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EXISTENCE, UNIQUENESS AND STABILITY RESULTS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS ON TIME SCALES

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ABSTRACT. This paper studies functional differential equations on time scales and develops the theory of existence and uniqueness of solutions by utilizing the induction principle and Gronwall's inequality on time scales. Further more, it establishes several criteria on uniform (asymptotic) stability and exponential stability using Lyapunov functions and Razumikhin technique. These criteria include some known results as special cases. Numerical examples are presented to illustrate the stability criteria.

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1. INTRODUCTION

Due to the wide existence of time-delays in many physical processes, functional differential equations have been used in various applied areas, such as control theory [1, 2], neural networks [3], economics, geosciences [4], among many others. Consequently, the theory of functional differential equations has attracted substantial attentions [5, 6]. On the other hand, functional difference equations are also used to model many real world phenomenon [7]. A large number of theoretical results on functional difference equations have been obtained [8, 9]. Stability is one of the most important issues in the study of both functional differential equations and functional difference equations. The Lyapunov-Razumikhin method is known as a very powerful and effective approach to investigate the stability properties of differential (difference) equations with time delay(see, e.g., [10, 11, 12]). This method has been successfully applied to analyze the stability of functional differential equations and design various controllers to stabilize continuous and discrete systems subjected to time-delay [13, 14, 15, 16, 17, 18].

From the mathematical modeling point of view, it is more realistic to model a dynamic process by a differential equation on hybrid time domain, which incorporates both continuous and discrete times, namely, time scale. The theory of time scales was first introduced by Stefan Hilger [19] in 1988 to unify the continuous and discrete analysis. While the theories of functional differential equations and delay difference

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equations are well developed, it is natural for us to investigate the functional differential equations on time scales. Recently, various non-delay dynamical systems on time scales have been studied(see, e.g., [20, 21, 22, 23]). However, as far as we know, the investigation of general functional differential equations on time scales is not much, and no Razumikhin-type results about the stability of functional differential equations on time scales have been reported. In this paper, we shall investigate the fundamental theory and stability of general retarded functional differential equations on time scales.

The rest of this paper is organized as follows. In the sequel of this section, we present fundamental concepts and basic results of the time scale theory. In Section 3, we formulate the problem and introduce some corresponding notations and definitions. In Section 4, local and global existence, uniqueness, and extended existence of solutions are presented and proved. In Section 5, several Razumikhin-type stability results are established based on the use of Lyapunov function method and Razumikhin technique. These results are used to study a class of linear delay differential equations on time scales. Some sufficient conditions which can be easily checked are derived to ensure the uniform asymptotic stability. The exponential stability is also studied in Section 6 by the Lyapunov-Razumikhin method. Two examples, along with numerical simulations, are given in Section 7 to show the effectiveness of the theoretical results. Finally, conclusions and some future directions are drawn in Section 8.

Notation. Throughout this paper, Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^+$ the set of nonnegative real numbers, $\mathbb{Z}$ the set of integers and $\mathbb{R}^n$ the set of the n-dimensional real space equipped with Euclidean norm $\|\cdot\|$. Let $\mathbb{R}^{n\times m}$ denote the set of $n\times m$ dimensional matrices. The superscript '$\mathtt{T}$' stands for the transpose of a matrix; $I\in\mathbb{R}^{n\times n}$ is the identity matrix. For symmetric matrix $P\in R^{n\times n}$, let $\lambda_{\max}(P)$ denote the largest eigenvalue of P.

2. PRELIMINARIES

In this section, we briefly introduce some basic definitions and results concerning time scales for later use. For more details about the theory of time scales, refer to [24, 25, 26].

Let $\mathbb{T}$ be an arbitrary nonempty closed subset of $\mathbb{R}$. We assume that $\mathbb{T}$ is a topological space with relative topology induced from $\mathbb{R}$. Then, $\mathbb{T}$ is called a time scale.

Definition 2.1. The mappings $\sigma,\rho:\mathbb{T}\to\mathbb{T}$ defined as

$$\sigma(t)=\inf\{s\in\mathbb{T}\mid s>t\}$$

$$\rho(t)=\sup\{s\in\mathbb{T}\mid s<t\}$$

are called forward and backward jump operators, respectively.

A non-maximal element $t \in \mathbb{T}$ is called right-scattered if $\sigma(t) > t$ and right-dense if $\sigma(t) = t$. A non-minimal element $t \in \mathbb{T}$ is called left-scattered if $\rho(t) < t$ and left-dense if $\rho(t) = t$. If $\mathbb{T}$ has a left-scattered maximum m, then we define $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$, otherwise, $\mathbb{T}^k = \mathbb{T}$.

Definition 2.2. The graininess function $\mu : \mathbb{T} \to \mathbb{R}^+$ is defined by

$$\mu(t) = \sigma(t) - t.$$

Definition 2.3. For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative $y^{\Delta}(t)$ of $y(t)$, to be the number (when it exists) with the property that for any $\varepsilon > 0$, there is a neigborhood U of t (i.e., $U = (t - \delta, t + \delta) \bigcap \mathbb{T}$ for some $\delta > 0$) such that

$$|y(\sigma(t)) - y(s) - y^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|, \quad \textit{for all } s \in U.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$. If f is continuous at each right-dense point and each left-dense point, f is said to be continuous on $\mathbb{T}$. If $a, b \in \mathbb{T}$, then we define the interval $[a, b]$ on $\mathbb{T}$ by $[a, b] := \{t \in \mathbb{T} \mid a \leq t \leq b\}$. Open intervals and half-open intervals can be defined similarly.

