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NONTRIVIAL SOLUTIONS OF A DIRICHLET BOUNDARY VALUE PROBLEM WITH IMPULSIVE EFFECTS

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ABSTRACT. By using variational methods and critical point theory, the authors establish criteria for the existence of nontrivial classical solutions to a class of Dirichlet boundary value problems with impulsive effects. The results are illustrated with two examples.

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1. INTRODUCTION

We consider the nonlinear Dirichlet boundary value problem consisting of

$$\begin{cases} -u''(x) = \lambda f(x, u(x)) + g(u(x)), & \text{a.e. } x \in [0, T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

together with the impulsive conditions

$$\Delta u'(x_j) = I_j(u(x_j)), \quad j = 1, \dots, p, \tag{1.2}$$

where λ is a positive parameter, $T > 0$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $x_0 = 0 < x_1 < \cdots < x_p < x_{p+1} = T$, and $\Delta u'(x_j)$ is defined by

$$\Delta u'(x_j) = u'(x_j^+) - u'(x_j^-) = \lim_{x \to x_j^+} u'(x) - \lim_{x \to x_j^-} u'(x).$$

Throughout this paper, we assume the following conditions hold without further mention:

(H1) $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with a Lipschitz constant $L \in (0, 4/T^2)$, i.e., $|g(t_1) - g(t_2)| \le L|t_1 - t_2|$ for all t_1, $t_2 \in \mathbb{R}$, and $g(0) = 0$;

(H2) The impulsive functions $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \dots, p$, are continuous and satisfy the condition $\sum_{j=1}^{p} (I_j(t_1) - I_j(t_2))(t_1 - t_2) \ge 0$ for all $t_1, t_2 \in \mathbb{R}$;

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(H3) I_j, $j = 1, \dots, p$, have sublinear growth, i.e., there exist constants $a_j > 0$, $b_j \geq 0$, and $\gamma_j \in [0, 1)$ such that $|I_j(t)| \leq a_j + b_j|t|^{\gamma_j}$ for every $t \in \mathbb{R}$ and $j = 1, \dots, p$.

The impulsive problem (1.1), (1.2) will be referred to as (IP) in the remainder of this paper.

By employing a consequence of a local minimum theorem due to Bonanno (see Lemma 2.1 below), we establish the existence of nontrivial classical solutions to the problem (IP). Under suitable algebraic assumptions on the nonlinear term f, we first prove the existence of at least one nontrivial classical solution, and then we obtain the existence of a second classical solution by further assuming that $f(t, 0) \neq 0$ for all $t \in [0, T]$ and asking that the following condition of Ambrosetti-Rabinowitz (AR) type holds:

(A1) there exist $s > \frac{2(4+LT^2)}{4-LT^2}$ and $R > 0$ such that

$$0 < sF(x, t) \leq tf(x, t) \quad \text{for all } |t| \geq R \text{ and } x \in [0, T].$$

The role of this condition is to ensure the boundedness of the Palais-Smale sequences for the Euler-Lagrange functional associated with the problem (IP). This is important in applications of critical point theory. The purpose of the condition $f(t, 0) \neq 0$ for all $t \in [0, T]$ is to avoid the existence of the trivial solution to the problem (IP).

Impulsive differential equations arise from real world processes that describe the dynamics in which sudden and discontinuous jumps occur. Because of the importance of such problems, they have been studied by many authors; for example, see [2, 14, 18, 23] and the references therein. In particular, the existence and multiplicity of solutions for impulsive differential equations have been examined in many works, and for an overview on this subject, we refer the reader to the monographs [6, 12] and the papers [1, 3, 4, 5, 8, 10, 16, 15, 17, 22, 24, 26, 27, 28]. We especially refer the reader to the paper [9] in which, using Lemma 2.1 below, existence results for two and three nontrivial solutions are established for a class of Sturm-Liouville problems with mixed conditions and the p-Laplacian.

2. **PRELIMINARIES**

Our main tool, Lemma 2.1 below, is a consequence of an existence result for a local minimum [7, Theorem 5.1] that in turn was inspired by Ricceri's variational principle [20]. In it, an inequality on the functionals Φ and Ψ is needed.

For a given nonempty set X and two functionals Φ, $\Psi : X \to \mathbb{R}$, we define

$$\vartheta(r_1, r_2) = \inf_{v \in \Phi^{-1}(r_1, r_2)} \frac{\sup_{u \in \Phi^{-1}(r_1, r_2)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}$$

and

$$\rho(r_1, r_2) = \sup_{v \in \Phi^{-1}(r_1, r_2)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(-\infty, r_1)} \Psi(u)}{\Phi(v) - r_1}$$

for all r_1, $r_2 \in \mathbb{R}$ with $r_1 < r_2$. In what follows, we let X^* denote the dual space of X.

Lemma 2.1 ([7, Theorem 5.1]). *Let X be a real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive, and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Let $I_\lambda = \Phi - \lambda\Psi$ and assume that there exist r_1, $r_2 \in \mathbb{R}$ with $r_1 < r_2$ such that*

$$\vartheta(r_1, r_2) < \rho(r_1, r_2).$$

Then, for each $\lambda \in \left(\frac{1}{\rho(r_1,r_2)}, \frac{1}{\vartheta(r_1,r_2)}\right)$, there exists $u_{0,\lambda} \in \Phi^{-1}(r_1, r_2)$ such that $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(r_1, r_2)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Let $X = H_0^1(0,T)$ and $H^2(0,T) = \{u \in C^1[0,T] : u'' \in L^2[0,T]\}$. The inner product

$$\prec u, v \succ = \int_0^T u'(x)v'(x)dx,$$

in X induces the norm

$$\|u\| = \left(\int_0^T |u'(x)|^2 dx\right)^{\frac{1}{2}}.$$

Definition 2.2. By a classical solution of the problem (IP), we mean a function $u \in \{u(x) \in H^1(0,T) : \ u(x) \in H^2(x_j, x_{j+1}), \ j = 0, 1, \ldots, p\}$ such that u satisfies (1.1) and (1.2). We say that a function $u \in X$ is a weak solution of the problem (IP) if

$$\int_0^T u'(x)v'(x)dx + \sum_{j=1}^{p} I_j(u(x_j))v(x_j) - \int_0^T g(u(x))v(x)dx - \lambda \int_0^T f(x, u(x))v(x)dx = 0$$

for every $v \in X$.

