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# ON THE UNIQUENESS OF THE LIMIT CYCLE FOR THE LIÉNARD EQUATION, VIA COMPARISON METHOD FOR THE ENERGY LEVEL CURVES

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**ABSTRACT.** The problem of uniqueness of limit cycles for the Liénard equation is investigated. Some sufficient conditions are presented which complement recent related results. The proofs are based on an energy level comparison method which guarantees that all the possible limit cycles intersect the lines $x = \alpha$ and $x = \beta$, being $\alpha < 0 < \beta$ the two nontrivial zeros of $F(x)$. Some examples illustrate the range of applicability of the main results.

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## 1. INTRODUCTION AND DISCUSSION ABOUT SOME UNIQUENESS RESULTS

In this paper we study the problem of uniqueness of limit cycles for the Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1.1}$$

(where $f, g : \mathbb{R} \to \mathbb{R}$) which is usually considered as the equivalent system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x)y - g(x) \end{cases} \tag{1.2}$$

in the phase-plane. Following a classical approach, our analysis will be performed through the study of the trajectories of the associated equivalent first order system in the *Liénard plane*:

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases} \tag{1.3}$$

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As is well known, when $f$ is continuous, equation (1.1) and system (1.3) are equivalent, by setting

$$F(x) := \int_0^x f(s)\,ds. \tag{1.4}$$

The intriguing problem of the uniqueness of limit cycles for (1.1) or (1.3) has been widely investigated in the literatures. Results in this direction, starting with the pioneering works of Van der Pol and Liénard, may be found in classical textbooks, like the the famous treatises of Lefschetz and Hale (see [9, 12] and the references therein).

A first general uniqueness result in this direction appears actually in the paper of Liénard [14], which is still a milestone. A more general case is considered by Levinson and Smith [13]. For an historical discussion on the "birth" of the relaxation oscillation theory, taking into account also the ideas of Poincaré, we refer to recent papers of Mawhin [15] and Ginoux [6, 7].

In the light of these classical works where the uniqueness is proved within the standard regularity and sign assumptions plus symmetry conditions on $f$ and $g$, we recall the following result which will be our starting point (cf. [9, 12]).

**Theorem 1.1.** *Let $f, g : \mathbb{R} \to \mathbb{R}$ with $f$ continuous and $g$ locally Lipsctitz continuous and satisfying $g(x)x > 0$ for all $x \neq 0$. Suppose, moreover, that $f$ is even (and so $F$ is odd) and $g$ is odd. Then equation (1.1) has at most one limit cycle if there exists $\bar{x} > 0$ such that $F(x)$ is strictly increasing for $x \leq -\bar{x}$ and $x \geq \bar{x}$ and, moreover, $F(-\bar{x}) = F(\bar{x}) = 0$ with $F(x)x < 0$ for $x \in ]-\bar{x}, \bar{x}[$, $x \neq 0$.*

We give a sketch of the proof, starting from well known results and putting in evidence some steps which will be useful in the following.

First of all, we consider the Duffing equation

$$\ddot{x} + g(x) = 0. \tag{1.5}$$

The Duffing equation is equivalent in both phase-plane and Liénard plane to the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -g(x) \end{cases} \tag{1.6}$$

and it is well known that the level curves of the function

$$E(x, y) := \frac{1}{2}y^2 + G(x), \tag{1.7}$$

where $G(x) = \int_0^x g(x)dx$, are its orbits.

Following the elegant and concise description of Lefschetz [12, p. 266], if we consider the level curve

$$\frac{1}{2}y^2 + G(x) = K, \tag{1.8}$$

in the dynamical interpretation as motion of a particle, the first term represents its kinetic energy and (1.7) expresses the law of conservation of energy as applied to the particle. For this reason, we may consider the level curves of the function $E(x, y)$ as energy levels (see [3] for a similar discussion).

