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POSITIVE SOLUTIONS OF SCALAR INTEGRAL EQUATIONS

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ABSTRACT. Under general conditions solutions are shared by the integral equations

$$x(t) = p(t) - \int_0^t A(t-s)f(s, x(s))ds$$

and

$$x(t) = p(t) - \int_0^t R(t-s)p(s)ds + \int_0^t R(t-s)\left[x(s) - \frac{f(s, x(s))}{J}\right]ds$$

where J is an arbitrary positive constant and $R(t)$ is the resolvent of $JA(t)$. A classical problem is to prove that there is a positive solution when $A(t)$ satisfies certain conditions including $A(t) > 0$ and when $f(t, x) > 0$ for $x > 0$. This requires $p(t) > 0$ and often $p(t) - \int_0^t R(t-s)p(s)ds > 0$. We offer a **constructive** method of manufacturing an infinite collection of functions, p, for which this holds. Any linear combination of any of these with positive coefficients will yield a function, p, also satisfying this property. We show that the property also holds for all functions near that set. We then give a brief treatment of the non-convolution case and offer a theorem stating that if there is a solution then it is positive.

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1. INTRODUCTION

There are many problems in applied mathematics which are modeled by the scalar integral equation

$$x(t) = p(t) - \int_0^t A(t-s)f(s, x(s))ds$$

where $A(t) > 0$, $f(t, x) > 0$ if $x > 0$, and $p(t) > 0$, together with continuity conditions. Frequently we seek a positive solution on $[0, \infty)$.

Suppose that $p(t)$ is sharply decreasing. Then everything is against us. The integral is growing and we reasonably expect it to catch up with $p(t)$.

But our intuition is completely wrong. So long as f does not promote non-uniqueness, everything depends on positivity of

$$z(t) := p(t) - \int_0^t R(t-s)p(s)ds$$

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where $R > 0$ is the resolvent of $A(t)$ and this equation for $z(t)$ is the linear variation of parameters formula for

$$z(t) = p(t) - \int_0^t A(t-s)z(s)ds.$$

Justification of these statements in the convolution case is found in the recent paper [5] and we will give a brief development in the last section here for non-convolution problems. We want the reader to see the direction we are taking so here is a brief summary. The argument goes as follows.

It is inherent in the basic theory of integral equations of this type [14, pp. 189–193] that each integral equation has two equivalent and complementary forms. The equations

$$x(t) = p(t) - \int_0^t A(t-s)f(s,x(s))ds$$

and

$$x(t) = p(t) - \int_0^t R(t-s)p(s)ds + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds$$

share solutions when J is an arbitrary positive constant and R is the resolvent of $JA(t)$. We arrange to make the sign in the last integral of the second equation positive. The result is that the first equation bounds the solution above by $p(t)$, while the second equation bounds the solution below by $z(t)$. If $z(t) > 0$, then we have a positive solution. The introduction of J and the fact that the solution is bounded above by $p(t)$ makes the change of sign in the last integral of the second equation feasible.

In view of the role of p in the second equation our task is clear. We must obtain a comprehensive set of functions, p, for which

$$p(t) - \int_0^t R(t-s)p(s)ds > 0.$$

In this note we do exactly that. Two very simple theorems are given which generate the "positive half" of a vector space of functions for which it is true. The method is constructive and many more suitable functions can be generated in the same way. These include positive non-decreasing functions and positive functions with a continuous derivative which do not decrease too rapidly. A given positive $p(t)$ with continuous derivative can be written as the product of a non-decreasing and non-increasing functions. If the non-increasing part does not decrease too rapidly, then this product also satisfies $z(t) > 0$.

Much has been written about positive solutions and the reader is referred to the bibliography [1], [2], and discussion in [5]. In the linear case there are many results concerning Laplace transforms for positivity of z, as seen throughout the comprehensive treatment [13]. These are exceedingly nice when the transform and its inverse are readily obtained in a recognizable form. But none of them give us a feeling of just how extreme p can be and still retain $z(t) > 0$. That same text gives several results on positivity of z in the non-convolution case and those will be given later.