Remark 2.3. Using standard methods, it is easy to verify that a weak solution of (IP) is also a classical solution (e.g., see [3, Lemma 5]).

Definition 2.4. Let X be a real reflexive Banach space. If any sequence $\{u_n\} \subset X$, such that $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence, then we say that J satisfies the Palais-Smale (PS) condition. Here, the sequence $\{u_n\}$ is called a (PS) sequence.

Next, we let

$$F(x,t) = \int_0^t f(x,\xi)d\xi \quad \text{for all } (x,t) \in [0,T] \times \mathbb{R},$$

$$G(t) = -\int_0^t g(\xi)d\xi \quad \text{for all } t \in \mathbb{R},$$

and define the functional $J : X \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\int_0^T (u'(x))^2dx + \sum_{j=1}^{p}\int_0^{u(x_j)} I_j(t)dt + \int_0^T G(u(x))dx - \lambda\int_0^T F(x, u(x))dx.$$

From the continuity of f, g, and I_j, we see that $J(u)$ is strongly continuous in X, $J \in C^1(X)$, and the Gâteaux derivative of J is

$$\begin{aligned}\langle J'(u), v\rangle &= \int_0^T u'(x)v'(x)dx + \sum_{j=1}^{p} I_j(u(x_j))v(x_j) \\ &\quad - \int_0^T g(u(x))v(x)dx - \lambda\int_0^T f(x, u(x))v(x)dx\end{aligned}$$

for every u, $v \in X$.

We need the following lemmas to prove our main result.

Lemma 2.5. *Assume that (A1) holds. Then, J satisfies the (PS) condition.*

Proof. Let $\{u_n\}$ be a sequence in X such that $\lim_{n\to\infty} J'(u_n) = 0$ and $J(u_n)$ is bounded. From (H3), we see that

$$|I_j(u_n(x_j))u_n(x_j)| \le a_j|u_n(x_j)| + b_j|u_n(x_j)|^{\gamma_j+1}. \tag{2.1}$$

Then, in view of the fact that

$$\|u\|_\infty = \max_{t\in[0,T]} |u(t)| \le \frac{\sqrt{T}}{2}\|u\| \quad \text{for } u \in X, \tag{2.2}$$

we have

$$\begin{aligned} I_j(u_n(x_j))u_n(x_j) &\le a_j|u_n(x_j)| + b_j|u_n(x_j)|^{\gamma_j+1} \\ &\le a_j\frac{\sqrt{T}}{2}\|u_n\| + b_j\left(\frac{\sqrt{T}}{2}\right)^{\gamma_j+1}\|u_n\|^{\gamma_j+1}. \end{aligned} \tag{2.3}$$

Again, by (H3), we have

$$\left|\int_0^{u(x_j)} I_j(x)dx\right| \le a_j|u(x_j)| + \frac{b_j}{\gamma_j+1}|u(x_j)|^{\gamma_j+1}.$$

Thus, from (2.2), it follows that

$$
\begin{aligned}
\left|\sum_{j=1}^{p}\int_0^{u(x_j)} I_j(x)dx\right| &\le \sum_{j=1}^{p}\left|\int_0^{u(x_j)} I_j(x)dx\right| \\
&\le \sum_{j=1}^{p}\left(a_j|u(x_j)|+\frac{b_j}{\gamma_j+1}|u(x_j)|^{\gamma_j+1}\right) \\
&\le \sum_{j=1}^{p}\left(a_j\frac{\sqrt{T}}{2}||u||+\frac{b_j}{\gamma_j+1}\left(\frac{\sqrt{T}}{2}\right)^{\gamma_j+1}||u||^{\gamma_j+1}\right). \qquad (2.4)
\end{aligned}
$$

From (A1), (2.3), and (2.4), we see that there exists $C>0$ such that

$$
\begin{aligned}
sJ(u_n)-\langle J'(u_n),u_n\rangle &= \left(\frac{s}{2}-1\right)\|u_n\|^2+\lambda\int_0^T[f(x,u_n(x))u_n(x)-sF(x,u_n(x))]dx \\
&\quad+\int_0^T[sG(u_n(x))+g(u_n(x))u_n(x)]dx \\
&\quad+s\sum_{j=1}^{p}\int_0^{u(x_j)} I_j(x)dx-\sum_{j=1}^{p}I_j(u_n(x_j))u_n(x_j) \\
&\ge \left(\frac{s}{2}-1\right)\|u_n\|^2-\frac{1}{4}LT^2\left(\frac{s}{2}+1\right)\|u_n\|^2 \\
&\quad-s\sum_{j=1}^{p}\left(a_j\frac{\sqrt{T}}{2}\|u_n\|+\frac{b_j}{\gamma_j+1}\left(\frac{\sqrt{T}}{2}\right)^{\gamma_j+1}\|u_n\|^{\gamma_j+1}\right) \\
&\quad-\sum_{j=1}^{p}\left(a_j\frac{\sqrt{T}}{2}\|u_n\|+b_j\left(\frac{\sqrt{T}}{2}\right)^{\gamma_j+1}\|u_n\|^{\gamma_j+1}\right)-C \\
&= \left[\frac{s}{2}\left(1-\frac{LT^2}{4}\right)-\left(1+\frac{LT^2}{4}\right)\right]\|u_n\|^2 \\
&\quad-s\sum_{j=1}^{p}\left(a_j\frac{\sqrt{T}}{2}\|u_n\|+\frac{b_j}{\gamma_j+1}\left(\frac{\sqrt{T}}{2}\right)^{\gamma_j+1}\|u_n\|^{\gamma_j+1}\right) \\
&\quad-\sum_{j=1}^{p}\left(a_j\frac{\sqrt{T}}{2}\|u_n\|+b_j\left(\frac{\sqrt{T}}{2}\right)^{\gamma_j+1}\|u_n\|^{\gamma_j+1}\right)-C.
\end{aligned}
$$

This implies that $\{u_n\}$ is bounded in X.

