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IMPULSIVE EFFECTS ON THE EXISTENCE OF SOLUTION FOR A FRACTIONAL DIFFUSION EQUATION

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ABSTRACT. Let α (< 1) be a positive constant. This article studies the following impulsive problem: for $n = 1, 2, 3, \ldots$

$$u_t - \left({}_{(n-1)T}D_t^{1-\alpha}u\right)_{xx} = \lambda f(u), \quad 0 < x < 1, (n-1)T < t \le nT^-,$$
$$u(x,0) = 0, \quad 0 \le x \le 1,$$
$$u(x,nT) = \sigma u(x,nT^-), \quad 0 \le x \le 1,$$
$$u(0,t) = 0 = u(1,t), \quad t > 0.$$

The number λ^* is called the critical value if the problem has a unique global solution u for $\lambda < \lambda^*$, and the solution quenches in a finite time for $\lambda > \lambda^*$. The existence of a unique λ^* is established.

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1. INTRODUCTION

Let σ, λ, T be positive constants, for $0 < \alpha < 1$, $L_a u = u_t - \left({}_aD_t^{1-\alpha}\right)_{xx}$ where ${}_aD_t^{1-\alpha}u$ denotes the Riemann-Liouville fractional derivative of u. We consider the following impulsive problem: for $n = 1, 2, 3, \ldots,$

$$\begin{cases} L_{(n-1)T}u = \lambda f(u), \quad 0 < x < 1, (n-1)T < t \le nT^-, \\ u(x,0) = 0, \quad 0 \le x \le 1, \\ u(x,nT) = \sigma u(x,nT^-), \quad 0 \le x \le 1, \\ u(0,t) = 0 = u(1,t), \quad t > 0, \end{cases} \tag{1.1}$$

where $f(u) > 0$, $f'(u) > 0$, $f''(u) \ge 0$ for $u \ge 0$, and $\lim_{u \to 1^-} f(u) = +\infty$. The Riemann-Liouville fractional derivative is given as

$${}_aD_t^p u(x,t) = \frac{1}{\Gamma(1-p)} \frac{d}{dt} \int_a^t (t-s)^{-p} u(x,s)d \qquad \text{for } 0 < p < 1,$$

$${}_aD_t^{-p} u(x,t) = \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1} u(x,s)ds \qquad \text{for } 0 < p.$$

For simplicity, we denote ${}_0D_t^p$ by D_t^p.

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A solution u of the problem (1.1) is said to quench if there is a $m \in \mathbb{N}$ and a positive number τ such that $(m-1)T < \tau \leq mT^-$ and

$$\max\{u(x,t) : 0 \leq x \leq 1\} \to 1 \quad \text{as } t \to \tau.$$

The number λ^* is called the critical value if the problem (1.1) has a unique global solution u for $\lambda < \lambda^*$, and the solution quenches in a finite time for $\lambda > \lambda^*$. If $\sigma = 1$, then there is no impulse; when $\sigma \neq 1$, the problem (1.1) describes an impulse proportional to u (with σ as the proportionality constant) being given at each time interval T.

The problems of thermal diffusion with subdiffusive properties are formulated in terms of fractional diffusion equations. The subdiffusive problem is described through the Riemann-Liouville fractional differential operator $D_t^{1-\alpha}$ which is used to model diffusive behavior of mean square displacement of Brownian motion evolves on a slower than normal time. By considering the solution $u(x,t)$ as the temperature of subdiffusive material, Kirk and Olmstead [5], and Olmstead and Roberts [9] studied the blow-up behavior of the solution of the problem with concentrated source.

Owing to short-term perturbations, many evolution processes at certain moments of time experience changes of state abruptly. Since the durations of the perturbations are negligible in comparison with the duration of each process, it is natural to assume that these perturbations act instantaneously in the form of impulses (cf. Lakshmikantham, Bainov, and Simeonov [6]). The impulsive effects on blow-up and quenching were first studied by Chan and Deng [1], Chan, Ke and Vatsala [2] for a semilinear heat equation, and Chan and Kong [3] for a degenerate semilinear equation. Liu [7] gave the quenching results for fast diffusive cases. In this paper, we study the impulsive effects on the existence of the solution. When $\sigma \leq 1$, the problem has a critical value λ^*, and when $\sigma > 1$, the solution always quenches.