2. THE SETTING

In the theory of scalar integral equations of the form

$$x(t) = p(t) - \int_0^t A(t-s)f(s,x(s))ds \tag{2.1}$$

there emerge two relations with widening application. For a linear equation

$$z(t) = p(t) - \int_0^t A(t-s)z(s)ds \tag{2.2}$$

there is a resolvent equation

$$R(t) = A(t) - \int_0^t A(t-s)R(s)ds \tag{2.3}$$

yielding the linear variation-of-parameters

$$z(t) = p(t) - \int_0^t R(t-s)p(s)ds. \tag{2.4}$$

Our entire focus is on the case in which

$$R(t) > 0 \tag{2.5}$$

and

$$z(t) = p(t) - \int_0^t R(t-s)p(s)ds > 0 \tag{2.6}$$

under which there are many important applications, not the least of which is its use in obtaining a positive solution of (2.1).

We will be concerned with conditions on $A(t)$ given in (A1), (A2), and (A3), below, under which it will be true that (2.6) follows trivially when $p(t)$ is nondecreasing, but the real challenge comes when $p(t)$ is approaching zero very quickly.

Verifying (2.5) and (2.6) is an old quest and perhaps it is most appropriate to begin with a result from Miller [14, pp. 209, 212–213, 224] and Gripenberg [12] in which there are three conditions.

(A1) The function $A \in C(0,\infty) \cap L^1(0,1)$.

(A2) Also, $A(t)$ is positive and non-increasing for $t > 0$.

(A3) For each $T > 0$ the function $A(t)/A(t+T)$ is non-increasing in t for $0 < t < \infty$.

These three conditions ensure the existence of R on $(0,\infty)$ with

$$0 < R(t) \leq \frac{A(t)}{1 + \int_0^t A(s)ds}, \tag{2.7}$$

$$\int_0^\infty A(s)ds = \infty \implies \int_0^\infty R(s)ds = 1, \tag{2.8}$$

and

$$\int_0^\infty A(s)ds = \alpha < \infty \implies \int_0^\infty R(s)ds = \frac{\alpha}{1+\alpha}. \tag{2.9}$$

Moreover, if A is completely monotone, so is R.

It is of immediate and fundamental importance that when R is the resolvent of any A satisfying (A1)–(A3) and when $p(t)$ is continuous on $[0, \infty)$, positive, and nondecreasing then (2.5) and (2.6) hold

$$R(t) > 0 \text{ and } p(t) - \int_0^t R(t-s)p(s)ds > 0. \tag{2.10}$$

The second relation follows from factoring $p(t)$ out of the integral and using $0 < \int_0^t R(s)ds < 1$ for $t > 0$.

With this in hand we turn to (2.6) and a result from Gripenberg et al. [13, pp. 234 and 263] starting with assumption (2.5) and certain integrability conditions. It is stated for a nonconvolution kernel $C(t, s)$ and it shows that if p is positive and if for $u \le v \le t$ we have

$$p(v)C(t, u) \le p(t)C(v, u) \tag{2.11}$$

then

$$p(t) - \int_0^t R(t, s)p(s)ds > 0.$$

Our interest is in the convolution case for many reasons, not the least of which is the wide application and the handy result (2.7), (2.8), (2.9). Thus, if we write (2.11) with a convolution kernel we have

$$p(v)A(t-u) \le p(t)A(v-u).$$

If we set $u = 0$ we have

$$p(v)A(t) \le p(t)A(v). \tag{2.12}$$

Any investigator of fractional differential equations of Riemann-Liouville type inverted as

$$x(t) = x^0 t^{q-1} - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))ds \tag{2.13}$$

immediately suggests letting $p(t) = A(t)$ and have the perfect identity

$$A(v)A(t) \le A(t)A(v). \tag{2.14}$$

We offer a **constructive** way of manufacturing a positive vector space of functions for which (2.5) and (2.6) are true.

3. A KEY RESULT AND AN ALGEBRA

Problem 1: We start from (2.2) with $p(t) = A(t+T)$ and T a positive number in

$$z(t) = A(t+T) - \int_0^t A(t-s)z(s)ds,$$

realizing that if $T = 0$ then this is the equation for $R(t)$ which is a positive solution. We then apply the linear variation of parameters formula and ask if it is true that we still have

$$z(t) = A(t+T) - \int_0^t R(t-s)A(s+T)ds > 0 \tag{3.1}$$

This is key and the answer is in the affirmative and there is an algebra suggested by (3.2) and (3.3) which we will develop.