Next, we show that $\{u_n\}$ converges strongly to some u in X. Since $\{u_n\}$ is bounded in X, there exists a subsequence of $\{u_n\}$ (denoted again by $\{u_n\}$) such that $\{u_n\}$ converges weakly to some u in X. Then, $\{u_n\}$ converges uniformly to u on $[0,T]$. Thus,

$$
u_n(0)\to u(0),\quad u_n(T)\to u(T),
$$

$$
\sum_{j=1}^{p}(I_j(u_n(x_j))-I_j(u(x_j)))(u_n(x_j)-u(x_j))\to 0,
$$

$$\int_0^T (g(u_n(x)) - g(u(x)))(u_n(x) - u(x))dx \to 0,$$

and

$$\int_0^T (f(x, u_n(x)) - f(x, u(x)))(u_n(x) - u(x))dx \to 0,$$

as $n \to \infty$. Since $\lim_{n\to+\infty} J'(u_n) = 0$ and $\{u_n\}$ converges weakly to $u \in X$, we have

$$\langle J'(u_n) - J'(u), u_n - u\rangle \to 0 \quad \text{as } n \to \infty.$$

Note that

$$\begin{aligned}\langle J'(u_n) - J'(u), u_n - u\rangle &= \int_0^T |u_n'(x) - u'(x)|^2 dx \\ &\quad + \sum_{j=1}^{p} (I_j(u_n(x_j)) - I_j(u(x_j)))(u_n(x_j) - u(x_j)) \\ &\quad - \int_0^T (g(u_n(x)) - g(u(x)))(u_n(x) - u(x))dx \\ &\quad - \lambda \int_0^T (f(x, u_n(x)) - f(x, u(x)))(u_n(x) - u(x))dx.\end{aligned}$$

Then, we have $||u_n - u|| \to 0$ as $n \to \infty$. Thus, $\{u_n\}$ converges weakly to u in X, and so the functional J satisfies the (PS) condition. □

The following lemma can be found in [11, Lemma 2.2].

Lemma 2.6. *Let $T : X \to X^*$ be the operator defined by*

$$T(u)v = \int_0^T u'(x)v'(x)dx + \sum_{j=1}^{p} I_j(u(x_j))v(x_j) - \int_0^T g(u(x))v(x)dx \tag{2.5}$$

for every u, $v \in X$. Then T admits a continuous inverse on X^*.

We now recall the following classic mountain pass lemma of Ambrosetti and Rabinowitz (see, for example, [13, Theorem 7.1]). Below, we denote by $B_r(u)$ the open ball centered at $u \in X$ with radius $r > 0$, $\overline{B}_r(u)$ its closure, and $\partial B_r(u)$ its boundary.

Lemma 2.7. *Let $(X, \|\cdot\|)$ be a real Banach space and $I \in C^1(X, \mathbb{R})$. Assume that I satisfies the (PS) condition and there exist $\bar{u}$, $\hat{u} \in X$ and $\rho > 0$ such that*

(I1) $\bar{u} \notin \overline{B}_\rho(\hat{u})$;
(I2) $\max\{I(\bar{u}), I(\hat{u})\} < \inf_{u\in\partial B_\rho(\hat{u})} I(u)$.

Then, I possesses a critical value which can be characterized as

$$c = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} I(\gamma(s)) \geq \inf_{u\in\partial B_\rho(u_0)} I(u),$$

where

$$\Gamma = \{\gamma \in C([0, 1], X) \ : \ \gamma(0) = u_0, \ \gamma(1) = u_1\}.$$

3. MAIN RESULTS

We begin by defining the constants (see [3])

$$C_1 = \frac{1}{2} - \sum_{j=1}^{p} \frac{b_j}{\gamma_j + 1} \left(\frac{\sqrt{T}}{2} \right)^{\gamma_j + 1},$$

$$C_2 = \frac{1}{2} + \sum_{j=1}^{p} \frac{b_j}{\gamma_j + 1} \left(\frac{\sqrt{T}}{2} \right)^{\gamma_j + 1},$$

and

$$C_3 = \frac{\sqrt{T}}{2} \sum_{j=1}^{p} a_j + \sum_{j=1}^{p} \frac{b_j}{\gamma_j + 1} \left(\frac{\sqrt{T}}{2} \right)^{\gamma_j + 1},$$

and the functions H_1, $H_2 : [0, \infty) \to \mathbb{R}$ by

$$H_1(t) = \left(C_1 - \frac{LT^2}{8} \right) t^2 - C_3 t \quad \text{and} \quad H_2(t) = \left(C_2 + \frac{LT^2}{8} \right) t^2 + C_3 t. \tag{3.1}$$