Definition 2.4. Let $f \in C_{rd}$. A function $g : \mathbb{T} \to \mathbb{R}$ is called the antiderivative of f on $\mathbb{T}$ if it is differentiable on $\mathbb{T}$ and satisfies $g^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}$. In this case, we define

$$\int_a^t f(s)\Delta s = g(t) - g(a), \; a, t \in \mathbb{T}.$$

We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd}\mathcal{R} = C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$, and the set of all positively regressive elements of $C_{rd}\mathcal{R}$ is denoted by $C_{rd}\mathcal{R}^+ = C_{rd}\mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in C_{rd}\mathcal{R} \mid 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Definition 2.5. We say that a function $m : \mathbb{T} \to \mathbb{R}$ is right-nondecreasing at a point $t \in \mathbb{T}$ provided

(i) if t is right-scattered, then $m(\sigma(t)) \geq m(t)$;
(ii) if t is right-dense, then there is a neighborhood U of t such that

$$m(s) \geq m(t), \text{ for all } s \in U \text{ with } s > t.$$

Similarly, we say that m is right-nonincreasing if above in (i) $m(\sigma(t)) \leq m(t)$ and in (ii) $m(s) \leq m(t)$. If m is right-nondecreasing (right-nonincreasing) at every $t \in \mathbb{T}$, we say that m is right-nondecreasing (right-nonincreasing) on $\mathbb{T}$.

The following lemmas will be used in the discussion of existence of solutions and stability of the functional differential equations on time scales proposed in Section 3.

Lemma 2.6 ([20])**.** *Let $m \in C_{rd}(\mathbb{T}, \mathbb{R})$. Then $m(t)$ is right-nondecreasing (right-nonincreasing) on $\mathbb{T}$ if and only if $D^+m^\Delta(t) \geq 0$ ($D^+m^\Delta(t) \leq 0$) for every $t \in \mathbb{T}$, where*

$$D^+m^\Delta(t) = \begin{cases} \frac{m(\sigma(t))-m(t)}{\mu(t)}, & \sigma(t) > t, \\ \limsup_{s\to t^+} \frac{m(s)-m(t)}{s-t}, & \sigma(t) = t. \end{cases}$$

Lemma 2.7 (Chain Rule [24])**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and the formula*

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^\Delta(t))dh \right\} g^\Delta(t)$$

holds.

Lemma 2.8 (Induction Principle [24])**.** *Let $t_0 \in \mathbb{T}$ and assume that*

$$\{S(t) : t \in [t_0, \infty)\}$$

is a family of statements satisfying:

(I) *The statement $S(t_0)$ is true.*
(II) *If $t \in [t_0, \infty)$ is right-scattered and $S(t)$ is true, then $S(\sigma(t))$ is also true.*
(III) *If $t \in [t_0, \infty)$ is right-dense and $S(t)$ is true, then there is a neighbourhood U of t such that $S(t)$ is true for all $s \in U \cap (t_0, \infty)$.*
(IV) *If $t \in (t_0, \infty)$ is left-dense and $S(t)$ is true for all $s \in [t_0, t)$, then $S(t)$ is true.*

Then $S(t)$ is true for all $t \in [t_0, \infty)$.

3. PROBLEM FORMULATION

For any $a, b \in \mathbb{R}$ and $a < b$, let $[a, b]_\mathbb{R}$ be the closed interval on $\mathbb{R}$. For any $\phi \in C([-\tau, 0]_\mathbb{R}, \mathbb{R}^n)$ with some $\tau > 0$, we define $\|\phi\|_\tau = \sup_{s\in[-\tau,0]_\mathbb{R}} \|\phi(s)\|$. In this paper, we define the operator $\theta : \mathbb{R} \to \mathbb{T}$ as follows

$$\theta(t) = \inf\{s \in \mathbb{T} \mid s \geq t\}.$$

It should be noticed that the operator θ is different from the forward jump operator σ, since these two operators have different domains.

Consider the retarded functional differential equations on time scale $\mathbb{T}$

$$\begin{cases} x^\Delta(t) &= f(t, x_t),\ t \in \mathbb{T}_0 = [t_0, \infty)_\mathbb{R} \bigcap \mathbb{T}^k, \\ x_{t_0} &= \phi,\ t_0 \in \mathbb{T}, \end{cases} \tag{3.1}$$

where $f : \mathbb{T} \times C([-\tau, 0]_{\mathbb{R}}, \mathbb{R}^n) \to \mathbb{R}^n$, $\phi \in C([-\tau, 0]_{\mathbb{R}}, \mathbb{R}^n)$, and $x_t : [-\tau, 0]_{\mathbb{R}} \to \mathbb{R}^n$ is defined by $x_t(s) = x(\theta(t+s))$ for $s \in [-\tau, 0]_{\mathbb{R}}$. One of the main focus of this paper is to study the existence and uniqueness of solutions to equation (3.1). Hence, a precise meaning of a solution to equation (3.1) is given in the following definition.

Definition 3.1. The map $x \in C([t_0 - \tau, t_0 + \gamma], \mathbb{R}^n)$ is said to be a solution of (3.1) on $[t_0, t_0 + \gamma]$ if $x(t)$ is an antiderivative of $f(t, x_t)$ on $[t_0, t_0 + \gamma] \cap \mathbb{T}^k$, and satisfies $x_t(s) = \phi(s)$ for all $s \in [-\tau, 0]_{\mathbb{R}}$.