Coming back to the Liénard system (1.3), if we evaluate the derivative of the energy function along the trajectories of system (1.3) we obtain

$$\dot{E}(x, y) = y\dot{y} + g(x)\dot{x} = -yg(x) + g(x)(y - F(x)) = -g(x)F(x).$$

This well known result in the dynamical interpretation shows that when $g(x)F(x) > 0$ we are losing energy, while when $g(x)F(x) < 0$ we are gaining energy, and in order to have the existence of the limit cycle it is necessary that $g(x)F(x)$ changes sign. Being $g(x)F(x) < 0$ for all $x \neq 0$ in the interval $]-\bar{x}, \bar{x}[$, we argue that no limit cycle lies entirely in the strip $[-\bar{x}, \bar{x}] \times \mathbb{R}$. Moreover, as the energy curve $E(x, y) = G(\bar{x}) = G(-\bar{x})$ intersects the $x$-axis at the points $(-\bar{x}, 0)$ and $(\bar{x}, 0)$ we can conclude that all possible limit cycles lie in the open region $E(x, y) > G(\bar{x})$ an therefore intersect both the lines $x = -\bar{x}$ and $x = \bar{x}$.

Now we consider a limit cycle $\Gamma$ and observe that

$$\oint_{\Gamma} -g(x(t))F(x(t))\, dt = 0.$$

This because if we consider a point $P = (x, y) \in \Gamma$ and follow the trajectory for its period $T$, we come back the the same point and therefore there is no gain or loss of energy. By contradiction, assume there are two limit cycles $\Gamma_1$ and $\Gamma_2$ with $\Gamma_1$ included in the open region bounded by $\Gamma_2$. Clearly,

$$\oint_{\Gamma_1} -g(x(t))F(x(t))\, dt = 0 = \oint_{\Gamma_2} -g(x(t))F(x(t))\, dt.$$

But, by some computations which can be found, for instance, in [12, p. 270–271] and we omit for sake of simplicity, it is possible to prove that actually,

$$\oint_{\Gamma_2} -g(x(t))F(x(t))\, dt < \oint_{\Gamma_1} -g(x(t))F(x(t))\, dt$$

and this gives the desired contradiction. In the simplest case $f(x)$ even and $g(x) = x$, this approach was first proposed by Liénard himself in [14], where the Liénard plane was introduced for the first time (see also [15, 16]), while, according to Lefschetz [12, p. 267], the more general form was first dealt with by Levinson and Smith [13].

We observe that the symmetry assumption, namely $f$ even and $g$ odd and, consequently, the symmetry of the two nontrivial zeros $\pm\bar{x}$ of $F(x)$, can be replaced with the assumption that all the limit cycles must intersect the lines $x = \alpha$ and $x = \beta$, where now $\alpha < 0 < \beta$ are the two nontrivial zeros of $F(x)$. We were not able to find the first result in which such observation was actually made in an explicit form and we refer to [19] and [1] for works where this approach was used. This will be the starting point of this discussion.

If we restrict our attention directly to system (1.3), even in the case in which $F$ is not defined by (1.4), we can consider some slightly more general assumptions. In particular, dealing with system (1.3) we assume the following standard conditions:

$(A)$ *$F, g : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous functions;*

$(B)$ *$F(0) = g(0) = 0$ and $g(x)x > 0$ for $x \neq 0$.*

$(C)$ *There exist $\alpha, \beta$ with $\alpha < 0 < \beta$ such that $F(x)$ is strictly increasing for $x \leq \alpha$ and $x \geq \beta$ and, moreover, $F(\alpha) = F(\beta) = 0$ with $F(x)x < 0$ for $x \in ]\alpha, \beta[$, $x \neq 0$.*

From $(A)$ and $(B)$, the uniqueness of the solutions of the initial value problems being granted, the origin is the unique equilibrium point and the trajectories move clockwise in the plane around the origin. Condition $(C)$ implies that we are gaining energy in the strip $[\alpha, \beta] \times \mathbb{R}$, while we are loosing energy outside.

Now we can summarize the previous discussion and state the following theorem.

**Theorem 1.2.** *Under the assumptions* $(A), (B), (C)$*, system* (1.3) *has at most one limit cycle provided that*

$(D)$ *all the limit cycles intersect both the lines $x = \alpha$ and $x = \beta$.*

At a first glance, assumption $(D)$ may look weak because it appears abstract. For this reason, we are looking for conditions which guarantee its validity in the setting of $(A), (B), (C)$, which are hypotheses that will be assumed throughout the paper.

A first result in this direction is the following corollary.