For $u \in X$, define the functionals Φ, $\Psi : X \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2} \int_0^T (u'(x))^2 dx + \sum_{j=1}^{p} \int_0^{u(x_j)} I_j(t) dt + \int_0^T G(u(x)) dx \tag{3.2}$$

and

$$\Psi(u) = \int_0^T F(x, u(x)) dx. \tag{3.3}$$

In the sequel, given a non-negative constant ν_1 and four positive constants ν_2, τ, η, and δ with η, $\delta < T/2$,

$$\frac{4}{T} \left(C_1 - \frac{LT^2}{8} \right) \nu_1^2 - \frac{2}{\sqrt{T}} C_3 \nu_1 \neq \frac{\eta + \delta}{\eta \delta} \left(C_2 + \frac{LT^2}{8} \right) \tau^2 + \sqrt{\frac{\eta + \delta}{\eta \delta}} C_3 \tau,$$

and

$$\frac{4}{T} \left(C_2 + \frac{LT^2}{8} \right) \nu_3^2 + \frac{2}{\sqrt{T}} C_3 \nu_3 \neq \frac{\eta + \delta}{\eta \delta} \left(C_2 + \frac{LT^2}{8} \right) \tau^2 + \sqrt{\frac{\eta + \delta}{\eta \delta}} C_3 \tau,$$

we let

$$a_\tau(\nu_1) = \frac{\int_0^T \sup_{t \in [-\nu_1, \nu_1]} F(x, t) dx - \int_\eta^{T - \delta} F(x, \tau) dx}{\frac{4}{T} \left(C_1 - \frac{LT^2}{8} \right) \nu_1^2 - \frac{2}{\sqrt{T}} C_3 \nu_1 - \frac{\eta + \delta}{\eta \delta} \left(C_2 + \frac{LT^2}{8} \right) \tau^2 - \sqrt{\frac{\eta + \delta}{\eta \delta}} C_3 \tau}$$

and

$$b_\tau(\nu_2) = \frac{\int_0^T \sup_{t \in [-\nu_3, \nu_3]} F(x, t) dx - \int_\eta^{T - \delta} F(x, \tau) dx}{\frac{4}{T} \left(C_2 + \frac{LT^2}{8} \right) \nu_2^2 + \frac{2}{\sqrt{T}} C_3 \nu_2 - \frac{\eta + \delta}{\eta \delta} \left(C_2 + \frac{LT^2}{8} \right) \tau^2 - \sqrt{\frac{\eta + \delta}{\eta \delta}} C_3 \tau},$$

provided $\nu_3 > 0$ satisfies

$$H_2\left(\frac{2}{\sqrt{T}}\nu_3\right) = \sup_{u\in\Phi(-\infty,r_2]} H_2(\|u\|) \tag{3.4}$$

and

$$r_2 = H_2\left(\frac{2}{\sqrt{T}}\nu_2\right) = \frac{4}{T}\left(C_2 + \frac{LT^2}{8}\right)\nu_2^2 + \frac{2}{\sqrt{T}}C_3\nu_2. \tag{3.5}$$

Remark 3.1. It is easy to see that if $u \in \Phi(-\infty, r_2]$, then $\|u\| < B < \infty$. Thus, (3.4) is well defined. Moreover, it is easy to check that $\nu_3 \geq \nu_2$.

We now present our main existence result.

Theorem 3.2. *Assume that $C_1 - \frac{LT^2}{8} > 0$ and there exist five positive constants ν_1, ν_2, τ, η, and δ with η, $\delta < T/2$, $\tau > \nu_1 > \frac{\sqrt{T}C_3}{2\left(C_1 - \frac{LT^2}{8}\right)}$, and $\sqrt{\frac{\eta+\delta}{\eta\delta}}\tau < \frac{2}{\sqrt{T}}\nu_2$ such that*

(A2) *$F(x,t) \geq 0$ for all $(x,t) \in ([0,\eta] \cup [T-\delta, T]) \times [0,\tau]$;*
(A3) *$b_\tau(\nu_2) < a_\tau(\nu_1)$.*

Then, for each $\lambda \in \Lambda_1 = \left(\frac{1}{a_\tau(\nu_1)}, \frac{1}{b_\tau(\nu_2)}\right)$, the problem (IP) has at least one nontrivial classical solution $u_1 \in X$ such that

$$r_1 < \frac{1}{2}\int_0^T (u_1'(x))^2 dx + \sum_{j=1}^{p} \int_0^{u_1(x_j)} I_j(t)dt + \int_0^T G(u_1(x))dx < r_2, \tag{3.6}$$

where

$$r_1 = \frac{4}{T}\left(C_1 - \frac{LT^2}{8}\right)\nu_1^2 - \frac{2}{\sqrt{T}}C_3\nu_1. \tag{3.7}$$

If we further assume that $f(t,0) \neq 0$ for all $t \in [0,T]$ and (A1) holds, then for each $\lambda \in \Lambda_1$, the problem (IP) has at least two distinct nontrivial classical solutions $u_1, u_2 \in X$ with u_1 satisfying (3.6).

Proof. Let the functionals Φ and Ψ be given in (3.2) and (3.3), respectively. It is well known that Ψ is a Gâteaux differentiable functional, is sequentially weakly lower semicontinuous, and its Gâteaux derivative at $u \in X$ is the functional $\Psi'(u) \in X^*$ defined by

$$\Psi'(u)(v) = \int_0^T f(x, u(x))v(x)dx \quad \text{for every } v \in X.$$

To show that $\Psi' : X \to X^*$ is a compact operator, let $u_n \to u \in X$ weakly in X as $n \to \infty$. Then, $u_n \to u$ strongly in $C([0,T])$. Since $f(x,\cdot)$ is continuous in $\mathbb{R}$ for every $x \in [0,T]$, we have $f(x,u_n) \to f(x,u)$ strongly as $n \to \infty$. By the Lebesgue dominated convergence theorem, we see that $\Psi'(u_n) \to \Psi'(u)$ strongly. Thus, Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by [25, Proposition 26.2].