A solution $u(x,t)$ of the problem (1.1) is a $C^{2,1}((0,1) \times ((n-1)T, nT)$ function for $n = 1, 2, \ldots$, which satisfies the initial and boundary conditions.

By using the transformation,

$$u_n(x, t-(n-1)T) = u(x,t), \quad (n-1)T < t \leq nT^-, \quad n = 1, 2, 3, \ldots,$$

and $Lu = u_t - ({}_0D_t^{1-\alpha}u)_{xx}$, the problem (1.1) can be written as

$$\begin{cases} Lu_n(x,t) = \lambda f(u_n(x,t)) \text{ in } (0,1) \times (0, T^-], \\ u_1(x,0) = 0, \quad 0 \leq x \leq 1, \\ u_{n+1}(x,0) = \sigma u_n(x, T^-), \quad 0 \leq x \leq 1, \\ u_n(0,t) = 0 = u_n(1,t), \quad 0 < t \leq T^-. \end{cases} \tag{1.2}$$

Thus, global existence of a solution of the problem (1.1) is now equivalent to existence of u_n for all positive integers n.

For ease of reference, we state the following theorem, which summarizes some results about fractional differential equations (cf. Podlubny [10, p.72–75]).

Theorem 1.1. (a) *For any real p and $m \in \mathbb{N}$ such that $m-1 \leq p < m$, $\beta > m-1$,*

$$D_t^p t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(-p+\beta+1)} t^{\beta-p}.$$

(b) *For $m, n \in \mathbb{N}$ and any real p and q such that $m-1 \leq p < m$, $n-1 \leq q < n$,*

$$D_t^p(D_t^q g(t)) = D_t^{q+p} g(t) - \sum_{j=1}^{n} (D_t^{q-j} g(t))\bigg|_{t=0} \frac{t^{-p-j}}{\Gamma(1-p-j)};$$

$$D_t^q(D_t^p g(t)) = D_t^{p+q} g(t) - \sum_{j=1}^{m} (D_t^{p-j} g(t))\bigg|_{t=0} \frac{t^{-q-j}}{\Gamma(1-q-j)}.$$

Hence $D_t^p(D_t^q g(t)) = D_t^q(D_t^p g(t))$ only if $D_t^{p-j} g(0) = 0$ and $D_t^{q-j} g(0) = 0$ for $j = 1, 2, \ldots, r$ where $r = \max\{m, n\}$.

(c) *For any nonnegative real numbers p and q, $D_t^p(D_t^{-q} g(t)) = D_t^{p-q} g(t)$, and*

$$D^{-p}(D_t^q g(t)) = D_t^{q-p} g(t) - \sum_{j=1}^{n} (D_t^{q-j} g(t))\bigg|_{t=0} \frac{t^{p-j}}{\Gamma(1+p-j)},$$

where $0 \leq n-1 \leq q < n$.

A weak form of maximum principle of the equation $Lu \geq 0$ is studied by Chan and Liu [4], and an extension for the operator $(L-c)u$ is given by Liu [8]. The next result will be used in our discussion.

Theorem 1.2. *If v satisfies $Lv - cv \geq 0$, $v(x,0) \geq 0$ for $x \in (0,1)$, $v(0,t) \geq 0$ and $v(1,t) \geq 0$ for $t \in (0,T]$, and $c \geq 0$ in $(0,1) \times (0,T]$, then $v(x,t) \geq 0$ on $[0,1] \times [0,T]$.*

2. EXISTENCE AND NONEXISTENCE OF SOLUTION

By using Theorem 1.2, we can show the positivity and uniqueness of the solution u_n of the problem (1.2).

Theorem 2.1. *The solution $u_n(x,t) \geq 0$, and the problem (1.2) has at most one solution on $[0,1] \times [0,T]$ for $n = 1, 2, \ldots$.*

Similar to the situation as in the heat equation, the increasing nature in the fractional diffusive case follows from Theorem 1.2.

Theorem 2.2. *The solution $u_n(x,t)$ is increasing with respect to t in $(0,1) \times (0,T)$ for $n = 1, 2, \ldots$.*

Proof. When $n = 1$, let $0 < \eta < T$ and $w(x,t) = u_1(x, t+\eta) - u_1(x,t)$.