**Corollary 1.3.** *Under the assumption* $(A), (B), (C)$*, system* (1.3) *has at most one limit cycle provided that*

$$(D1) \qquad G(\alpha) = G(\beta).$$

This because the level curve $E(x, y) = G(\alpha) = G(\beta)$ works precisely as the energy line $E(x, y) = G(-\bar{x}) = G(\bar{x})$ considered before. Related results in this direction, with some modifications in $G$ and $F$ can be found in [2] and the problem was further generalized to more general systems, like, for instance

$$\begin{cases} \dot{x} = \ell(x)[\phi(y) - F(x)] \\ \dot{y} = -m(y)g(x) \end{cases} \tag{1.9}$$

(see, for instance, [1, 18, 24]).

When assumption $(D1)$ is not fulfilled, an interesting result is due to Hayashi [10] who assumes $G(\alpha) < G(\beta)$ and the existence of a point $\bar{x} \in ]\alpha, 0[$ such that

$$\frac{1}{2}F(\bar{x})^2 \geq G(\beta) - G(\bar{x}).$$

Such result will be discussed with some more details in the following section.

In this paper we start from the result of Hayashi and give alternative conditions, which, in some sense complete and complements those in [10]. Moreover, in the final part, in the light of the classical counterexample of Duff and Levinson which provides a result of multiplicity of limit cycles, we show further examples of uniqueness/notuniqueness for the modified equation

$$\begin{cases} \dot{x} = y - \lambda F(x) \\ \dot{y} = -g(x) \end{cases} \tag{1.10}$$

which stress the fact that the multiplicity results may occur only for $\lambda$ small. The case of one-sided condition on $F(x)$ will be considered as well.

Our approach to gain uniqueness via condition $(D)$, is based on a comparison method with the energy level lines of the associated Duffing equation. Such an approach which has been already mentioned above, will be called *method of energy*, following [3].

## 2. MAIN RESULTS

In view of the results presented in the previous section and focusing on Theorem 1.2 we start from the following situation.

$$G(\alpha) < G(\beta) \tag{$D2$}$$

(the opposite case can be treated in the same way).

Following the paper of Hayashi [10, Theorem 1] we observe that if there is a point $\bar{x}$ with $\alpha < \bar{x} < 0$ such that

$$\frac{1}{2}F(\bar{x})^2 \geq G(\beta) - G(\bar{x}), \tag{2.1}$$

then there is at most one limit cycle. This because for any point $P = (\alpha, y)$ with $y \geq 0$, the positive semi-trajectory starting from $P$ is forced, by the graph of the function $F$, to intersect the level line of energy $G(\beta)$ at some point $x < \bar{x}$ in the strip $\alpha \leq x \leq 0$. In virtue of the sign assumption on $g(x)F(x)$ discussed above, such a trajectory is from now on bounded away from this level energy until it crosses the line $x = \beta$. See Figure 1.