Theorem 3.1. *Let $T > 0$. If $A(t)$ satisfies (A1)–(A3) then* (3.1) *holds. Moreover, if p is continuous, positive, and non-decreasing then*

$$p(t) - \int_0^t R(t-s)p(s)ds > 0,$$

(which is (2.6)*)*

$$y(t) := A(t+T) + p(t) - \int_0^t R(t-s)[A(s+T) + p(s)]ds > 0, \tag{3.2}$$

and

$$u(t) := p(t)A(t+T) - \int_0^t R(t-s)[p(s)A(s+T)]ds > 0. \tag{3.3}$$

It follows that for each p which is continuous, positive, and non-decreasing and each pair of positive constants c_1 and c_2 then the function $\psi = c_1 p(t) + c_2 A(t+T)$ satisfies

$$\psi(t) - \int_0^t R(t-s)\psi(s)ds > 0. \tag{3.4}$$

Proof. Given $A(t)$ we find the resolvent from

$$R(t) = A(t) - \int_0^t A(t-s)R(s)ds = A(t) - \int_0^t R(t-s)A(s)ds$$

satisfying the properties (2.7)–(2.9). In particular, $R(t) > 0$. From this we have

$$\begin{aligned} 0 < R(t+T) &= A(t+T) - \int_0^{t+T} R(t+T-s)A(s)ds \\ &= A(t+T) - \int_{-T}^t R(t-s)A(s+T)ds \\ &= A(t+T) - \int_0^t R(t-s)A(s+T)ds - \int_{-T}^0 R(t-s)A(s+T)ds \\ &< A(t+T) - \int_0^t R(t-s)A(s+T)ds. \end{aligned}$$

This verifies (3.1). The proof of (3.2) with p non-decreasing follows from the addition of our first relation in this theorem to the one just obtained.

To verify (3.3) simply factor out $p(t)$ from the integral and use the above display as follows:

$$u(t) \geq p(t)A(t+T) - p(t)\int_0^t R(t-s)A(s+T)ds.$$

The same argument shows that (2.6) holds. The final conclusion is direct. □

Problem 2: The conditions (A1)–(A3) and $f(t,x) > 0$ for $x > 0$ characterize a wide class of problems from applied mathematics, but $p(t)$ is of a different category. We must be prepared to deal with given positive functions, $p(t)$. We need a class of functions g which are non-increasing with

$$g(t) - \int_0^t R(t-s)g(s)ds > 0.$$

These functions would then enter into positive linear combinations with continuous, positive, non-decreasing functions $p(t)$ and with $A(t+T)$ to yield a function L satisfying

$$L(t) - \int_0^t R(t-s)L(s)ds > 0.$$

Such combinations could then go a long way toward approximating a given positive function $p(t)$. Theorem 3.4 deals with approximations.

Main note Here is a theorem which does exactly that and it will then lead us to positive functions with continuous derivatives which will satisfy our basic relation (2.6). Two parts are given. Frequently we are only interested in a positive solution on a very long, but finite, interval $[0, E]$. This can result in a significant simplification when we come to a stability result at the end of the section. For every function, $p(t)$, yielding a positive solution, there will be a neighborhood of that function. Every function in that neighborhood will also generate a positive solution. This means that a $p(t)$ in our given equation (2.1) need only be **approximated** by the functions $p(t)$ which we generate. Once we approximate it, we have an exact measure of the sufficiency of the approximation.

Theorem 3.2. *Let A satisfy (A1)–(A3) with $A'(t)$ continuous on $(0,\infty)$ and $A'(t) < 0$.*

(i) For a given $E > 0$ let $g : [0, E] \to (0,\infty)$ have a continuous derivative and suppose there is a $T > 0$ such that

$$0 \leq t \leq E \implies 0 \geq g'(t) > A'(t+T), \quad g(0) > A(T). \tag{3.5}$$

Then

$$0 \leq t \leq E \implies g(t) - \int_0^t R(t-s)g(s)ds > 0. \tag{3.6}$$

(ii) If $g : [0,\infty) \to (0,\infty)$ has a continuous derivative and if there is a $T > 0$ with

$$0 \le t < \infty \implies 0 \ge g'(t) > A'(t+T), \quad g(0) > A(T) \tag{3.7}$$

then

$$0 \le t < \infty \implies g(t) - \int_0^t R(t-s)g(s)ds > 0. \tag{3.8}$$

Proof. We will prove (3.8) and the reader will see that (3.6) follows from a similar argument. Under the assumptions of (ii) we will show that the function

$$h : [0,\infty) \to (0,\infty) \tag{3.9}$$

with

$$g(t) = h(t)A(t+T) \tag{3.10}$$

is a non-decreasing function.