Since $u_1 \in C^{2,1}((0,1)\times(0,T))$, the function $w(x,t) \in C^{2,1}((0,1)\times(0,T-\eta))$ and satisfies the equation

$$Lw = \lambda f'(\xi)\cdot w,$$

where ξ lies between $u_1(x,t+\eta)$ and $u_1(x,t)$.

Since $u_n(x,t) \geq 0$ for $t \geq 0$, we have $w(x,0) = u_1(x,\eta) - u_1(x,0) \geq 0$. From the boundary conditions, we get $w(0,t) = 0 = w(1,t)$. It follows from Theorem 1.2 that $w(x,t) \geq 0$ on $[0,1]\times[0,T-\eta)$. Hence $u_1(x,t)$ is increasing with respect to t in $(0,1)\times(0,T]$.

Assume that $u_n(x,t)$ is increasing with respect to t in $(0,1)\times(0,T)$. Let $w(x,t) = (u_{n+1})_t(x,t)$. Then $w(x,t)$ satisfies

$$Lw = \lambda f'(u_{n+1})\cdot w.$$

Since $(u_n)_t(x,t) \geq 0$ for $t \in (0,T)$, we have $(u_{n+1})_t(x,0) = \sigma(u_n)_t(x,T) \geq 0$. At $x = 0$ and $x = 1$, since $u_{n+1}(0,t) = 0 = u_{n+1}(1,t)$ for $t \in (0,T)$, we have $w(0,t) = 0 = w(1,t)$. From Theorem 1.2 again, we have $w(x,t) \geq 0$. This implies that $(u_{n+1})_t(x,t) \geq 0$. Therefore, it follows from the mathematical induction that u_n is increasing with respect to t for $n = 1, 2, \ldots$. □

Since T denotes the time-step for the solution u experience the impulsive effects, shorter the time-step implies more frequency the impulses take effect on the solution. The next theorem gives the relation.

Theorem 2.3. *Let $\tilde{u}_n$ and $\hat{u}_n$ be the solutions corresponding to the problem* (1.2) *on $[0,1]\times[0,T_1]$ and $[0,1]\times[0,T_2]$ respectively. If $T_1 < T_2$, then $\tilde{u}_n \leq \hat{u}_n$ for any $n = 1, 2, \ldots$ on $[0,1]\times[0,T_1]$.*

Proof. From the uniqueness of the solution, $\tilde{u}_1 = \hat{u}_1$ on $[0,1]\times[0,T_1]$. The function $w_2(x,t) = \tilde{u}_2(x,t) - \hat{u}_2(x,t)$ satisfies

$$Lw_2 = \lambda f'(\xi) w_2, \text{ in } (0,1)\times(0,T_1),$$

where ξ lies between $\tilde{u}_2(x,t)$ and $\hat{u}_2(x,t)$. From Theorem 2.2, we have $(\hat{u}_1)_t \geq 0$, then the relation $\tilde{u}_2(x,0) = \sigma\tilde{u}_1(x,T_1) = \sigma\hat{u}_1(x,T_1) \leq \sigma\hat{u}_1(x,T_2) = \hat{u}_2(x,0)$ gives that $w_2(x,0) \geq 0$. Since on the boundary, $w_2(0,t) = 0 = w_2(1,t)$, it follows from Theorem 1.2 that $w_2(x,t) \geq 0$. Hence $\tilde{u}_2(x,t) \leq \hat{u}_2(x,t)$ on $[0,1]\times[0,T_1]$.

Next we assume that $\tilde{u}_n(x,t) \leq \hat{u}_n(x,t)$ on $[0,1]\times[0,T_1]$. From Theorem 2.2 again, we have $\hat{u}_n(x,t)$ is increasing with respect to t, then $\tilde{u}_{n+1}(x,0) = \sigma\tilde{u}_n(x,T_1) = \sigma\hat{u}_n(x,T_1) \leq \sigma\hat{u}_n(x,T_2) = \hat{u}_{n+1}(x,0)$. It follows a similar argument as in the case for $n = 1$ that $\tilde{u}_{n+1}(x,t) \leq \hat{u}_{n+1}(x,t)$. By induction, the result follows. □

We next prove that u_n increases as λ increases.