For the continuous-time functional differential equations, i.e., (3.1) on time scale $\mathbb{T} = \mathbb{R}$, it is well known that if $x \in C([t_0 - \tau, t_0 + \gamma], \mathbb{R}^n)$ then x_t is continuous function of t for $t \in [t_0, t_0 + \gamma]$. However, the continuity of x_t could be destroyed when different time scales and time delays are considered. To see how the continuity of x_t is destroyed, consider the time scale $\mathbb{T} = [0, 1]_{\mathbb{R}} \cup [2, 3]_{\mathbb{R}}$, $t_0 = 1$, and the function

$$x(t) = \begin{cases} 1, & \text{if } x \in [0, 1]_{\mathbb{R}}, \\ 2, & \text{if } x \in [2, 3]_{\mathbb{R}}. \end{cases}$$

It can be seen that x is rd-continuous on $\mathbb{T}$. Suppose $t_1 = 2$ and $t_2 \in (2, 3)_{\mathbb{R}}$, then we will investigate the continuity of x_t at t_1.

- If $0 < \tau < 1$, then $\theta(t_1 + s) = 2$ and $\theta(t_2 + s) = 2$ for all $s \in [-\tau, 0]_{\mathbb{R}}$. Thus, $|x(\theta(t_1 + s)) - x(\theta(t_2 + s))| = |x(2) - x(2)| = 0$ for $s \in [-\tau, 0]_{\mathbb{R}}$, i.e., $\|x_{t_1} - x_{t_2}\|_\tau = 0$, which implies that x_t is rd-continuous at $t_1 = 2$.
- If $\tau = 1$, then $\theta(t_1+s) = \begin{cases} 1, & s = -\tau \\ 2, & \text{otherwise} \end{cases}$, and $\theta(t_2+s) = 2$ for all $s \in [-\tau, 0]_{\mathbb{R}}$. Therefore, $|x(\theta(t_1 + s)) - x(\theta(t_2 + s))| = \begin{cases} 1, & s = -\tau \\ 0, & \text{otherwise} \end{cases}$, i.e., $\|x_{t_1} - x_{t_2}\|_\tau = 1$, which implies that x_t is not rd-continuous since t_1 is right-dense and left-scattered.

To generalize the classical fundamental results for continuous-time functional differential equations, additional conditions on time scales and the size of time-delay are required to guarantee the continuity of x_t. We prove this in the following lemma.

Lemma 3.2. *Assume*

(i) $x \in C_{rd}([t_0 - \tau, t_0 + \alpha], \mathbb{R}^n)$*;*
(ii) $\theta(t + s)$ *is right-dense for any* $s \in [-\tau, 0]_{\mathbb{R}}$*, if* t *is a right-dense point,*

then, x_t *is a rd-continuous function of* t *for* $t \in [t_0, t_0 + \alpha]$.

Proof. To show x_t is rd-continuous at t, we need to prove x_t is continuous at all right-dense points, and the left limit of x_t exists and is finite at all left-dense points.

Let t^* denote a left-dense but right-scattered point, then the left limit of x exists and is finite at t^*. Define a function $\bar{x} : [t_0 - \tau, t_0 + \alpha] \times \mathbb{R}^n$ as follows

$$\bar{x}(t) = \begin{cases} x(t^-), & \text{if } t \text{ is left-dense,} \\ x(t), & \text{otherwise.} \end{cases}$$

Then, $\bar{x}$ is different from x only when t is left-dense point, which implies that $\bar{x}$ is continuous on $[t_0 - \tau, t_0 + \alpha]$. For any given $h < 0$, there exists a $\tilde{s} \in [-\tau, 0]_{\mathbb{R}}$ such that

$$\begin{aligned} \|x_{t_*+h} - \bar{x}_{t_*}\|_\tau &= \sup_{s\in[-\tau,0]_{\mathbb{R}}} |x(\theta(t_* + h + s)) - \bar{x}(\theta(t_* + s))| \\ &= |x(\theta(t_* + h + \tilde{s})) - \bar{x}(\theta(t_* + \tilde{s}))|. \end{aligned}$$

If $\theta(t^* + \tilde{s})$ is left-scattered, then

$$\begin{aligned} \lim_{h\to 0^-} |x(\theta(t^* + h + \tilde{s})) - \bar{x}(\theta(t^* + \tilde{s}))| &= \lim_{h\to 0^-} |x(\theta(t^* + \tilde{s})) - \bar{x}(\theta(t^* + \tilde{s}))| \\ &= \lim_{h\to 0^-} |\bar{x}(\theta(t^* + \tilde{s})) - \bar{x}(\theta(t^* + \tilde{s}))|, \end{aligned}$$

which is zero and implies that $\lim_{h\to 0^-} \|x_{t^*+h} - \bar{x}_{t^*}\|_\tau = 0$. On the other hand, if $\theta(t^* + \tilde{s})$ is left-dense, then the left limit of x at $\theta(t^* + \tilde{s})$ exits which is $\bar{x}(\theta(t^* + \tilde{s}))$ according to the definition of function $\bar{x}$. Then, $\lim_{h\to 0^-} |x(\theta(t^* + h + \tilde{s})) - \bar{x}(\theta(t^* + \tilde{s}))| = \lim_{h\to 0^-} |\bar{x}(\theta(t^* + h + \tilde{s})) - \bar{x}(\theta(t^* + \tilde{s}))| = 0$. Hence, the left limit of x_t at t^* exists, and is given by $\bar{x}_{t^*}$. Clearly, $\|\bar{x}_{t^*}\|_\tau$ is bounded.

Therefore, the left limit of x_t at left dense point exists and is finite. Next, we shall show that x_t is continuous at right-dense points.