We also know that Φ is a Gâteaux differentiable functional whose Gâteaux derivative at $u \in X$ is the functional $\Phi'(u) \in X^*$ given by

$$\Phi'(u)(v) = \int_0^T u'(x)v'(x)dx + \sum_{j=1}^{p} I_j(u(x_j))v(x_j) - \int_0^T g(u(x))v(x)dx \text{ for every } v \in X.$$

Moreover, Lemma 2.6 implies that Φ' admits a continuous inverse on X^*. Since Φ' is monotonic, Φ is sequentially weakly lower semicontinuous (see [25, Proposition 25.20]).

Since $\|u\|^{\gamma_j+1} \le \|u\|^2$ for every $\|u\| \ge 1$ and $\|u\|^{\gamma_j+1} \le \|u\|$ for every $\|u\| < 1$, we have that $\|u\|^{\gamma_j+1} \le \|u\| + \|u\|^2$ for all $u \in X$. Condition (H1) implies

$$|g(t)| \le L|t| \quad \text{for all} \quad t \in \mathbb{R}, \tag{3.8}$$

Hence, from (2.2), (2.4), and (3.8), we see that

$$\left(C_1 - \frac{LT^2}{8}\right)\|u\|^2 - C_3\|u\| \le \Phi(u) \le \left(C_2 + \frac{LT^2}{8}\right)\|u\|^2 + C_3\|u\| \tag{3.9}$$

and Φ is coercive.

Let

$$w(x) = \begin{cases} \frac{\tau}{\eta}x, & x \in [0, \eta], \\ \tau, & x \in [\eta, T-\delta], \\ \frac{\tau}{\delta}(T-x), & x \in [T-\delta, T]. \end{cases} \tag{3.10}$$

It is clear that $w \in X$ and

$$\|w\| = \sqrt{\frac{\eta+\delta}{\eta\delta}}\tau. \tag{3.11}$$

Since $C_1 - \frac{LT^2}{8} > 0$ and $C_3 > 0$, we have $H_1(t) \le 0$ for $t \in \left[0, \frac{C_3}{C_1 - \frac{LT^2}{8}}\right]$, $H_1(t)$ is strictly increasing on $\left[\frac{C_3}{2(C_1 - \frac{LT^2}{8})}, \infty\right)$, and $H_1(t) \ge 0$ on $\left[\frac{C_3}{C_1 - \frac{LT^2}{8}}, \infty\right)$. Now η, $\delta \in (0, T/2)$ implies $\frac{1}{\eta} + \frac{1}{\delta} > \frac{4}{T}$. Since $\tau > \nu_1 > \frac{\sqrt{T}C_3}{2\left(C_1 - \frac{LT^2}{8}\right)} > 0$, we have $\sqrt{\frac{\eta+\delta}{\eta\delta}}\tau > \sqrt{\frac{\eta+\delta}{\eta\delta}}\nu_1 > \frac{2}{\sqrt{T}}\nu_1$. Moreover, $\nu_1 > \frac{\sqrt{T}C_3}{2\left(C_1 - \frac{LT^2}{8}\right)}$ implies $\frac{2}{\sqrt{T}}\nu_1 > \frac{C_3}{C_1 - \frac{LT^2}{8}}$. Hence, from (3.11), we see that $\|w\| > \frac{2}{\sqrt{T}}\nu_1 > \frac{C_3}{C_1 - \frac{LT^2}{8}}$, which, together with the fact that $H_1(t)$ is strictly increasing on $\left[\frac{C_3}{C_1 - \frac{LT^2}{8}}, \infty\right)$, implies

$$H_1(\|w\|) > H_1\left(\frac{2}{\sqrt{T}}\nu_1\right) > H_1\left(\frac{C_3}{C_1 - \frac{LT^2}{8}}\right) = 0.$$

Thus, $r_1 = H_1\left(\frac{2}{\sqrt{T}}\nu_1\right)$, and so

$$\left(C_1 - \frac{LT^2}{8}\right)\|w\|^2 - C_3\|w\| > r_1 > 0. \tag{3.12}$$

With H_2 and r_2 defined by (3.1) and (3.5), we see that $H_2(t)$ is strictly increasing on $\left[-\frac{C_3}{2\left(C_1-\frac{LT^2}{8}\right)}, \infty\right)$. Since $0 < \sqrt{\frac{\eta+\delta}{\eta\delta}}\tau < \frac{2}{\sqrt{T}}\nu_2$, we have $H_2\left(\sqrt{\frac{\eta+\delta}{\eta\delta}}\tau\right) < r_2 = H_2(\frac{2}{\sqrt{T}}\nu_2)$, i.e.,

$$\left(C_2 + \frac{LT^2}{8}\right)\|w\|^2 + C_3\|w\| < r_2. \tag{3.13}$$

Hence, from (3.9), (3.12), and (3.13), we obtain

$$r_1 < \Phi(w) < r_2.$$

Since $H_1(t)$ is strictly increasing on the interval $\left[\frac{C_3}{C_1-\frac{LT^2}{8}}, \infty\right)$, $\frac{2}{\sqrt{T}}\nu_1 > \frac{C_3}{C_1-\frac{LT^2}{8}}$, and $r_1 = H_1\left(\frac{2}{\sqrt{T}}\nu_1\right)$, we see that

$$\left[\frac{C_3}{C_1 - \frac{LT^2}{8}},\ \frac{2}{\sqrt{T}}\nu_1\right] \subseteq \{t \in [0,\infty)\ :\ H_1(t) \le r_1\} \tag{3.14}$$

and

$$\left(\frac{2}{\sqrt{T}}\nu_1,\ \infty\right) \cap \{t \in [0,\infty)\ :\ H_1(t) \le r_1\} = \emptyset. \tag{3.15}$$

Recalling that $H_1(t) \le 0$ for $t \in \left[0, \frac{C_3}{C_1-\frac{LT^2}{8}}\right]$, we have

$$\left[0, \frac{C_3}{C_1 - \frac{LT^2}{8}}\right] \subseteq \{t \in [0,\infty)\ :\ H_1(t) \le r_1\}. \tag{3.16}$$