Note first that $g(0) = h(0)A(T)$ and from (3.7) $h(0) > 1$. From (3.10) we have

$$h'(t)A(t+T) + h(t)A'(t+T) = g'(t)$$

so that with $A(t+T) > 0$ we have

$$h'(t) + \frac{A'(t+T)}{A(t+T)}h(t) = \frac{g'(t)}{A(t+T)} \tag{3.11}$$

or by (3.7)

$$\begin{aligned} h'(t) &= \frac{g'(t)}{A(t+T)} - \frac{A'(t+T)}{A(t+T)}h(t) \\ &> \frac{A'(t+T)}{A(t+T)} - \frac{A'(t+T)}{A(t+T)}h(t) \\ &= \frac{A'(t+T)}{A(t+T)}(1-h(t)). \end{aligned} \tag{3.12}$$

As $h(0) > 1$, we see from the final line of (3.12) that $h'(0) > 0$ so h is increasing and from (3.12) we see that h will continue to increase. Recall that g was a given non-increasing function and from (3.11) with $g(0) = h(0)A(T) > A(T)$ we see that $h(0)$ is a fixed positive number so $h(t)$ is now uniquely determined with $h(t)$ increasing. That is, h satisfies a linear first order ordinary differential equation with continuous coefficients. Finally, (3.8) now follows from (3.1) and (3.3) with strict inequality since h is positive. □

Remark 3.3. With this theorem we see that (3.2) and (3.3) can be rewritten with $A(t+T)$ replaced by $g(t)$. An example of suitable $A(t)$ is $(t+T)^{q-1}$ with $0 < q < 1$. Thus, for $p(t)$ continuous, positive, and non-decreasing the functions $p(t)$, $g(t)$ and

$(t+T)^{q-1}$ can be taken as basis vectors so that for positive constants c_i we can form functions

$$L(t) = c_1 p(t) + c_2 g(t) + c_3 (t+T)^{q-1}$$

which all satisfy the relation

$$L(t) - \int_0^t R(t-s)L(s)ds > 0. \tag{3.13}$$

Problem 3: We now take Theorem 3.2 as a lemma toward an entirely different kind of problem. For example, suppose we have

$$p(t) = 2 + \sin t.$$

There are well defined and regular intervals on which $p(t)$ is increasing and the same for decreasing. With patience we can write

$$p(t) = a(t)g(t) \tag{3.15}$$

where $a(t)$ is non-decreasing, while $g(t)$ is non-increasing. We would then apply Theorem 3.2 to $g(t)$, writing it as

$$g(t) = h(t)A(t+T) \tag{3.16}$$

as in (3.10). Finally, we have

$$p(t) = a(t)h(t)A(t+T) \tag{3.17}$$

where $a(t)h(t)$ is now non-decreasing and (3.3) shows that our $p(t)$ will satisfy (2.6).

Here are details of one way of writing (3.15) when $p(t)$ is positive and has a continuous derivative. Let $0 < \beta < T$ and define

$$a(t) = \frac{p(0)}{A(T-\beta)} \exp \int_0^t [p'(s)_+/p(s)]ds$$

and

$$g(t) = A(T-\beta) \exp - \int_0^t [p'(s)_-/p(s)]ds$$

where

$$p'(t)_+ = \max[p'(t), 0] \text{ and } p'(t)_- = \max[-p'(t), 0]$$

so that

$$p'(t) = p'(t)_+ - p'(t)_-.$$

This decomposition was given in [7].

Notice that $g(0) = A(T-\beta) > A(T)$ in (3.7). Thus, to satisfy Theorem 3.2 we require that

$$g'(t) > A'(t+T)$$

where

$$0 \geq g'(t) = -A(T-\beta)[p'(t)_-/p(t)] \exp - \int_0^t [p'(s)_-/p(s)]ds > A'(t+T).$$

Organization

We can organize the results as follows. Let

$$\mathcal{U} = \left\{ u : [0,\infty) \to (0,\infty), u(t) - \int_0^t R(t-s)u(s)ds > 0 \right\}.$$

Let $A(t)$ satisfy (A1)–(A3) and let $R(t)$ be the fixed resolvent for that $A(t)$.