Denotes t_* a right-dense point, then x is continuous at t_*, and $\theta(t_* + s)$ is right-dense for any $s \in [-\tau, 0]_{\mathbb{R}}$. For a given small $h > 0$, there exists a $\bar{s} \in [-\tau, 0]_{\mathbb{R}}$ such that

$$\begin{aligned} \|x_{t_*+h} - x_{t_*}\|_\tau &= \sup_{s\in[-\tau,0]_{\mathbb{R}}} |x(\theta(t_* + h + s)) - x(\theta(t_* + s))| \\ &= |x(\theta(t_* + h + \bar{s})) - x(\theta(t_* + \bar{s}))|. \end{aligned}$$

Thus, the continuity of x at $\theta(t_*+\bar{s})$ implies that $\lim_{h^+\to 0} \|x_{t_*+h}-x_{t_*}\|_\tau = 0$. Similarly, we have $\lim_{h^-\to 0} \|x_{t_*+h} - x_{t_*}\|_\tau = 0$ if t_* is also left-dense. Therefore, x_t is continuous at t if it is right-dense point.

Based on the above discussion, we see that x_t is rd-continuous in t. □

In Lemma 3.2, if x is a solution of (3.1) on $[t_0, t_0+\alpha]$, then $x : [t_0-\tau, t_0+\alpha] \to \mathbb{R}^n$ is a continuous function. Hence, from the proof of Lemma 3.2, we can see that x_t is a continuous function of t on $[t_0, t_0 + \alpha]$, which coincides with the classical result for $\mathbb{T} = \mathbb{R}$ (Lemma 2.1 in [5]). Another main objective of this paper is to find sufficient conditions to ensure system (3.1)'s stability property, which is formulated in the following definition. We assume that $f(t, 0) \equiv 0$ for all $t \in \mathbb{T}$, so that system (3.1) admits the trivial solution.

Definition 3.3. The trivial solution of system (3.1) is said to be

(S1) **stable** if for every $\varepsilon > 0$ and $t_0 \in \mathbb{T}$, there exists some $\delta = \delta(\varepsilon, t_0) > 0$ such that $\|\phi\|_\tau \leq \delta$ implies that $\|x(t)\| < \varepsilon$ for all $t \geq t_0$, where $t \in \mathbb{T}$, and $x(t) = x(t, t_0, \phi)$.
(S2) **uniformly stable** if δ in (S_1) is independent of t_0.
(S3) **asymptotically stable** if it is stable and there is a positive constant $c = c(t_0)$ such that $\|x(t)\| \to 0$ as $t \to \infty$, for all $\|\phi\|_\tau < c$.
(S4) **uniformly asymptotically stable** if it is uniformly stable and for any $\eta > 0$, there exist $\delta = \delta(\eta) > 0$ and $T = T(\eta) > 0$ such that $\|\phi\|_\tau < \delta$ implies $\|x(t)\| < \eta$, for all $t \in [t_0 + T, \infty)_\mathbb{T}$.
(S5) **globally exponentially stable** if, for any initial data $x_{t_0} = \phi$, there exist constants $\alpha > 0$, $M \geq 1$ such that $\|x(t, t_0, \phi)\| \leq M\|\phi\|_\tau e^{-\alpha(t-t_0)}$, for all $t \geq t_0$.

Definition 3.4. Given a function $V \in C_{rd}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$, the upper right-hand delta derivative of V with respect to system (3.1) is defined by

$$D^+V^\Delta(t, x(t)) = \begin{cases} \frac{V(\sigma(t),x(\sigma(t)))-V(t,x(t))}{\mu(t)}, & \sigma(t) > t, \\ \limsup_{s\to t^+} \frac{V(s,x(t)+(s-t)f(t,x_t))-V(t,x(t))}{s-t}, & \sigma(t) = t, \end{cases}$$

where $C_{rd}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n) = \{V : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n \mid V(t, x)$ is rd-continuous in t, and continuous in x for all $(t, x) \in \mathbb{T} \times \mathbb{R}^n\}$.

4. EXISTENCE AND UNIQUENESS RESULTS

If x_t is rd-continuous in t and $f(t, \psi)$ is assumed to be rd-continuous in its first variable and continuous in its second variable, then the composite function $f(t, x_t)$ is also rd-continuous in t. Since the continuity of x_t can be guaranteed by Lemma 3.2, we are in the position to generalize the existence and uniqueness results for continuous-time functional differential equations to the results for functional differential equations on general time scales. The induction principle on time scales (Lemma 2.8) will be applied to establish the generalized results. Throughout this section, we assume that all conditions of Lemma 3.2 are satisfied, constant $\alpha \in \mathbb{T}$, and denote $\Omega = [t_0, \alpha) \times C([-\tau, 0]_\mathbb{R}, \mathbb{R}^n)$.

Theorem 4.1 (Local Existence)**.** *Let $f : \Omega \to \mathbb{R}^n$ be rd-continuous in its first variable and continuous in its second variable, then for each $\phi \in C([-\tau, 0]_\mathbb{R}, \mathbb{R}^n)$, there exists a solution $x(t) = x(t; t_0, \phi)$ of the initial value problem* (3.1) *on $[t_0 - \tau, \beta)$ for some $\beta \in (t_0, \alpha]$.*

Proof. For any $r \in [t_0, \beta)$, define the following mapping

$$f^{r]}(t, x_t) = \begin{cases} f(t, x_t), & \text{if } t \in [t_0, r); \\ f(r^-, x_{r^-}), & \text{if } t = r, \end{cases}$$

where

$$x_{r^-}(s) = \begin{cases} x_r(s), & \text{if } s \in [-\tau, 0); \\ x(r^-), & \text{if } s = 0, \end{cases}$$

We will apply the induction principle (Lemma 2.8) to the statement $A(r)$ defined as follows:

$A(r)$: the initial value problem (IVP)

$$\begin{cases} x^\Delta &= f^{r]}(t, x_t), \text{ for } t \in [t_0, r], \\ x_{t_0} &= \phi, \end{cases} \tag{4.1}$$

has a solution $x^{r]}(t)$ on $[t_0, r]$.