From (3.14)–(3.16), we have

$$\{t \in [0,\infty)\ :\ H_1(t) \le r_1\} = \left[0,\ \frac{2}{\sqrt{T}}\nu_1\right]. \tag{3.17}$$

For any $u \in \Phi^{-1}(-\infty, r_1]$, from (3.9), we observe that

$$H_1(\|u\|) \le \Phi(u) \le r_1.$$

Hence, using (3.17), we have $\|u\| \le \frac{2\nu_1}{\sqrt{T}}$. Then, in view of (2.2), we see that

$$\Phi^{-1}(-\infty, r_1] \subseteq \{u \in X\ :\ ||u||_\infty \le \nu_1\}.$$

Thus,

$$\begin{aligned}\sup_{u\in\Phi^{-1}(-\infty,r_1]} \Psi(u) &= \sup_{u\in\Phi^{-1}(-\infty,r_1]} \int_0^T F(x,u(x))dx \\ &\le \int_0^T \sup_{t\in[-\nu_1,\nu_1]} F(x,t)dx.\end{aligned}$$

Since $0 \leq w(x) \leq \tau$ for each $x \in [0,T]$, condition (A2) implies

$$\int_0^\eta F(x,u_1(x))dx + \int_{T-\delta}^T F(x,u_1(x))dx \geq 0. \tag{3.18}$$

Therefore,

$$\begin{aligned}
\rho(r_1,r_2) &\geq \frac{\Psi(w) - \sup_{u\in\Phi^{-1}(-\infty,r_1]}\Psi(u)}{\Phi(w)-r_1} \\
&\geq \frac{\Psi(w) - \int_0^T \sup_{t\in[-\nu_1,\nu_1]} F(x,t)dx}{\Phi(w)-r_1} \\
&\geq \frac{\int_\eta^{T-\delta} F(x,\tau)dx - \int_0^T \sup_{t\in[-\nu_1,\nu_1]} F(x,t)dx}{\frac{\eta+\delta}{\eta\delta}(C_2+\frac{LT^2}{8})\tau^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}}C_3\tau - \frac{4}{T}(C_1-\frac{LT^2}{8})\nu_1^2 + \frac{2}{\sqrt{T}}C_3\nu_1} \\
&= a_\tau(\nu_1).
\end{aligned}$$

On the other hand, if we let $r_3 = \sup_{u\in\Phi(-\infty,r_2]} H_2(\|u\|)$, then, for any $u \in \Phi(-\infty,r_2]$, $H_2(\|u\|) \leq r_3$. This, together with (3.4) and the fact that H_2 is increasing on $[0,\infty)$, implies that $\|u\| \leq \frac{2\nu_3}{\sqrt{T}}$. Thus, from (2.2), it follows that

$$\Phi^{-1}(-\infty,r_2] \subseteq \{u\in X : \|u\|_\infty \leq \nu_3\}.$$

Then,

$$\begin{aligned}
\sup_{u\in\Phi^{-1}(-\infty,r_2]} \Psi(u) &= \sup_{u\in\Phi^{-1}(-\infty,r_2]} \int_0^T F(x,u(x))dx \\
&\leq \int_0^T \sup_{t\in[-\nu_3,\nu_3]} F(x,t)dx.
\end{aligned}$$

Thus,

$$\begin{aligned}
\vartheta(r_1,r_2) &\leq \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u) - \Psi(u_1)}{r_2-\Phi(u_1)} \\
&\leq \frac{\int_0^T \sup_{t\in[-\nu_3,\nu_3]} F(x,t)dx - \Psi(u_1)}{r_2-\Phi(u_1)} \\
&\leq \frac{\int_0^T \sup_{t\in[-\nu_3,\nu_3]} F(x,t)dx - \int_\eta^{T-\delta} F(x,\tau)dx}{\frac{4}{T}(C_1-\frac{LT^2}{8})\nu_2^2 + \frac{2}{\sqrt{T}}C_3\nu_2 - \frac{\eta+\delta}{\eta\delta}(C_2+\frac{LT^2}{8})\tau^2 - \sqrt{\frac{\eta+\delta}{\eta\delta}}C_3\tau} \\
&= b_\tau(\nu_2).
\end{aligned}$$

Hence, from (A3), we have $\vartheta(r_1,r_2) \leq b_\tau(\nu_2) < a_\tau(\nu_1) \leq \rho(r_1,r_2)$. Therefore, applying Lemma 2.1, we obtain that for each $\lambda \in \Lambda_1$, the functional $J = \Phi - \lambda\Psi$ has at least one critical point $u_1 \in X$ that is a local minimum of J such that $r_1 < \Phi(u_1) < r_2$. That is, (3.6) holds.

Now, we establish the existence of a second local minimum of J distinct from u_1. Without loss of generality, we may assume that u_1 is a strict local minimum for the functional J in X. Then, there exists $\rho > 0$ such that $\inf_{\|u-u_1\|=\rho} J(u) > J(u_1)$. Choose $\tilde{u} \in X \setminus \{0\}$. From (A1), there exist a, $b > 0$ such that

$$\begin{aligned} J(t\tilde{u}) &= \frac{1}{2}\int_0^T (t\tilde{u}'(x))^2 dx + \sum_{j=1}^p \int_0^{t\tilde{u}(x_j)} I_j(t)dt + \int_0^T G(t\tilde{u}(x))dx - \lambda \int_0^T F(x, t\tilde{u}(x))dx \\ &\le \frac{t^2}{2}\int_0^T (\tilde{u}'(x))^2 dx + \sum_{j=1}^p \int_0^{t\tilde{u}(x_j)} I_j(t)dt + \frac{Lt^2}{2}\int_0^T |\tilde{u}(x)|^2 dx \\ &\quad - \lambda t^s a \int_0^T |\tilde{u}(x)|^s dx + Tb \to -\infty \quad \text{as } t \to \infty. \end{aligned}$$