Theorem 2.4. *Let $u_n(x,t;\lambda)$ denotes the solution of the problem* (1.2)*. If $\lambda_1 > \lambda_2$, then*

$$u_n(x,t;\lambda_1) \geq u_n(x,t;\lambda_2).$$

Proof. Since $u_1(x,0;\lambda_1) = u_1(x,0;\lambda_2) = 0$, $u_1(x,t;\lambda_1) = 0 = u_1(x,t;\lambda_2) = 0$ on the boundary of $[0,1]$, and

$$L(u_1(x,t;\lambda_1) - u_1(x,t;\lambda_2)) = \lambda_1 f(u_1) - \lambda_2 f(u_1) \geq \lambda_1 f'(\xi)(u_1(x,t;\lambda_1) - u_1(x,t;\lambda_2)),$$

we have $u_1(x,t;\lambda_1) \geq u_1(x,t;\lambda_2)$. By the principle of mathematical induction, the theorem is then proved. □

Let $\varphi(x)$ be the nonnegative solution of the eigenvalue problem

$$\varphi'' = -\mu\varphi, \text{ for } 0 < x < 1,$$

$\varphi(0) = 0 = \varphi(1)$ with $\int_0^1 \varphi(x)dx = 1$.

Let $F_n(t) = \int_0^1 u_n(x,t)\varphi(x)dx$. We now prove the existence of λ^*.

Theorem 2.5. *The solution u_n of the problem* (1.2) *quenches in a finite time when λ is large enough.*

Proof. Apply the operator $D_t^{-1+\alpha}$ on (1.2). It follows from Theorem 1.1c and $u_1(x,0) = 0$ that u_1 satisfies

$$\begin{aligned} D_t^\alpha u_1 = D_t^{-1+\alpha}(u_1)_t &= \left(D_t^{-1+\alpha} D_t^{1-\alpha} u_1\right)_{xx} + \lambda\left(D_t^{-1+\alpha} f(u_1)\right) \\ &= (u_1)_{xx} + \lambda\left(D_t^{-1+\alpha} f(u_1)\right). \end{aligned}$$

Next, multiplying $\varphi(x)$ on both sides of the equation and integrating with respect to x from 0 to 1, we obtain

$$D_t^\alpha\left(\int_0^1 \varphi(x)u_1(x,t)dx\right) = -\mu\int_0^1 \varphi(x)u_1(x,t)dx + \lambda D_t^{-1+\alpha}\left(\int_0^1 f(u_1(x,t))\varphi(x)dx\right),$$

or

$$D_t^\alpha F_1(t) = -\mu F_1(t) + \lambda D_t^{-1+\alpha}\left(\int_0^1 f(u_1(x,t))\varphi(x)dx\right). \tag{2.1}$$

Next we give an estimation of the last term on the right-hand side of (2.1). Since $f(u) > 0$, $f'(u) > 0$ and $f''(u) \geq 0$ for $u \geq 0$, there is $k_1 > 0$ such that $f(u_1) > k_1$. Then $\displaystyle\int_0^1 f(u_1(x,t))\varphi(x)dx \geq k_1$. By using the definition of Riemann derivative and Theorem 1.1a, we get

$$\begin{aligned} D_t^{-1+\alpha}\left(\int_0^1 f(u_1(x,t))\varphi(x)dx\right) &\geq \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}k_1 ds \\ &= \frac{k_1}{\Gamma(1-\alpha)(1-\alpha)}t^{1-\alpha}. \end{aligned}$$

Hence, from (2.1), $F_1(t)$ satisfies

$$D_t^\alpha F_1(t) \geq -\mu F_1(t) + \frac{\lambda k_1}{\Gamma(1-\alpha)(1-\alpha)} t^{1-\alpha}.$$

Since $F_1(t) \leq 1$, this gives

$$F_1(t) \geq D_t^{-\alpha}\left(-\mu + \frac{\lambda k_1}{\Gamma(1-\alpha)(1-\alpha)} t^{1-\alpha}\right).$$

It follows from a direct computation that

$$D_t^{-\alpha}\left(-\mu + \frac{\lambda k_1}{\Gamma(1-\alpha)(1-\alpha)} t^{1-\alpha}\right) = \frac{1}{\Gamma(\alpha)}\left[-\frac{\mu}{\alpha} t^\alpha + \frac{\lambda k_1 \Gamma(\alpha)\Gamma(2-\alpha)}{\Gamma(1-\alpha)\Gamma(2)(1-\alpha)} t\right].$$