Note that once we have shown this, the claim of the local existence result follows.

(I) The statement $A(t_0)$ is trivially true since $x^{t_0]}(t_0) = \phi(0)$.

(II) Let r be right-scattered and $A(r)$ be true, i.e., the IVP (4.1) has a solution $x^{r]}(t)$ on $[t_0, r]$. Define a function $x^{\sigma(r)]}$ as follows:

$$x^{\sigma(r)]}(t) = \begin{cases} x^{r]}(t), & \text{if } t \in [t_0, r], \\ x^{r]}(r) + \mu(r) f^{r]}(r, x_r), & \text{if } t = \sigma(r). \end{cases}$$

Then $x^{\sigma(r)]}(t)$ is a solution of IVP (4.1) on $[t_0, \sigma(r)]$.

(III) Let r be right-dense and $A(r)$ be true. Then there exists $\delta > 0$ such that $[r, r+\delta]_{\mathbb{R}} \subset [t_0, \alpha)$, and by the classical existence result for continuous-time functional differential equations, the following IVP has a solution $y(t)$

$$\begin{cases} y^\Delta &= y'(t) = f(t, y_t), \text{ for } t \in [r, r+\delta], \\ y_r &= x_r^{r]}, \end{cases}$$

on $[r, r+\bar{\delta}]$ for some $0 < \bar{\delta} \le \delta$. Then for any $s \in [r, r+\delta]$ the mapping defined by

$$x^{s]}(t) = \begin{cases} x^{r]}(t), & \text{if } t \in [t_0, r], \\ y(t), & \text{if } r < t \le s, \end{cases}$$

is a solution of IVP(4.1) on $[t_0, s]$, i.e., $A(s)$ is true.

(IV) Let r be left-dense and $A(s)$ be true for all $s \in [t_0, r)$. Then there exists $\varepsilon > 0$ such that $[r - \varepsilon, r]_{\mathbb{R}} \subset [t_0, \alpha)$. For any $s \in [r - \varepsilon/2, r)$ the solution of IVP(4.1) has a solution $x^{s]}(t)$ on $[t_0, s]$ defined by

$$x^{s]}(t) = x(t_0) + \int_{t_0}^{t} f^{s]}(\xi, x_\xi)\Delta\xi, \ t \in [t_0, s].$$

It can be seen that s is left-dense and right-dense for any $s \in [r - \varepsilon/2, r)$, then the rd-continuity of $f(t, x_t)$ as a function of t implies that

$$x^{s]}(t) = x(t_0) + \int_{t_0}^{r-\varepsilon/2} f^{s]}(\xi, x_\xi)\Delta\xi + \int_{r-\varepsilon/2}^{t} f^{s]}(\xi, x_\xi) d\xi,$$

$$= \quad x(t_0) + \int_{t_0}^{r-\varepsilon/2} f(\xi, x_\xi)\Delta\xi + \int_{r-\varepsilon/2}^{t} f(\xi, x_\xi)d\xi,$$

for $t \in [t_0, s]$ and $s \in [r - \varepsilon/2, r)$. Since r is left-dense and f is rd-continuous, the limit $\lim_{t\to r^-} f(t, x_t)$ exists and is finite. Define a mapping

$$x^{r]}(t) = \begin{cases} x^{(r-\varepsilon/2)]}(t), & \text{if } t \in [t_0, r - \varepsilon/2], \\ x^{(r-\varepsilon/2)]}(r - \varepsilon/2) + \int_{r-\varepsilon/2}^{t} f(\xi, x_\xi)d\xi, & \text{if } r - \varepsilon/2 < t < r, \\ x^{(r-\varepsilon/2)]}(r - \varepsilon/2) + \int_{r-\varepsilon/2}^{r} f^{r]}(\xi, x_\xi)d\xi, & \text{if } t = r, \end{cases}$$

then $x^{r]}(t)$ is a solution of IVP(4.1) on $[t_0, r]$, i.e., $A(r)$ is true.

As an application of Lemma 2.8, the proof is complete. □

To show the uniqueness of solutions, we will need the following local Lipschitz condition on function f.

Definition 4.2. The function $f : \Omega \to \mathbb{R}^n$ is said to be locally Lipschtiz on Ω, if for any given $(\bar{t}, \bar{\psi}) \in \Omega$, there exist positive constants a and b such that

$$\Xi = ([\bar{t} - a, \bar{t} + a]_{\mathbb{R}} \cap [t_0, \alpha)) \times \{\psi \in C([-\tau, 0]_{\mathbb{R}}, \mathbb{R}^n) \mid \|\psi - \bar{\psi}\|_\tau \le b\}$$

is a subset of Ω and f is Lipschtiz on Ξ.

Now we are in the position to give a uniqueness result.

Theorem 4.3 (Uniqueness)**.** *Let $f : \Omega \to \mathbb{R}^n$ be rd-continuous in its first variable, continuous in its second variable and locally Lipschitz on its domain. Then, for any $\phi \in C([-\tau, 0]_{\mathbb{R}}, \mathbb{R}^n)$, there exists a unique solution $x(t) = x(t; t_0, \phi)$ of the initial value problem* (3.1) *on $[t_0 - \tau, \beta)$ for some $\beta \in (t_0, \alpha]$.*

Proof. The existence of solution can be obtained from Theorem 4.1. Next, we will use the method of proof by contradiction to show the uniqueness of the solution.

