{3}$, $f_1(x,y) = \frac{1}{4}e^{-2/(x+y+1)}$, $w = 0$ and $q(x,y,s,t,u) = \frac{x+y+1}{3} st\cos(t+u)$, then equation (1.1) reduces to the following form of integral equation

$$u(x,y) = \frac{1}{4}e^{-2/(x+y+1)} + \frac{x+y+1}{9}\int_0^x \int_0^y st\cos(t+u(s,t))dt\,ds, \tag{4.2}$$

where $(x,y) \in J = [0,1]\times[0,1]$. It is clearly seen that assumptions (A_1) and (A_2) are satisfied. We only check that (A_3) also holds. Take $r = 1$ then we obtain $X_2 \leq 1$ and

$$\begin{aligned}&\sup\{|f_1(x,y)| : (x,y) \in J\} + \sup\{|f_2(x,y,u,w,v)| : (x,y) \in J, -1 \leq u \leq 1,\\&\quad -1 \leq v \leq 1\}\\&\leq \sup\left\{\left|\frac{1}{4}e^{-2/(x+y+1)}\right| : (x,y) \in [0,1]\times[0,1]\right\} + \sup\left\{\frac{1}{3}|v| : -1 \leq v \leq 1\right\} \leq 1.\end{aligned}$$

Example 4.3. Consider the following nonlinear functional integral equation:

$$\begin{aligned}u(x,y) = &\left[\frac{1}{8}\int_0^x\int_0^y \left\{\frac{y}{2}\sin u(s,t) + 5xy\ln(1+|u(s,t)|)\right\}dt\,ds\right]\\&\cdot\left[\frac{1}{5}\int_0^1\int_0^1 \left\{\frac{x}{2}\sin(t+u(s,t)) + (4x+y)\arctan\left(\frac{|u(s,t)|}{1+|u(s,t)|}\right)\right\}dt\,ds\right]\end{aligned} \tag{4.3}$$

where $(x,y) \in [0,1]\times[0,1]$.

Observe that equation (4.3) is a special case of equation (1.2). Let us take $F, G : J \times \mathbb{R} \to \mathbb{R}$ and $f, g : J \times J \times \mathbb{R} \to \mathbb{R}$, here $J = [0,1]\times[0,1]$ and comparing (4.3) with equation (1.2), we get

$$F(x,y,z) = \frac{1}{8}z, \quad G(x,y,z) = \frac{1}{5}z.$$

It is also seen that these functions are continuous and satisfies the hypothesis (H_2) with $a_1 = \frac{1}{8}, b_1 = \frac{1}{5}$. Further,

$$|F(x,y,0)| = 0, |G(x,y,0)| = 0.$$

Also, the functions f and g satisfies assumption (H_3) with $n(x,y) = \frac{y}{2} + 5xy$ and $n^*(x,y) = \frac{9x}{2} + y$. Moreover,

$$|f(x,y,s,t,u)| = \left|\frac{y}{2}\sin u + 5xy\ln(1+|u|)\right| \leq \frac{1}{2} + 5|u|,$$
$$|g(x,y,s,t,u)| = \left|\frac{x}{2}\sin(t+u) + (4x+y)\arctan\left(\frac{|u|}{1+|u|}\right)\right| \leq \frac{1}{2} + 5|u|.$$

It is observed that $\xi = \frac{1}{2}$, $\eta = 5$, $m = 0$, $a = 1$, $b = 1$ and $ab\eta > 1$. In this case, we have

$$k = \max\left\{\frac{1}{8}, \frac{1}{5}\right\} = \frac{1}{5}.$$

Finally, we see that

$$4\xi^{'}\eta^{'} = \frac{12}{25} < 1.$$

Hence all the hypotheses from (H_1)–(H_6) are satisfied. Applying the result obtained in Theorem 3.2, we deduce that equation (4.3) has at least one solution in Banach algebra E.

Example 4.4. Consider the following functional integral equation:

$$u(x,y) = \left[\frac{1}{7}\int_0^x\int_0^y \left\{x + \frac{e^{xy}}{1+xy}\sin|u(s,t)|\right\} dt\, ds\right] \cdot \left[\frac{1}{4}\int_0^1\int_0^1 \left\{y + \frac{e}{2}\ln(1+|u(s,t)|)\right\} dt\, ds\right] \tag{4.4}$$

where $(x,y) \in [0,1]\times[0,1]$. In this example one can easily verify that all the assumptions of our existence Theorem 3.2 are satisfied, hence equation (4.4) has at least one solution in Banach algebra E.

5. CONCLUSION

The solutions of integral equations have a major role in the field of science and engineering. In this paper, we have discussed about the existence of the solutions of nonlinear functional-integral equations in two independent variables in Banach algebra by using a strategy which is different from other authors approach. Since the nonlinear functional-integral equation involving with two independent variables on Banach algebra, this will be useful for several researchers to show the existence of solutions of these kind of integral equations. By employing some necessary restrictions on nonlinear terms and on the basis of Theorem 3.1 and Theorem 3.2, we obtain the existence results. Also, the applicability of our results is illustrated by some examples.

REFERENCES

[1] S. Abbas and M. Benchohra, Fractional order integral equations of two independent variables, *Appl. Math. Comput.*, 227:755–761, 2014.

[2] R. P. Agarwal, N. Hussain and M. A. Taoudi, Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations, *Abstr. Appl. Anal.*, 2012:15pp, 2012. Article ID 245872.

[3] R. P. Agarwal, D. O'Regan and P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic, Dordrecht, 1999.

[4] M. A. Al-Thagafi and N. Shahzad, Krasnosel'skii-type fixed-point results, *J. Nonlinear Convex Anal.*, 14:483–491, 2013.