Thus

$$F_1(t) \geq \frac{1}{\Gamma(\alpha)}\left[-\frac{\mu}{\alpha} t^\alpha + \frac{\lambda k_1 \Gamma(\alpha)\Gamma(2-\alpha)}{\Gamma(1-\alpha)\Gamma(2)(1-\alpha)} t\right] = \frac{1}{\Gamma(\alpha)}\left[-\frac{\mu}{\alpha} t^\alpha + \lambda k_1 \Gamma(\alpha) t\right].$$

By taking $\lambda > (1/\Gamma(\alpha) + \mu T^\alpha/\alpha) / (k_1\Gamma(\alpha)T)$, then $F_1(T_0) > 1$ for some $T_0 \leq T$. This showed that $u_1(x,t)$ reaches 1 in a finite time less than T. This complete the proof. □

Next, we claim that the solution exists globally for λ is small.

Theorem 2.6. *For $\sigma \leq 1$. If λ is small, then the solution u_n exist and bounded above by 1 for $n = 1, 2, \ldots$ on $[0,1] \times [0,T]$.*

Proof. Let $w(x,t) = (1-\beta t)x(1-x)$ where β is a positive number with $\beta T < 1$. The initial data gives $w(x,0) = x(1-x) \geq 0$ for $0 \leq x \leq 1$. On the boundary, we have $w(0,t) = 0 = w(1,t)$. At $t = T$, we have $w(x,0) - \sigma w(x,T) = x(1-x) - \sigma(1-\beta T)x(1-x) \geq x(1-x)(1-(1-\beta T)) \geq 0$.

It follows from the definition that $w_t = -\beta x(1-x)$ and

$$(D_t^{1-\alpha} w)_{xx} = -2(D_t^{1-\alpha}(1-\beta t)) = -\frac{2}{\Gamma(\alpha)} t^{-1+\alpha} + \frac{2\beta}{\Gamma(1+\alpha)} t^\alpha.$$

Hence

$$\begin{aligned} w_t - (D_t^{1-\alpha})_{xx} - \lambda f(w) &= -\beta x(1-x) + \frac{2}{\Gamma(\alpha)} t^{-1+\alpha} - \frac{2\beta}{\Gamma(1+\alpha)} t^\alpha - \lambda f(w) \\ &\geq -\beta\left[x(1-x) + \frac{2T^\alpha}{\Gamma(1+\alpha)}\right] + \frac{2}{\Gamma(\alpha)} T^{-1+\alpha} - \lambda f\left(\frac{1}{4}\right). \end{aligned}$$

Let us pick λ be small such that $\lambda < 2/(T^{1-\alpha}\Gamma(\alpha) f(1/4))$. Then

$$\frac{2}{\Gamma(\alpha)} T^{-1+\alpha} - \lambda f\left(\frac{1}{4}\right) > 0.$$

Furthermore take β be small such that

$$\frac{2}{\Gamma(\alpha)} T^{-1+\alpha} - \lambda f\left(\frac{1}{4}\right) > \beta\left[x(1-x) + \frac{2T^\alpha}{\Gamma(1+\alpha)}\right].$$

Therefore $w_t - (D_t^{1-\alpha} w)_{xx} - \lambda f(w) > 0$. Then by Theorem 1.2, we get $u_n \le w$ for $(x,t) \in [0,1] \times [0,T]$ for $n = 1, 2, \ldots$. Since $\max w(x,t) = 1/4 < 1$, this shows that $u_n < 1$. □

Then combining the Theorems 2.4, 2.5, and 2.6, we have the following result.

Theorem 2.7. *For $\sigma \le 1$, the problem* (2.1) *has a unique critical λ^*.*

Differ from the heat or fast diffusive cases, the fractional operator $D_t^{1-\alpha}$ gives a slower diffusive situation. Hence, when $\sigma > 1$, the impulsive effect guarantee the solution u_n quenches for any λ and time-step T.