Suppose that for some $\beta \in (t_0, \alpha]$ there exist two distinct solutions x and y of (3.1) on $[t_0 - \tau, \beta)$. Let $t_1 = \inf\{t \in (t_0, \beta) \mid x(t) \ne y(t)\}$. Then $t_0 < t_1 < \beta$, and $x(t) = y(t)$ for $t \in [t_0 - \tau, t_1)$. To construct the contradiction, we consider the following two cases of t_1.

Case I: t_1 is right-scattered. Then, according to the continuity of the solution x and the definition of t_1, we have that t_1 must be left-scattered and $x(t_1) \ne y(t_1)$, $x(t) = y(t)$ for all $t \in [t_0 - \tau, \rho(t_1)]$. Thus,

$$\begin{aligned} x(t_1) &= x(\rho(t_1)) + \mu(\rho(t_1))x^\Delta(\rho(t_1)) \\ &= x(\rho(t_1)) + \mu(\rho(t_1))f(\rho(t_1), x_{\rho(t_1)}) \\ &= y(\rho(t_1)) + \mu(\rho(t_1))f(\rho(t_1), y_{\rho(t_1)}) \\ &= y(t_1), \end{aligned}$$

which is a contradiction.

Case II: t_1 is right-dense. Then the definition of t_1 implies that $x(t_1) = y(t_1)$. Since f is locally Lipschitz, there exists $a, b > 0$ such that $t_0 + a \in \mathbb{T}$, the set $\Theta = [t_1, t_1 + a] \times \{\psi \in C([-\tau, 0]_{\mathbb{R}}, \mathbb{R}^n) \mid \|\psi - x_{t_1}\|_\tau \le b\}$ is contained in $[t_0, \beta) \times C([-\tau, 0]_{\mathbb{R}}, \mathbb{R}^n)$, and f is Lipschitz on Θ with Lipschitz constant L. By Lemma 3.2 there exits $\delta \in (0, a]$ such that $t_1 + \delta \in \mathbb{T}$ and both (t, x_t) and (t, y_t) belong to Θ for $t_1 \le t \le t_1 + \delta$. Thus, for a given $t^* \in [t_1, t_1 + \delta]$, there exists $s^* \in [-\tau, 0]_\tau$ such that

$$\begin{aligned} \|x_{t^*} - y_{t^*}\|_\tau &= \sup_{s \in [-\tau,0]_\tau} |x(\theta(t^* + s)) - y(\theta(t^* + s))| \\ &= |x(\theta(t^* + s^*)) - y(\theta(t^* + s^*))| \\ &\le \left\| \int_{t_0}^{\theta(t^*+s^*)} [f(s, x_s) - f(s, y_s)]\Delta s \right\|. \end{aligned} \tag{4.2}$$

If $\theta(t^* + s^*) \le t_1$, then (4.2) implies $\|x_{t^*} - y_{t^*}\|_\tau = 0$, and then $x(t) = y(t)$ for $t \in [t_1, t^*]$, which is a contradiction to the choice of t_1. If $\theta(t^* + s^*) > t_1$, then we can get from (4.2) that

$$\begin{aligned} \|x_{t^*} - y_{t^*}\|_\tau &\le \left\| \int_{t_0}^{\theta(t^*+s^*)} [f(s, x_s) - f(s, y_s)]\Delta s \right\| \\ &= \left\| \int_{t_1}^{\theta(t^*+s^*)} [f(s, x_s) - f(s, y_s)]\Delta s \right\| \\ &\le \int_{t_1}^{\theta(t^*+s^*)} L\|x_s - y_s\|_\tau \Delta s. \end{aligned}$$

From this and the Gronwall's inequality (Theorem 6.4 in [24]) it follows that $\|x_{t^*} - y_{t^*}\|_\tau = 0$ contradicting the definition of t_1.

From the discussion of Case I and Case II, we can conclude the uniqueness of the solution. □

In order to introduce an extended existence result, the following definition is required.

Definition 4.4. We say $f : \Omega \to \mathbb{R}^n$ is quasi-bounded, if f is bounded on every set of the form $[t_0, \beta] \times C([-\tau, 0]_{\mathbb{R}}, B)$ where $\beta \in (t_0, \alpha)$ and B is a closed bounded subset of $\mathbb{R}^n$.

Theorem 4.5 (Extended Existence)**.** *Let $f : \Omega \to \mathbb{R}^n$ be rd-continuous in its first variable, continuous in its second variable, locally Lipschitz on its domain and quasi-bounded. Then, for each $\phi \in C([-\tau, 0]_{\mathbb{R}}, \mathbb{R}^n)$, there is $\beta \in (t_0, \alpha]$ such that*

(a) *the initial value problem* (3.1) *has an unique noncontinuable solution $x(t) = x(t; t_0, \phi)$ on $[t_0 - \tau, \beta)$; and*
(b) *if $\beta < \alpha$ then for every closed bounded set $A \in \mathbb{R}^n$, $x(t) \notin A$ for some $t \in (t_0, \beta)$.*

The proof of Theorem 4.5 is similar to the proof of the result for continuous-time equations, and thus omitted. It worths noting that the continuity of f does not imply

f to be bounded on closed bounded subsets of Ω. A counter example can be found on page 44 of [5] for the special case $\mathbb{T} = \mathbb{R}$.

Theorem 4.6 (Global Existence)**.** *Let $f : \Omega \to \mathbb{R}^n$ be rd-continuous in its first variable, continuous in its second variable and locally Lipschitz on its domain. If*

$$\|f(t, \psi)\| \le M(t) + N(t)\|\psi\|_\tau, \text{ on } \Omega, \tag{4.3}$$

where $M, N : [t_0, \alpha) \to \mathbb{R}$ are rd-continuous, positive functions, then the unique noncontinuable solution of (3.1) *exists on the entire interval* $[t_0, \alpha)$.