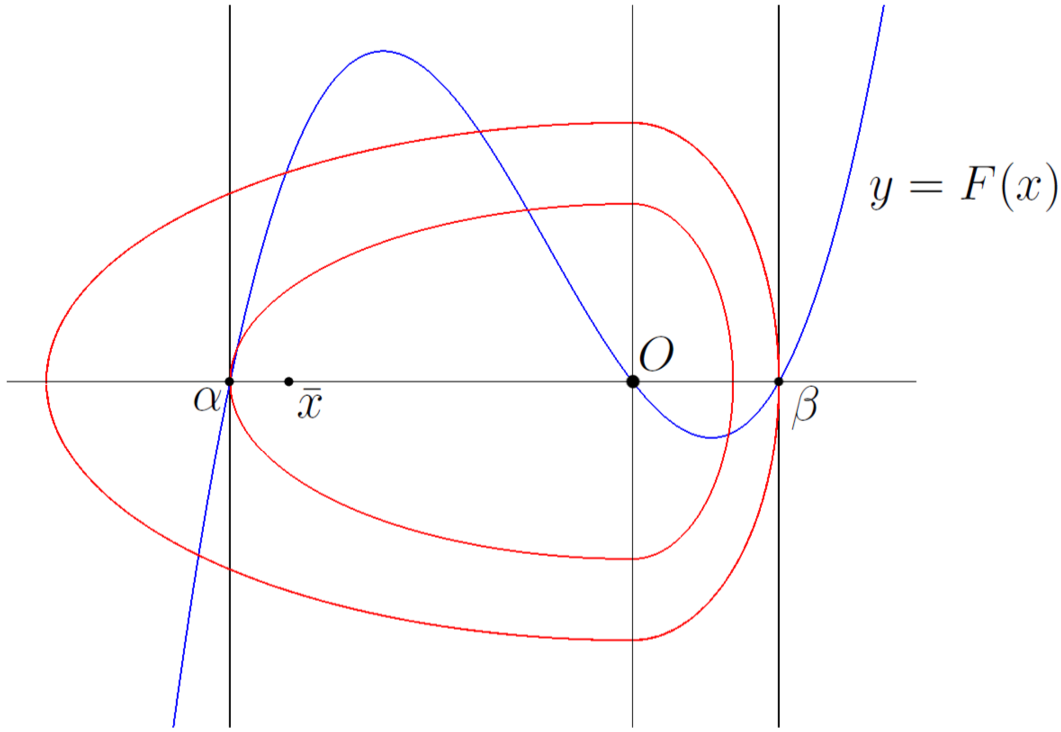


FIGURE 1. A typical example in which the Hayashi result can be applied.

For this reason, we suppose that

$$\frac{1}{2}F(x)^2 < G(\beta) - G(x), \quad \forall\, x:\, \alpha \le x \le 0.$$

Arguing as above, we consider the positive semi-trajectory $\gamma^+$ starting at a point $(\alpha, y_0)$ with $y_0 \ge 0$ such that

$$\frac{1}{2}y_0^2 \le G(\beta) - G(\alpha).$$

Such a trajectory will be bounded below by the level line of energy $G(\alpha)$ in the strip $[\alpha, \alpha_1] \times \mathbb{R}^+$, where $\alpha_1 \in ]0, \beta[$ is such that $G(\alpha) = G(\alpha_1)$. On the other hand, in general, we cannot guarantee that $\gamma^+$ intersects the line $x = \beta$ without other assumptions. For this reason, we adopt a different approach to achieve such intersection and we present the following result.

**Theorem 2.1.** *Assume conditions* $(A), (B), (C), (D2)$ *and suppose there exist* $\varepsilon, c$ *with* $0 < \varepsilon < c \le \alpha_1$ *such that*

$$\int_\varepsilon^c \frac{g(x)}{-F(x)}\, dx \le \sqrt{2(G(\alpha) - G(\varepsilon))} - \sqrt{2(G(\beta) - G(c))}$$

*holds. Then system* $(1.3)$ *has at most a limit cycle.*

*Proof.* Let $0 < \varepsilon < c \le \alpha_1$. Since in the strip $[\alpha, \alpha_1] \times \mathbb{R}^+$, our trajectory is a graph of a function $y = y(x)$ we can evaluate the slope of $y$ with respect to $x$ as

$$y'(x) = \frac{dy(x)}{dx} = \frac{-g(x)}{y - F(x)},$$

so that

$$y(c) = y(\varepsilon) - \int_\varepsilon^c \frac{g(x)}{y(x) - F(x)}\, dx \ge y(\varepsilon) - \int_\varepsilon^c \frac{g(x)}{-F(x)}\, dx$$

(in fact, note that $y(x) \ge 0$ and $F(x) < 0$ in $[\varepsilon, c]$). If

$$\frac{1}{2} y(c)^2 \ge G(\beta) - G(c),$$

we have that $\gamma^+$ intersects the vertical line $x = c$ at a value above the level line of energy $G(\beta)$ and then it will intersect the line $x = \beta$. Thus we need to prove that

$$y(\varepsilon) \ge \sqrt{2(G(\beta) - G(c))} + \int_\varepsilon^c \frac{g(x)}{-F(x)}\, dx.$$

Using the fact that, in the $x$-interval considered, the trajectory is above the level line of energy $G(\alpha)$, we know that

$$\frac{1}{2} y(\varepsilon)^2 + G(\varepsilon) \ge G(\alpha), \tag{2.2}$$

that is

$$y(\varepsilon) \ge \sqrt{2(G(\alpha) - G(\varepsilon))}.$$

In conclusion, if there exist $\varepsilon, c$ with $0 < \varepsilon < c \le \alpha_1$ such that

$$\int_\varepsilon^c \frac{g(x)}{-F(x)}\, dx \le \sqrt{2(G(\alpha) - G(\varepsilon))} - \sqrt{2(G(\beta) - G(c))}, \tag{2.3}$$

the intersection property is granted. □

We observe that, in order to have (2.3) satisfied, it is necessary to assume that $|F(x)|$ is sufficiently large in a compact subinterval of $]0, \alpha_1]$. Such situation appears in Figure 2. Roughly speaking, we can compare the two figures and observe that while $F(x)$ "does the job" for $\alpha < x < 0$ in Figure 1 (in the frame of Hayashi result), now we have a "dual case" In Figure 2. For simplicity we omit the computations, but it is possible to see that for the example in Figure 2 the positive semi-trajectory starting at the point $(\alpha, 0)$ actually intersects the vertical line $x = \beta$. Clearly, the same occurs for any positive semi-trajectory starting at a point $(\alpha, y_0)$ with $y_0 > 0$.