1. $\mathcal{P}$ is the set of all positive non-decreasing functions $p : [0,\infty) \to (0,\infty)$. See from Thm. 3.1 that $p \in \mathcal{U}$ and also cp if $c > 0$.

We will be forming "linear combinations" and we offer the following convention. The "constants" in those combinations will be functions $cp(t)$ where c is any positive real number and $p(t)$ is any positive continuous non-decreasing function. This combination will be mentioned in 3., 4., and 5. below.

2. $\mathcal{M} = \{A(t+T_\alpha) \colon T_\alpha > 0\}$. See from Theorem 3.1 that $A(t+T_\alpha) \in \mathcal{U}$.

3. $\mathcal{G}$ is the set of all functions $g : [0,\infty) \to (0,\infty)$ with $T > 0$ and

$$0 \geq g'(t) > A'(t+T), \quad g(0) > A(T).$$

See from Theorem 3.2 that $p \in \mathcal{P}$ and $g \in \mathcal{G}$ implies that all positive linear combinations (using Item 1. above) of $p(t)g(t)$ and $p(t)+g(t)$ are in $\mathcal{U}$.

4. $\mathcal{Q}$ is the set of all positive functions $q : [0,\infty) \to (0,\infty)$ with continuous derivatives for which there are positive constants T_α and $\beta < T_\alpha$ so that for

$$\begin{aligned} q(t) &= \frac{q(0)}{A(T_\alpha - \beta)} \exp \int_0^t [q'(s)_+/q(s)]ds \times A(T_\alpha - \beta) \exp - \int_0^t [q'(s)_-/q(s)]ds \\ &=: p^*(t)q^*(t) \end{aligned}$$

satisfies

$$(q^*)'(t) = -A(T_\alpha - \beta)[q'(t)_-/q(t)] \exp - \int_0^t [q'(s)/q(s)]ds > A'(t+T_\alpha).$$

See Problem 3. All positive linear combinations (in the sense of Item 1) of these functions are in $\mathcal{U}$.

5. Finally, all positive linear combinations (in the sense of Item 1) of all of the above functions are also in $\mathcal{U}$.

Stability

We have seen how to construct a substantial set of functions for which (2.6) holds. Yet, we cannot expect to be able to obtain a given function, $p(t)$, occurring in (2.1). Moreover, we frequently only need to be sure that (2.6) will hold on a very long interval $[0,E]$. We now show that any function q residing "close" to some p satisfying (2.6) on $[0,E]$ will also satisfy (2.6) on $[0,E]$.

Theorem 3.4. *Let $p, q : [0, E] \to (0, \infty)$ be continuous and suppose there is a number $\mu > 0$ with*

$$z(t) = p(t) - \int_0^t R(t-s)p(s)ds \geq \mu.$$

Let

$$Z(t) = q(t) - \int_0^t R(t-s)q(s)ds$$

with the norm on $[0, E]$ satisfying

$$\|p - q\| < \frac{\mu - \delta}{2}$$

where $\delta \in (0, \mu)$. Then $Z(t) > \delta$ for $0 \leq t \leq E$.

Proof. For $0 \leq t \leq E$ we have

$$\begin{aligned} |z(t) - Z(t)| &= |p(t) - q(t) + \int_0^t R(t-s)[q(s) - p(s)]ds| \\ &\leq \|p - q\| \left(1 + \int_0^t R(s)ds\right) \\ &< 2\|p - q\| < \mu - \delta. \end{aligned}$$

Thus, for each $t \in [0, E]$ we have

$$\begin{aligned} |z(t) - Z(t)| < \mu - \delta &\implies \delta - \mu < Z(t) - z(t) < \mu - \delta \\ &\implies z(t) + \delta - \mu < Z(t) < \mu - \delta + z(t) \\ &\implies \mu + \delta - \mu < z(t) + \delta - \mu < Z(t) \\ &\implies \delta < Z(t). \end{aligned}$$

□

In the second display above we note that the $1 + \int_0^E R(s)ds$ was replaced by 2 since we cannot evaluate the integral well. However, this means that if $z(t) > \mu$ on $[0, \infty)$ then our estimate of $Z(t) > \delta$ holds on $[0, \infty)$.