Proof. Theorem 4.5 implies that for any $\phi \in C([-\tau, 0]_\mathbb{R}, \mathbb{R}^n)$ there is $\beta \in (t_0, \alpha]$ such that (3.1) has a unique noncontinuable solution x on $[t_0 - \tau, \beta)$, since inequality (4.3) guarantees that f is quasi-bounded.

To construct a contradiction, suppose $\beta < \alpha$. Then, there exist positive constants $\bar{M}$ and $\bar{N}$ such that $M(t) \le \bar{M}$ and $N(t) \le \bar{N}$ for all $t \in [t_0, \beta]$. Integrating both sides of (3.1) yields

$$\|x(t)\| \le \|\phi\|_\tau + \int_{t_0}^t \bar{M}\Delta s + \int_{t_0}^t \bar{N}\|x_s\|_\tau \Delta s, \text{ for } t \in [t_0, \beta],$$

which implies that

$$\|x_t\|_\tau \le \|\phi\|_\tau + \bar{M}(\beta - t_0) + \int_{t_0}^t \bar{N}\|x_s\|_\tau \Delta s, \text{ for } t \in [t_0, \beta].$$

Then, using Gronwall's inequality,

$$\|x(t)\| \le \|x_t\|_\tau \le [\|\phi\|_\tau + \bar{M}(\beta - t_0)]\Big(1 + \bar{N}\int_{t_0}^\beta e_{\bar{N}}(\beta, \sigma(s))\Delta s\Big),$$

on $t \in [t_0, \beta]$, where $e_{\bar{N}}(\beta, \sigma(s))$ is the exponential function on time scales (see Definition 2.30 in [24]). This shows that $x(t)$ remains in a closed bounded set which contradicts the extended existence result in Theorem 4.5.

Therefore, $\beta = \alpha$, i.e., the solution $x(t)$ exists on the entire interval $[t_0, \alpha)$. □

5. UNIFORM STABILITY RESULTS

In this section, the uniform (asymptotic) stability of system (3.1) is investigated using Lyapunov functions in the spirit of Razumikhin. Two Razumikhin-type stability criteria are established. Let

$$\mathcal{K} = \{g \in C(\mathbb{R}^+, \mathbb{R}^+) \mid g \text{ is nondecreasing in } s, g(0) = 0, \text{ and } g(s) > 0 \text{ for } s > 0\}$$

Theorem 5.1. *Suppose $u, v \in \mathcal{K}$ and $w \in C(\mathbb{R}^+, \mathbb{R}^+)$. If there exists a Lyapunov function $V \in C_{rd}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+)$ such that*

(i) *$u(\|x\|) \le V(t, x) \le v(\|x\|)$ and $V(t, x)$ is locally Lipschitz in x for each right-dense point $t \in \mathbb{T}$;*

(ii) $D^+V^\Delta(t,x) \le -w(\|x(\sigma(t))\|)$ *if* $V(\theta(t+s), x(\theta(t+s))) \le V(\sigma(t), x(\sigma(t)))$, $s \in [-\tau, 0]_{\mathbb{R}}$,

then the trivial solution of system (3.1) *is uniformly stable.*

Proof. For any $\varepsilon > 0$, choose $\delta > 0$ such that $v(\delta) < u(\varepsilon)$. We shall prove that for any solution $x(t) = x(t, t_0, \phi)$ of system (3.1), $\|\phi\|_\tau < \delta$ implies that $\|x(t)\| < \varepsilon$ for all $t \ge t_0$.

When $t = t_0$, we have

$$V(t_0, x(t_0)) \le v(\|x(t_0)\|) < v(\delta) < u(\varepsilon)$$

i.e., $\|x(t_0)\| < \varepsilon$.

Next, we shall show

$$V(t, x(t)) \le v(\delta), \text{ for all } t \ge t_0.$$

Suppose it is not true, then there exists some $t > t_0$ such that $V(t,x) > v(\delta)$. Let $t^* = \inf\{t \ge t_0 \mid V(t, x(t)) > v(\delta)\}$, then we have $V(t^*, x(t^*)) \ge v(\delta)$ and $V(t,x) \le v(\delta)$ for $t_0 \le t < t^*$.

If t^* is left-dense, by the definition of t^*, we know that $V(t^*, x(t^*)) = v(\delta)$ and t^* is right-dense. Then

$$V(\theta(t^*+s), x(\theta(t^*+s))) \le V(t^*, x(t^*)) = V(\sigma(t^*), x(\sigma(t^*))), \text{ for } s \in [-\tau, 0]_{\mathbb{R}}.$$

It follows from condition (ii) that $D^+V^\Delta(t^*, x(t^*)) < 0$, which is a contradiction to the definition of t^*.

If t^* is left-scattered, then $V(t^*, x(t^*)) \ge \delta$ and $V(t,x) \le v(\delta)$ for $t_0 \le t \le \rho(t^*)$. By setting $\bar{t} = \rho(t^*)$, we have

$$V(\theta(\bar{t}+s), x(\theta(\bar{t}+s))) \le v(\delta) \le V(\sigma(\bar{t}), x(\sigma(\bar{t}))), \text{ for } s \in [-\tau, 0]_{\mathbb{R}}.$$

By condition (ii), we have $D^+V^\Delta(\bar{t}, x(\bar{t})) \le -w(\|x(t^*)\|) < 0$. Since $\bar{t}$ is right-scattered, it follows that

$$D^+V^\Delta(\bar{t}, x(\bar{t})) = \frac{V(\sigma(\bar{t}), x(\sigma(\bar{t}))) - V(\bar{t}, x(\bar{t}))}{\mu(\bar{t})} < 0,$$

i.e., $V(t^*, x(t^*)) < V(\bar{t}, x(\bar{t})) = V(\rho(t^*), x(\rho(t^*)))$, which is a contradiction to the definition of t^*.