Theorem 2.8. *The solution u_n quenches for any positive numbers λ and T when $\sigma > 1$.*

Proof. We use a similar argument as in the proof of Theorem 2.5, that is, applying the operator $D_t^{-1+\alpha}$ on the equation, multiplying $\varphi(x)$ on both sides of the equation and integrating with respect to x from 0 to 1. Then, we get

$$D_t^\alpha \left(\int_0^1 \varphi(x) u_n(x,t) dx \right) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \int_0^1 \varphi(x) u_n(x,0) dx - \mu \int_0^1 \varphi(x) u_n(x,t) dx + \lambda D_t^{-1+\alpha} \left(\int_0^1 f(u_n(x,t)) \varphi(x) dx \right).$$

By using the Jensen's inequality and the fact that $F_n(t) < 1$, we get

$$D_t^\alpha F_n(t) \ge \frac{t^{-\alpha}}{\Gamma(1-\alpha)} F_n(0) - \mu + \lambda D_t^{-1+\alpha} f(F_n(t)).$$

Applying the operator $D_t^{-\alpha}$ on the above inequality, it follows from Theorem 1.1c that

$$F_n(t) \ge F_n(0) - \mu \cdot \frac{t^\alpha}{\alpha \Gamma(\alpha)} + \lambda f(F_n(0)) \cdot t.$$

In particular, at $t = T$,

$$F_n(T) \ge F_n(0) - \mu \cdot \frac{T^\alpha}{\alpha \Gamma(\alpha)} + \lambda T f(F_n(0)),$$

for $n = 2, 3, \ldots$. By using $u_n(x,0) = \sigma u_{n-1}(x,T)$ and $F_n(0) = \sigma F_{n-1}(T)$ for $n = 2, 3, \ldots$, we get

$$F_n(T) \ge \sigma F_{n-1}(T) - \mu \cdot \frac{T^\alpha}{\alpha \Gamma(\alpha)} + \lambda T f(\sigma F_{n-1}(T)).$$

Next we claim that $\sigma F_{n-1}(T) - \mu \cdot T^\alpha / \alpha \Gamma(\alpha) > 0$ for some n. Suppose that this is not true, that is $\sigma F_{n-1}(T) \le \mu T^\alpha / \alpha \Gamma(\alpha)$ for $n = 2, 3, \ldots$. From Theorem 2.2 $F_n(T) \ge F_n(0)$ for any $n = 2, 3, \ldots$. Then we get

$$F_{n-1}(T) \ge F_{n-1}(0) = \sigma F_{n-2}(T) \ge \sigma F_{n-2}(0) = \sigma^2 F_{n-3}(T) \ge \cdots.$$

This concludes that $F_{n-1}(T) \geq \sigma^{n-1}F_1(T)$ for $n = 2, 3, \ldots$. From the above inequality, we obtain

$$\sigma^n F_1(T) \leq \mu \cdot \frac{T^\alpha}{\alpha\Gamma(\alpha)}.$$

For $\sigma > 1$, as $n \to \infty$, we have $\sigma^n \to \infty$ which leads to a contradiction. This shows that there is n^* such that $\sigma F_{n^*-1}(T) - \mu \cdot T^\alpha/\alpha\Gamma(\alpha) > 0$. Since $F_{n-1}(T) \leq F_n(T)$, we get $\sigma F_{n-1}(T) > \mu \cdot T^\alpha/\alpha\Gamma(\alpha)$ for $n \geq n^*$. Therefore,

$$F_n(T) \geq \sigma F_{n-1}(T) - \mu \cdot \frac{T^\alpha}{\alpha\Gamma(\alpha)} + \lambda T f(\sigma F_{n-1}(T)) \geq \lambda T f(\sigma F_{n-1}(T)),$$

for $n \geq n^*$.

From $f > 0$, $f' > 0$ and $f'' \geq 0$, there is $k_2 > 0$ such that $f(u) \geq k_2 u$. The above inequality becomes

$$F_n(T) \geq \lambda T k_2 \sigma F_{n-1}(T) \tag{2.2}$$

for $n \geq n^*$. Using the inequality (2.2) repeatedly, we obtain

$$F_n(T) \geq \lambda T \sigma^{n-n^*} k_2 F_{n^*}(T)$$

for $n > n^*$. There exists some N such that $\lambda T \sigma^{N-n^*} k_2 F_{n^*}(T) > 1$. Hence $F_N(T) > 1$, this implies u_N reaches 1 in a finite time. □

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