4. COMPLEMENTARY EQUATIONS, POSITIVE SOLUTIONS

As noted in the introduction, one of the most useful facts in the study of qualitative properties of integral equations is that

$$x(t) = p(t) - \int_0^t A(t-s)f(s, x(s))ds$$

and

$$x(t) = p(t) - \int_0^t R(t-s)p(s)ds + \int_0^t R(t-s)\left[x(s) - \frac{f(s, x(s))}{J}\right]ds$$

share solutions when J is an arbitrary positive constant and R is now the resolvent of $JA(t)$. This follows from a study of the classical introduction to integral equations beginning with the resolvent, the linear variation of parameters formula, and the non-linear variation of parameters formula. We follow the presentation of Miller [14, pp. 189–191] and translate it into the pair of equations in the non-convolution case which is more complicated because we do not place any restrictions on the kernel other than continuity and positivity of the resolvent.

We explained in the introduction just how this pair could be used to show that there is a positive solution and the reader may find many applications in [3], [4], [5], [6], [8], [9], [10], and [11].

In this section we show just how critical (2.6) is for the nonlinear equation (2.1) to have a positive solution. We are going to state this for a non-convolution problem. It would require much space to fill in all of the details for non-convolution problems and we would again refer to Miller to see what is needed. We will state our result in the form that if there is a solution, then it is positive. In this way, the reader can supply whatever assumptions are needed to ensure that a solution exists. All necessary conditions are present if (A1)–(A3) hold. In [5] we use Schaefer's fixed point theorem to show existence of a solution in case $A(t) = t^{q-1}$ where $0 < q < 1$ as this is the case in many applied problems and in the inversion of fractional differential equations of both Caputo and Riemann-Liouville type.

Consider the scalar equation

$$x(t) = p(t) - \int_0^t C(t,s)f(s,x(s))ds \tag{4.1}$$

in which $C : (0,\infty) \times (0,\infty) \to (0,\infty)$ is continuous as are $p : [0,\infty) \to (0,\infty)$ and $f : [0,\infty) \times \Re \to \Re$ with $f(t,x) > 0$ if $x > 0$. We suppose that for each $E > 0$ there is a $J > 0$ such that

$$\left[0 \le t \le E, \quad 0 \le x \le \max_{0\le t\le E} p(t)\right] \implies 0 < \frac{f(t,x)}{Jx} < 1. \tag{4.2}$$

Let

$$R(t,s) = JC(t,s) - \int_s^t JC(t,u)R(u,s)du$$

and assume that R is continuous and **positive** for $0 < s < t < \infty$. Thus, R is the resolvent of $JC(t,s)$ and the solution of

$$Z(t) = p(t) - \int_0^t JC(t,s)Z(s)ds \tag{4.3}$$

is

$$Z(t) = p(t) - \int_0^t R(t,s)p(s)ds, \tag{4.4}$$

as seen in Miller [14, p. 191].

Conditions in the non-convolution case under which both $R(t,s)$ and $Z(t)$ are positive are found in Gripenberg et al. [13, pp. 259–264] and Miller [14, p. 217]. We have, of course, given extensive discussion when (A1)–(A3) hold in the convolution case.

It is important to note here that the linear variation of parameters result (4.4) depends on change of order of integration

$$\int_0^t R(t,u)\Big[\int_0^u C(u,s)Z(s)ds\Big]du = \int_0^t \Big[\int_s^t R(t,u)C(u,s)du\Big]Z(s)ds$$

as seen in Miller [14, p. 190]. A similar upcoming relation is needed in the nonlinear variation of parameters formula.

In the proof of the following theorem we will show that (4.1) can be written in an equivalent way which has been shown in recent years to reveal many interesting properties of the integral equation. It becomes clear that (4.1) and its complementary equation can profitably be written together whenever properties are being studied.

Theorem 4.1. *Let* (4.2) *hold, let* (4.1) *have a continuous solution* x *on an interval* $[0, E]$*, and let* $Z(t)$ *in* (4.4) *be continuous and positive on that interval. Suppose also that for the continuous solution* x *the order of integration of*

$$\int_0^t R(t,s)\int_0^s C(s,u)f(u,x(u))duds$$

can be changed. Then that solution is positive.

Proof. The change of order of integration is used in the variation of parameters which we will be using and it occurs in [14, pp. 190, 192]. Sufficient conditions are given in the Hobson-Tonelli test [15, p. 93].