[5] J. Banaś and A. Chlebowicz, On existence of integrable solutions of a functional integral equation under Carathéodory conditions, *Nonlinear Anal.*, 70:3172–3179, 2009.

[6] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces, Lecture Notes in Pure and Applied Mathematics*, Vol. 60, Marcel Dekker, New York, 1980.

[7] J. Banaś and M. Lecko, Fixed points of the product of operators in Banach algebra, *Panamer. Math. J.*, 12:101–109, 2002.

[8] J. Banaśand B. Rzepka, An application of a measure of noncompactness in the study of asymptotic stability, *Appl. Math. Lett.*, 16:1–6, 2003.

[9] J. Banaś and B. Rzepka, On existence and asymptotic stability of solutions of a nonlinear integral equation, *J. Math. Anal. Appl.*, 284:165–173, 2003.

[10] J. Banaś and B. Rzepka, On local attractivity and asymptotic stability of solutions of a quadratic Volterra integral equation, *Appl. Math. Comput.*, 213:102–111, 2009.

[11] J. Banaś and K. Sadarangani, Solutions of some functional-integral equations in Banach algebra, *Math. Comput. Modelling*, 38:245–250, 2003.

[12] T. A. Burton and B. Zhang, Fixed point and stability of an integral equation: nonuniqueness, *Appl. Math. Lett.*, 17:839–846, 2004.

[13] J. Caballero, A. B. Mingarelli and K. Sadarangani, Existence of solutions of an integral equation of Chandrasekhar type in the theory of radiative transfer, *Electron. J. Differential Equations*, 2006:1–11, 2006.

[14] C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, New York, 1990.

[15] M. A. Darwish, On Solvability of some quadratic functional-integral equation in Banach algebra, *Commun. Appl. Anal.*, 11:441–450, 2007.

[16] M. A. Darwish and J. Banaś, Existence and characterization of solutions of nonlinear Volterra-Stieltjes integral equations in two variables, *Abstr. Appl. Anal.* 2014:11pp, 2014. Article ID 618434.

[17] Deepmala and H. K. Pathak, A study on some problems on existence of solutions for nonlinear functional-integral equations, *Acta Math. Sci.*, 33:1305–1313, 2013.

[18] Deepmala and H. K. Pathak, Study on existence of solutions for some nonlinear functional-integral equations with applications, *Math. Commun.*, 18:97-107, 2013.

[19] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985.

[20] L. S. Goldenštein and A.S. Markus, On the measure of non-compactness of bounded sets and of linear operators, *Stud. Algebra Math. Anal.*, (Russian), Izdat, "Karta Moldovenjaske", Kishinev, 45–54, 1965.

[21] D. Guo, V. Lakshmikantham and X. Z. Liu, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer, Dordrecht, 1996.

[22] S. Hu, M. Khavani and W. Zhuang, Integral equations arising in the kinetic theory of gases, *Appl. Anal.*, 34:261–266, 1989.

[23] M. Kazemi and R. Ezzati, Existence of solution for some nonlinear two-dimensional Volterra integral equations via measures of noncompactness, *Appl. Math. Comput.*, 275:165–171, 2016.

[24] K. Kuratowski, Sur les espaces completes, *Fundam. Math.*, 15:301–335, 1934.

[25] K. Maleknejad and M. Alizadeh, Sinc approximation for numerical solution of integral equation arising in conductor-like screening model for real solvent, *J. Comput. Appl. Math.*, 251:81–92, 2013.

[26] K. Maleknejad, R. Mollapourasl and K. Nouri, Study on existence of solutions for some nonlinear functional-integral equations, *Nonlinear Anal.*, 69:2582–2588, 2008.

[27] S. Micula, A spline collocation method for Fredholm-Hammerstein integral equations of the second kind in two variables, *Appl. Math. Comput.*, 265:352–357, 2015.

[28] L. N. Mishra, R. P. Agarwal and M. Sen, Solvability and asymptotic behavior for some nonlinear quadratic integral equation involving Erdélyi-Kober fractional integrals on the unbounded interval, *Progr. Fract. Differ. Appl.*, Vol. 2, No. 3, 2016.

[29] L. N. Mishra and M. Sen, On the concept of existence and local attractivity of solutions for some quadratic Volterra integral equation of fractional order, *Appl. Math. Comput.*, 285:174–183, 2016.

[30] L. N. Mishra, M. Sen and R. N. Mohapatra, On existence theorems for some generalized nonlinear functional-integral equations with applications, *Filomat*, accepted on March 21, 2016, in press.

[31] L. N. Mishra, H. M. Srivastava and M. Sen, Existence results for some nonlinear functional-integral equations in Banach algebra with applications, *Int. J. Anal. Appl.*, 11:1–10, 2016.

[32] L. T. P. Ngoc and N. T. Long, On a nonlinear Volterra-Hammerstein integral equation in two variables, *Acta Math. Sci.*, 33: 484–494, 2013.

[33] D. O'Regan, Existence results for nonlinear integral equations, *J. Math. Anal. Appl.*, 192:705–726, 1995.

[34] B. G. Pachpatte, On Fredholm type integral equation in two variables, *Differ. Equ. Appl.*, 1:27–39, 2009.

[35] B. G. Pachpatte, *Multidimensional Integral Equations and Inequalities*, Atlantis Press, Paris, 2011.

[36] W. V. Petryshyn, Structure of the fixed points sets of k-set-contractions, *Arch. Ration. Mech. Anal.*, 40:312–328, 1970/1971.

[37] A. M. Wazwaz, *Linear and Nonlinear Integral Equations, Methods and Applications*, Higher Education Press, Beijing, Springer-Verlag, NewYork, 2011.

[38] H. Zhang and F. Meng, On certain integral inequalities in two independent variables for retarded equations, *Appl. Math. Comput.*, 203:608–616, 2008.