The above result can be refined in the following way. Suppose that there is a point $\bar{x}$ with $\alpha < \bar{x} < 0$ such that

$$\frac{1}{2} F(\bar{x})^2 \ge G(\alpha) - G(\bar{x}). \tag{2.4}$$

In this case, arguing in the same way as in the proof of Theorem 2.1 we have the following result.

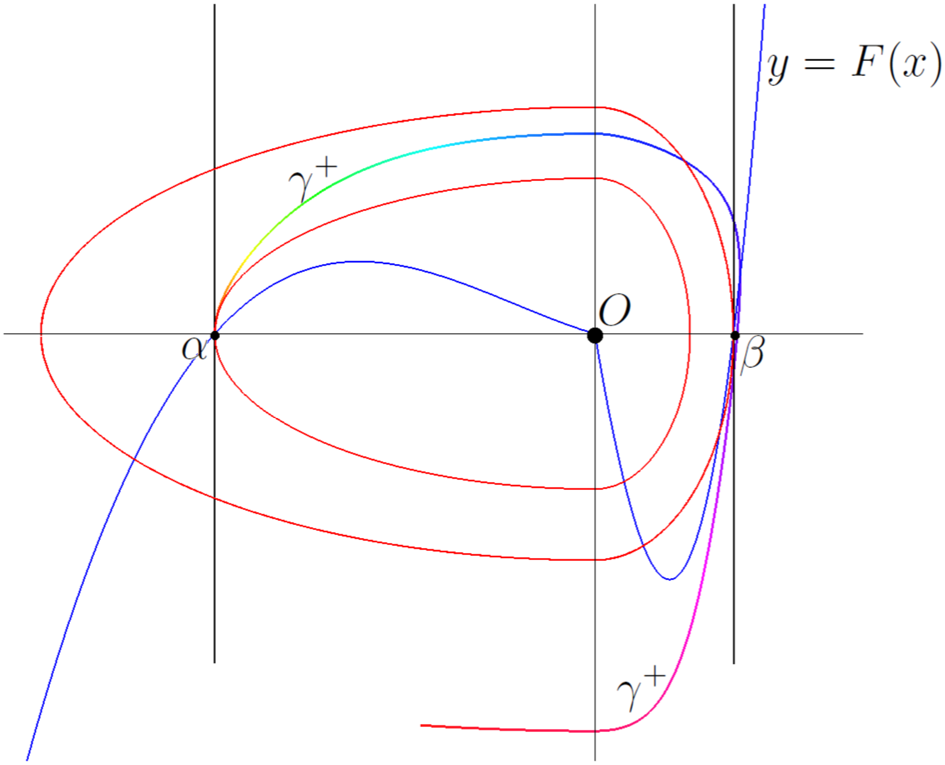


FIGURE 2. A typical example in which Hayashi theorems does not applies but the result is still true in virtue of Theorem 2.1. Indeed the semi-trajectory $\gamma^+$ intersects the vertical line $x = \beta$. For this example we have taken $g(x) = x$ and $F(x) = x^3 + 0.9x^2 - 0.73x$ for $x \leq 0$, while $g(x) = 16x$ and $F(x) = \lambda(x^3 + 0.9x^2 - 0.73x)$ for $x \geq 0$. The computations, which have been performed using Maple software, show that the intersection property holds for $\lambda = 20$.