As $x(0) = p(0) > 0$, there is a maximal interval $[0, E^*)$ on which $x(t) > 0$ (so $f(t, x(t)) > 0$). By way of contradiction, we suppose that $E^* \le E$ and that $x(E^*) = 0$. As $C(t, s)$ and $f(s, x(s))$ are positive on $(0, E^*)$ from (4.1) we have $0 < x(t) \le p(t)$ on $[0, E^*)$.

We now prepare to transform (4.1) into

$$x(t) = p(t) - \int_0^t R(t,s)p(s)ds + \int_0^t R(t,s)\Big[x(s) - \frac{f(s,x(s))}{J}\Big]ds. \tag{4.5}$$

Here are the details. Write (4.1) as

$$\begin{aligned} x(t) &= p(t) - \int_0^t C(t,s)[Jx(s) - Jx(s) + f(s,x(s))]ds \\ &= p(t) - \int_0^t JC(t,x)x(s) + \int_0^t JC(t,s)\Big[x(s) - \frac{f(s,x(s))}{J}\Big]ds. \end{aligned}$$

The linear part is (4.3) and by the linear variation of parameters formula we have (4.4). The non-linear variation of parameters formula now gives us (4.5).

By (4.1) so long as $x(t) > 0$, then $x(t) \leq p(t)$ so we examine (4.5) at $t < E^*$:

$$x(t) = Z(t) + \int_0^t R(t,s)x(s)\left[1 - \frac{f(s,x(s))}{Jx(s)}\right]ds.$$

The integrand is positive on $[0, E^*)$ by (4.2) and so the integral is positive at $t = E^*$. As $Z(t)$ is also positive, we have a contradiction to $x(E^*) = 0$. □

REFERENCES

[1] Ravi P. Agarwal, Donal O'Regan, and Patricia J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht, 1999.

[2] Ravi P. Agarwal, Donal O'Regan, and Patricia J. Y. Wong, *Constant Sign Solutions of Systems of Integral Equations*, Springer, Cham, 2013.

[3] L. C. Becker, T. A. Burton, and I. K. Purnaras, Complementary equations: A fractional differential equation and a Volterra integral equation, *Electron. J. Qual. Theory Differ. Equ.*, No. 12:1-24, 2015.

[4] L. C. Becker, T. A. Burton, and I. K. Purnaras, An inversion of a fractional differential equation and fixed points, *Nonlinear Dynamics and Systems Theory* **15** No. 4:242-271, 2015.

[5] L. C. Becker, T. A. Burton, and I. K. Purnaras, Integral and fractional equations, positive solutions, and Schaefer's fixed point theorem, *Opuscula Math.* **36**, no. 4:431-458, 2015.

[6] T. A. Burton, Fractional differential equations and Lyapunov functionals, *Nonlinear Anal.: TMA* **74**:5648-5662, 2011.

[7] T. A. Burton and R. C. Grimmer, On the asymptotic behavior of solutions of $x'' + a(t)f(x) = e(t)$, *Pacific J. Math.* **41**, no. 1:43-55, 1972.

[8] T. A. Burton and Bo Zhang, Fractional equations and generalizations of Schaefer's and Krasnoselskii's fixed point theorems, *Nonlinear Anal.: TMA* **75**:6485-6495, 2012.

[9] T. A. Burton and Bo Zhang, Fixed points and fractional differential equations: Examples, *Fixed Point Theory* **14**, no. 2:313-326, 2013.

[10] T. A. Burton and Bo Zhang, L^p-solutions of fractional differential equations, *Nonlinear Studies* **19**, no. 2:161-177, 2012.

[11] T. A. Burton and Bo Zhang, A Schauder-type fixed point theorem, *J. Math. Anal. Appl.* **417**:552-558, 2014.

[12] Gripenberg, G, On positive, nonincreasing resolvent of Volterra equations, *J. Differential Equations* **30**:380-390, 1978.

[13] Gripenberg, G, Londen, S-O., and Staffans, O., *Volterra Integral and Functional Equations*, Cambridge Univ. Press, Cambridge, 1990.

[14] Miller, R. K., *Nonlinear Volterra Integral Equations*, Benjamin, Menlo Park, CA, 1971.

[15] Natanson, I. P., *Theory of Functions of a Real Variable, Vol. II*, Frederick Ungar Publishing Co., New York, 1961.