Thus, there exists $\tilde{t}$ large enough so that $J(\tilde{t}\tilde{u}) < \inf_{\|u-u_1\|=\rho} J(u)$. Hence, in view of Lemma 2.5, all the conditions of Lemma 2.7 are satisfied with $\hat{u} = u_1$ and $\bar{u} = \tilde{t}\tilde{u}$, so there is a critical point u_2 of J such that $J(u_2) > J(u_1)$. Clearly, the condition that $f(t, 0) \neq 0$ for all $t \in [0, T]$ implies that u_2 is nontrivial. Finally, taking into account Remark 2.3 and the fact that the weak solutions of the problem (IP) are exactly the critical points of the functional J, completes the proof of the theorem. □

Remark 3.3. From the proof of Theorem 3.2 (see (3.9)), we have

$$H_1(\|u\|) \le \Phi(u) \le H_2(u) \quad \text{for } u \in X.$$

We now apply Theorem 3.2 to the case where the function f in (IP) is separable. Let $\alpha \in L^1([0, T])$ be such that $\alpha(x) \ge 0$ a.e. $x \in [0, T]$, $\alpha \not\equiv 0$, and let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $h(t) \ge 0$ on $[0, \infty]$ and $h(0) \neq 0$. Set

$$H(t) = \int_0^t h(\xi)d\xi \quad \text{for all } t \in \mathbb{R}.$$

Corollary 3.4. *Assume that* $C_1 - \frac{LT^2}{8} > 0$ *and there exist five positive constants* ν_1, ν_2, τ, η, *and* δ *with* η, $\delta < T/2$, $\tau > \nu_1 > \frac{\sqrt{T}C_3}{2\left(C_1 - \frac{LT^2}{8}\right)}$, *and* $\sqrt{\frac{\eta+\delta}{\eta\delta}}\tau < \frac{2}{\sqrt{T}}\nu_2$ *such that:*

(A4) $\overline{b}_\tau(\nu_2) < \overline{a}_\tau(\nu_1)$, *where*

$$\overline{a}_\tau(\nu_1) = \frac{\|\alpha\|_{L^1([0,T])}H(\nu_1) - \|\alpha\|_{L^1([\eta,T-\delta])}H(\tau)}{\frac{4}{T}\left(C_1 - \frac{LT^2}{8}\right)\nu_1^2 - \frac{2}{\sqrt{T}}C_3\nu_1 - \frac{\eta+\delta}{\eta\delta}\left(C_2 + \frac{LT^2}{8}\right)\tau^2 - \sqrt{\frac{\eta+\delta}{\eta\delta}}C_3\tau},$$

$$\overline{b}_\tau(\nu_2) = \frac{\|\alpha\|_{L^1([0,T])}H(\nu_3) - \|\alpha\|_{L^1([\eta,T-\delta])}H(\tau)}{\frac{4}{T}\left(C_2 + \frac{LT^2}{8}\right)\nu_2^2 + \frac{2}{\sqrt{T}}C_3\nu_2 - \frac{\eta+\delta}{\eta\delta}\left(C_2 + \frac{LT^2}{8}\right)\tau^2 - \sqrt{\frac{\eta+\delta}{\eta\delta}}C_3\tau},$$

and ν_3 *satisfies* (3.4).

Then, for each $\lambda \in \Lambda_3 = \left(\frac{1}{a_\tau(\nu_1)}, \frac{1}{b_\tau(\nu_2)}\right)$, *the problem*

$$\begin{cases} -u''(x) = \lambda\alpha(x)h(u(x)) + g(u(x)), & a.e.\ x \in [0, T], \\ \Delta u'(x_j) = I_j(u(x_j)), & j = 1, 2, \ldots, p, \\ u(0) = u(T) = 0, \end{cases} \tag{3.19}$$

has at least one nontrivial classical solution $u_1 \in X$ *satisfying* (3.6).

If we further assume that

(A5) *there exist* $s > \frac{2(4+LT^2)}{4-LT^2}$ *and* $R > 0$ *such that*

$$0 < sH(t) \le th(t) \quad \text{for all } |t| \ge R,$$

then, for each $\lambda \in \Lambda_3$, *the problem* (3.19) *has at least two distinct nontrivial classical solutions* u_1, $u_2 \in X$ *with* u_1 *satisfying* (3.6).

We conclude this paper with two examples to illustrate our results.

Example 3.5. Consider the problem

$$\begin{cases} -u''(x) = \lambda\left(\frac{3}{2}u^{1/2}+1\right) + \frac{1}{8}u, \quad \text{a.e. } x \in [0,4], \\ u(0) = u(4) = 0, \\ \Delta u'(x_1) = 1, \quad 0 < x_1 < 4. \end{cases} \tag{3.20}$$

We claim that there exists $\lambda^* > 0$ such that, for each $\lambda > \lambda^*$, the problem (3.20) has at least one nontrivial classical solution.

In fact, with $T = 4$, $p = 1$, $\alpha(x) = 1$, $h(u) = \frac{3}{2}u^{1/2} + 1$, $g(u) = \frac{1}{8}u$, and $I_1(u) = 1$, we see that problem (3.20) is of the form of the problem (3.19) and the covering assumptions (H1)–(H3) are satisfied. In particular, in (H1), we can take $L = \frac{1}{8}$, and in (H3), we can choose $a_1 = 1$ and $b_1 = \gamma_1 = 0$. From the definition of C_1, C_2, and C_3, we have $C_1 = C_2 = \frac{1}{2}$ and $C_3 = 1$. Clearly, $C_1 - \frac{LT^2}{8} = \frac{1}{4} > 0$.