**Theorem 2.2.** *Assume* $(A), (B), (C), (D2)$ *and suppose that there is a point* $\bar{x}$ *with* $\alpha < \bar{x} < 0$ *such that* (2.4) *holds. If there exist* $\varepsilon, c$ *with* $0 < \varepsilon < c \leq \alpha_1$ *such that*

$$\int_\varepsilon^c \frac{g(x)}{-F(x)}\, dx \leq \sqrt{F(\bar{x})^2 + 2(G(\bar{x}) - G(\varepsilon))} - \sqrt{2(G(\beta) - G(c))},$$

*then system* (1.3) *has at most a limit cycle.*

*Proof.* We proceed with the same argument as above and observe that in virtue of (2.4) we can replace condition (2.2) with

$$\frac{1}{2}y(\varepsilon)^2 + G(\varepsilon) \geq \frac{1}{2}F(\bar{x})^2 + G(\bar{x}).$$

From this inequality we get the conclusion. □

**Remark 2.3.** Notice that condition (2.4) allows to treat some cases not contained in (2.1). Indeed, we can consider situations in which

$$G(\alpha) < \frac{1}{2}F(\bar{x})^2 + G(\bar{x}) < G(\beta).$$

On the other hand, if (2.1) holds, then we can get the expected already known uniqueness result by showing that the condition of Theorem 2.2 is satisfied by choosing $\varepsilon$ and $c$ small enough. Geometrically, this means that if we are above the level energy $G(\beta)$ for some $\bar{x}$ in the interval $[\alpha, 0]$, we remain above the same level also for $x$ positive near zero.

It remains to treat the case in which $\gamma^+$ does not intersect the line $x = \beta$. In this case $\gamma^+$ intersects the $x$-axis at a point $(x_1, 0)$ with $\alpha_1 \leq x_1 < \beta$. Note that $\dot{x} > 0$ as long as $y > F(x)$. Therefore, $\gamma^+$ intersects the vertical isocline, that is $y = F(x)$, at a point $(x_2, F(x_2))$ with $\alpha_1 \leq x_1 < x_2 < \beta$. Now we take a point $x_m$ of minimum for the function $F(x)$ in the interval $[0, \alpha_1]$ and consider the minimum value $y_m := F(x_m) = \min_{[0,\alpha_1]} F(x)$. A straightforward calculation shows that, once that $y < F(x)$, the trajectory cannot intersect anymore the graph $y = F(x)$ in the strip $[0, \alpha_1] \times \mathbb{R}^-$.

Consider the energy level at the point $(x_m, y_m)$, namely

$$E_m := \frac{1}{2} y_m^2 + G(x_m).$$

Moreover, suppose that

$$E_m > G(\alpha).$$

This means that the graph $y = F(x)$ intersects the level line of energy $G(\alpha)$ in the strip $[0, \alpha_1] \times \mathbb{R}^-$. Such an assumption is indeed in the line of (2.1) or (2.4), where $F(x)$ was "doing a similar job" on the strip $[\alpha, 0] \times \mathbb{R}^+$.

Arguing as before, we can say that $\gamma^+$ will intersect the line $x = \alpha$ at a point $(\alpha, y^*)$ below the level line of energy $E_m$ so that

$$y^* \leq y_\alpha := -\sqrt{2(E_m - G(\alpha))}.$$

Our goal now is to find conditions ensuring that $\gamma^+$, after having intersected the line $x = \alpha$ at the point $(\alpha, y^*)$, will intersect again the line $x = \alpha$ at a point $(\alpha, y^{**})$ above the level line of energy $G(\beta)$. Indeed, if this happens, the semi-trajectory $\gamma^+$, after the point $(\alpha, y^{**})$ will remain above the level line of energy $G(\beta)$ in the strip $[\alpha, \beta]$ and eventually will intersect the line $y = \beta$. To this purpose, we consider the modified energy function

$$W(x, y) := \frac{1}{2}(y + k)^2 + G(x),$$

where $k > 0$ is a constant to be determined. Evaluation of the derivative along the trajectories yields to

$$\dot{W}(x, y) = -g(x)(y + k) + g(x)(y - F(x)) = -g(x)(k + F(x)).$$

Therefore, we can conclude that $\dot{W}(x, y) \geq 0$ on the half plane $x \leq \alpha$ provided that

$$F(x) \geq -k, \quad \forall\, x \leq \alpha. \tag{2.5}$$

Let $(\alpha, y_\alpha^+)$ be the second point where the level line $W(x,y) = W(\alpha, y_\alpha)$ intersects the vertical line $x = \alpha$. By the above discussion, we need to impose that

$$y_\alpha^+ \geq \sqrt{2(G(\beta) - G(\alpha))}, \tag{2.6}$$

which implies the fact that the the energy level $E(\alpha, y^{**})$ is greater or equal to $G(\beta)$. By definition of $W(x,y)$, which has the level lines symmetric with respect to the horizontal line $y = -k < 0$, we know that $y_\alpha^+ = -y_\alpha + 2k$, therefore, (2.6) is satisfied when

$$\sqrt{2(E_m - G(\alpha))} \geq 2k + \sqrt{2(G(\beta) - G(\alpha))}.$$

We observe that in this argument we implicitly assumed that the part of the level line $W(x,y) = W(\alpha, y_\alpha)$ for $x \leq \alpha$ intersects the line $y = -k$ and thus it is an arc connecting the two points $(\alpha, y_\alpha)$ and $(\alpha, y_\alpha^+)$. It is well known that this fact cannot be in general guaranteed, unless we have the additional assumption on the divergence of $G(x)$, namely, $G(-\infty) = +\infty$. However, this assumption is not needed as long as we are interested in proving the *uniqueness* (and not the *existence*) of the limit cycle. Indeed, if the level line of $W$ passing through $(\alpha, y_\alpha)$ does not intersect the line $y = -k$, then, using also the boundedness of $F(x)$ for $x \leq \alpha$, we get that $\gamma^+$ does not intersect the $x$-axis and therefore system (1.3) has no limit cycle. Moreover, we observe it is not necessary to assume that condition (2.5) for every $x \leq \alpha$, but it suffices to restrict ourselves to the interval $]\mu, \alpha]$ where

$$G(\mu) = W(\alpha, \sqrt{2(G(\beta) - G(\alpha))}). \tag{2.7}$$

In other words, $(\mu, -k)$ is the intersection point of the level curve of $W$ passing through $(\alpha, \sqrt{2(G(\beta) - G(\alpha))})$ with the horizontal line $y = -k$ and this intersection gives the desired amplitude of the interval.

Summarizing the above result as a formal statement of a theorem, we can give the following.

**Theorem 2.4.** *Assume conditions* $(A), (B), (C), (D2)$ *and suppose that* $\gamma^+$ *does not intersect the vertical line* $x = \beta$ *before having intersected the* $x$*-axis. Then the trajectory* $\gamma^+$ *either does not intersect the* $x$*-axis in* $x < 0$ *or it eventually intersects the line* $x = \beta$*, provided that*

$$F(x) \geq -k, \quad \forall\, x \in\, ]\mu, \alpha]$$

*with* $\mu$ *defined by* (2.7) *and*

$$2k \leq \sqrt{F(x_m)^2 + 2(G(x_m) - G(\alpha))} - \sqrt{2(G(\beta) - G(\alpha))}.$$

*Proof.* The proof follows from the previous discussion and we just remark that if $\gamma^+$ does not intersects the $x$-axis in $x < 0$ it is necessary that $G(-\infty) < +\infty$, while, if $G(-\infty) = +\infty$, then $\gamma^+$ intersects $x = \beta$. $\square$

Notice that the above theorem shows that all the possible limit cycles intersect the strip $\alpha < x < \beta$ and the following corollary holds.

**Corollary 2.5.** *Assume conditions* $(A), (B), (C), (D2)$ *and suppose*

$$F(x) \geq -k, \quad \forall\, x \in ]\mu, \alpha]$$

*with* $\mu$ *defined by* (2.7) *and*

$$2k \leq \sqrt{F(x_m)^2 + 2(G(x_m) - G(\alpha))} - \sqrt{2(G(\beta) - G(\alpha))}.$$

*Then system* (1.3) *has at most a limit cycle.*

**Remark 2.6.** The assumptions of Theorem 2.4 and hence of Corollary 2.5 are restrictive if compared with those of Theorem 2.1, because we require the boundedness of $F(x)$ for $x \leq \alpha$. on the other hand, no assumption on $F(x)$ is required in the strip $\alpha < x < 0$, as in Hayashi result. Hence one can say that these three results complement each other. If, like in Theorem 2.2, we assume that (2.4) holds, then the result of Theorem 2.4 can be refined in the same way. In fact, if we assume that $\gamma^+$ does not intersect the line $x = \beta$, we can estimate a point $x^+$ such that $\gamma^+$ intersect the positive $x$-axis at a point $(x_1, 0)$ with

$$\alpha_1 < x^+ \leq x_1 < \beta.$$

In this manner, the interval where to look for the minimum of $F(x)$ is now $[0, x^+]$ instead of $[0, \alpha_1]$ and the minimum can be improved.
Finally, we also observe that we took a minimum point of $F(x)$ for sake of simplicity. What really matter is only the existence of a point $\tilde{x} \in [0, \alpha_1]$ (or, respectively $\tilde{x} \in [0, x^+]$) which is not necessarily the minimum such that

$$\sqrt{F(\tilde{x})^2 + 2(G(\tilde{x}) - G(\alpha))} - \sqrt{2(G(\beta) - G(\alpha))} \geq 2k.$$