Choose $\eta = \delta = 1$ and $\nu_1 > \frac{\sqrt{T}C_3}{2\left(C_1 - \frac{LT^2}{8}\right)} = 4$. Note that

$$\overline{a}_\tau(\nu_1) = \frac{4\left(\nu_1^{3/2} + \nu_1\right) - 2\left(\tau^{3/2} + \tau\right)}{\frac{1}{4}\nu_1^2 - \nu_1 - \frac{3}{2}\tau^2 - \sqrt{2}\tau}.$$

Then, there exists $\tau > \nu_1$ large enough so that $\overline{a}_\tau(\nu_1) > 0$.

For large $\nu_2 > 0$, let r_2 be defined by (3.5). Then, for $u \in \Phi^{-1}(-\infty, r_2]$, by (3.1) and Remark 3.3, we have

$$H_1(\|u\|) = \frac{1}{4}\|u\|^2 - \|u\| \le \phi(u) \le r_2.$$

Thus,

$$\|u\| \le 2\left(1 + \sqrt{1 + r_2}\right). \tag{3.21}$$

Again, by (3.1) and Remark 3.3, we see that

$$H_2(\|u\|) - \phi(u) \le H_2(\|u\|) - H_1(\|u\|) = \frac{1}{4}\|u\|^2 + 2\|u\|. \tag{3.22}$$

In the remainder of this example, let D_i, $i = 1, \dots, 6$, denote some appropriate constants. For $u \in \Phi^{-1}(-\infty, r_2]$, from (3.21) and (3.22), it follows that

$$\begin{aligned} H_2(\|u\|) &\le r_2 + \frac{1}{4}\left(2\left(1+\sqrt{1+r_2}\right)\right)^2 + 4\left(1+\sqrt{1+r_2}\right) \\ &\le D_1 r_2 + D_2. \end{aligned}$$

Then, in view of (3.4) and (3.5), we have

$$H_2(\nu_3) = H_2\left(\frac{2\nu_3}{\sqrt{T}}\right) \le D_1 r_2 + D_2 = D_1\left(\frac{3}{4}\nu_2^2 + \nu_2\right) + D_2 \le D_3\nu_2^2 + D_4,$$

i.e.,

$$\frac{3}{4}\nu_3^2 + \nu_3 \le D_3\nu_2^2 + D_4.$$

This, together with Remark 3.1, implies that

$$\nu_2 \le \nu_3 \le D_5\nu_2 + D_6. \tag{3.23}$$

Hence,

$$\begin{aligned} &\frac{4\left(\nu_2^{3/2} + \nu_2\right) - 2\left(\tau^{3/2} + \tau\right)}{\frac{3}{4}\nu_2^2 + \nu_2 - \frac{3}{2}\tau^2 - \sqrt{2}\tau} \\ &\le \overline{b}_\tau(\nu_2) = \frac{4\left(\nu_3^{3/2} + \nu_3\right) - 2\left(\tau^{3/2} + \tau\right)}{\frac{3}{4}\nu_2^2 + \nu_2 - \frac{3}{2}\tau^2 - \sqrt{2}\tau} \\ &\le \frac{4\left((D_5\nu_2 + D_6)^{3/2} + (D_5\nu_2 + D_6)\right) - 2\left(\tau^{3/2} + \tau\right)}{\frac{3}{4}\nu_2^2 + \nu_2 - \frac{3}{2}\tau^2 - \sqrt{2}\tau}. \end{aligned}$$

Thus $\overline{b}_\tau(\nu_2) \ge 0$ for ν_2 large enough and $\overline{b}_\tau(\nu_2) \to 0$ as $\nu_2 \to \infty$.

Note that $\frac{1}{\overline{b}_\tau(\nu_2)} \to \infty$ as $\nu_2 \to \infty$. Then, the claim follows from the first part of Corollary 3.4.

Remark 3.6. In Example 3.5, since $\overline{a}_\tau(\nu_1) > 0$ and $\overline{b}_\tau(\nu_2) \to 0$ as $\nu_2 \to \infty$, then we can fix a ν_2 large enough so that $\sqrt{\frac{\eta+\delta}{\eta\delta}}\tau < \frac{2}{\sqrt{T}}\nu_2$ and $0 \le \overline{b}_\tau(\nu_2) < \overline{a}_\tau(\nu_1)$.

Example 3.7. Let ν_2 be fixed as in Remark 3.6. Then, $0 \le \overline{b}_\tau(\nu_2) < \overline{a}_\tau(\nu_1)$. Choose $\zeta > 0$ large enough so that $\zeta > D_5\nu_2 + D_6$, where D_5 and D_6 are the constants given in Example 3.5. Let $h : \mathbb{R} \to \mathbb{R}$ be continuous satisfying

$$h(t) = \begin{cases} \frac{3}{2}t^{1/2} + 1, & |t| \le \zeta \\ 3t^8, & |t| > 2\zeta. \end{cases}$$

Consider the problem

$$\begin{cases} -u''(x) = \lambda h(u) + \frac{1}{8}u, & \text{a.e. } x \in [0,4], \\ u(0) = u(4) = 0, \\ \Delta u'(x_1) = 1, \quad 0 < x_1 < 4. \end{cases} \tag{3.24}$$

We claim that there exists $\overline{\lambda} > \underline{\lambda} > 0$ such that, for each $\lambda \in (\underline{\lambda}, \overline{\lambda})$, the problem (3.24) has at least two nontrivial classical solutions.

In fact, since $h(t) = \frac{3}{2}t^{1/2} + 1$ for $|t| \leq \zeta$, from the reasoning in Example 3.5 and the choice of ζ, we see that all the assumptions of the first part of Corollary 3.4 are satisfied. Moreover, since $h(t) = 3t^8$ for $|t| > 2\zeta$, condition (A5) also holds. Thus, the claim follows from the second part of Corollary 3.4.

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