## 3. EXAMPLES AND REMARKS

Arguing as in [21], we can modify equation (1.1) as

$$\ddot{x} + \lambda f(x)\dot{x} + g(x) = 0, \tag{3.1}$$

being $\lambda$ a positive real parameter. The corresponding Liénard system (1.3) takes the form

$$\begin{cases} \dot{x} = y - \lambda F(x) \\ \dot{y} = -g(x). \end{cases} \tag{3.2}$$

Let $x_0$ be a point of maximum of $F(x)$ in the interval $[\alpha, 0]$ which can be easily found, being a zero of $f(x)$. Now we increase $\lambda$ until $\lambda F(x_0)$ intersects the energy level $G(\beta)$ at some $\hat{\lambda}$ which can be determined with straightforward calculations. In the light of the theorem of Hayashi [10], we get the following result.

**Proposition 3.1.** *There exists $\hat{\lambda} > 0$ such that for every $\lambda > \hat{\lambda}$, equation* (3.1) *has at most one limit cycle.*

We observe that actually $x_0$ may be not the optimal choice. Indeed, $\lambda F(x)$ may intersect the energy level $G(\beta)$ at a different point in the interval $[\alpha, 0]$, giving in this way a smaller value for $\hat{\lambda}$, but $x_0$ is the easiest choice for a concrete evaluation of $\hat{\lambda}$. We also observe that we can choose to modify the function $F$ only for $x < 0$.

The last result is perhaps useful to better understand some counterexamples of multiple limit cycles. For instance, we can consider a situation in which for $\lambda$ small there are three limit cycles, the first and the third stable and the second one unstable. Consider this as a starting equation with $\lambda = 1$. Increasing the value of $\lambda$ at a certain moment we reach a threshold value for which there will be only one stable limit cycle (the existence is granted by the conditions on the examples). This means that for a certain value of $\bar{\lambda} \leq \hat{\lambda}$, one stable limit cycle collapses with the unstable one, giving a neutral limit cycle which then disappears for $\lambda > \bar{\lambda}$. Reversing the movement of $\lambda$ we will have a bifurcation phenomenon, different from the well known Hopf bifurcation. For the "movement" and bifurcation of limit cycles, we recall the classical paper of Duff [4], the results of Perko [17] and also [22].

Let us discuss in details this situation. We start from the well known example of Duff and Levinson [5], namely, the system

$$\begin{cases} \dot{x} = y - \lambda\left(\frac{64}{35\pi}x^7 - \frac{112}{5\pi}x^5 + \frac{196}{3\pi}x^3 - \frac{C}{2}x^2 - \frac{36}{\pi}x\right) \\ \dot{y} = -x. \end{cases} \tag{3.3}$$

In [5], using the Poincaré small parameter method, the Authors proved that such a system has at least three limit cycles provided that $C > 0$ is large enough $\lambda > 0$ is sufficiently small. On the other hand, in [10], Hayashi (using his uniqueness result) proved that for $C = 54$, the system (3.3) has a unique limit cycle for every $\lambda \geq \lambda_1 \simeq 2.58483$. In the same light, following [2], one can consider the system

$$\begin{cases} \dot{x} = y - \lambda\frac{x}{\pi}\left(-\frac{4}{81} + \frac{196}{81}\frac{x^2}{3} - \frac{112}{9}\frac{x^4}{5} + \frac{64}{5}\frac{x^6}{7} + \frac{1}{200}x + \frac{1}{2}x^3\right) \\ \dot{y} = -x \end{cases} \tag{3.4}$$

and observe that $F(x)$ has three real zeros $\alpha < 0 < \beta$ and it is monotone increasing outside $]\alpha, \beta[$. As in the Duff-Levinson example, system (3.4) has at least three limit cycles for $\lambda > 0$ sufficiently small but, using again Maple software, it is possible to see that it has exactly one limit cycle for $\lambda \geq \lambda_1 \simeq 141.515778$.

We just observe that these examples can be (partially) improved, by enlarging $\lambda$ only for $x$ positive.

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