Hence, $V(t,x) \le v(\delta)$ for $t \ge t_0$. By condition (i), we have

$$u(\|x(t)\|) \le V(t, x(t)) \le v(\delta) < u(\varepsilon), \ t \ge t_0,$$

i.e., $\|x(t)\| < \varepsilon$ for all $t \ge t_0$. □

If impose a stronger assumption on condition (ii) of Theorem 5.1, then we have the following uniform asymptotic stability result.

Theorem 5.2. *Suppose* $u, v \in \mathcal{K}$, $w \in C(\mathbb{R}^+, \mathbb{R}^+)$ *and* $w(s) > 0$ *if* $s > 0$. *If there exist a continuous nondecreasing function* $P(s) > 0$ *for* $s > 0$ *and a Lyapunov function* $V \in C_{rd}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+)$ *such that*

(i) $u(\|x\|) \le V(t,x) \le v(\|x\|)$ *and* $V(t,x)$ *is locally Lipschitz in* x *for each right-dense point* $t \in \mathbb{T}$;

(ii) $D^+V^\Delta(t,x) \le -w(\|x(\sigma(t))\|)$ *if* $V(\theta(t+s), x(\theta(t+s))) < P(V(\sigma(t), x(\sigma(t))))$, $s \in [-\tau, 0]_{\mathbb{R}}$,

then the trivial solution of system (3.1) *is uniformly asymptotically stable.*

Proof. By Theorem 5.1, we know that, for any given $H > 0$, we can choose $\delta > 0$ such that $v(\delta) < u(H)$ and $\|\phi\|_\tau < \delta$ implies that

$$\|x(t)\| < H, \text{ for } t \ge t_0,$$

and

$$V(t,x) \le v(\delta) < u(H).$$

Suppose $\varepsilon \in (0, \inf\{s \in \mathbb{R}^+ \mid u(s) = v(\delta)\})_{\mathbb{R}}$ be arbitrary, then $u(\varepsilon) < v(\delta)$. We need to show there is a number $T = T(\varepsilon, \delta) > 0$ such that, for any $t_0 \in \mathbb{T}$ and $\|\phi\|_\tau < \delta$, the solution $x(t) = x(t, t_0, \phi)$ of system (3.1) satisfies $\|x(t, t_0, \phi)\| < \varepsilon$, for $t \ge t_0 + T$. This will be true if we show that $V(t,x) < u(\varepsilon)$ for $t \ge t_0 + T$.

From the property of the function P, there is a number $a > 0$ such that $P(s) - s > a$ for $u(\eta) \le s \le v(\delta)$ where $\eta > 0$ satisfying $u(\eta) < u(\varepsilon)$. Let N be the first nonnegative integer such that $u(\eta) + Na \ge v(\delta)$ and let

$$\gamma = \inf_{s_0 \le s \le H} w(s), \text{ where } s_0 = \sup\{s \in \mathbb{R}^+ \mid v(s) = u(\eta)\}.$$

Set $t_k = t_0 + k(\tau + \frac{v(\delta)}{\gamma} + \bar{\mu})$, $k = 0, 1, 2, \dots, N$, where $\bar{\mu}$ is the upper bound of the graininess function μ. We now claim that

$$V(t,x) \le u(\eta) + (N-k)a, \text{ for } t \ge \theta(t_k), \text{ and } k = 0, 1, 2, \dots, N. \tag{5.1}$$

Trivially, (5.1) holds for $k = 0$. Suppose now for some $k (0 \le k < N)$, (5.1) holds. We want to show that

$$V(t,x) \le u(\eta) + (N-k-1)a \text{ for } t \ge \theta(t_{k+1}). \tag{5.2}$$

To this end, we first claim that there must be some $\bar{t} \in [\theta(t_k + \tau), \theta(t_{k+1})]$ such that

$$V(\bar{t}, x(\bar{t})) \le u(\eta) + (N-k-1)a. \tag{5.3}$$

Suppose it is not true, then for all $t \in [\theta(t_k + \tau), \theta(t_{k+1})]$, we would have

$$V(t, x(t)) > u(\eta) + (N-k-1)a. \tag{5.4}$$

On the other hand, by our assumption,

$$V(t, x(t)) \leq u(\eta) + (N-k)a, \text{ for } t \geq \theta(t_k),$$

i.e.,

$$u(\eta) + (N-k-1)a < V(t, x(t)) \leq u(\eta) + (N-k)a, \text{ for } t \in [\theta(t_k+\tau), \theta(t_{k+1})].$$

Then, we have, for $\theta(t_k+\tau) \leq t \leq \rho(\theta(t_k))$,

$$\begin{aligned} P(V(\sigma(t), x(\sigma(t)))) > V(\sigma(t), x(\sigma(t))) &> u(\eta) + (N-k-1)a + a \\ &= u(\eta) + (N-k)a \\ &\geq V(\theta(t+s), x(\theta(t+s))), \ s \in [-\tau, 0]_{\mathbb{R}} \end{aligned}$$

By condition (ii), we have, for $t \in [\theta(t_k+\tau), \rho(\theta(t_{k+1}))]$,

$$D^+V^{\Delta}(t,x) \leq -w(\|x(\sigma(t))\|) \leq -\gamma < 0.$$

Therefore, for $t \in [\theta(t_k+\tau), \rho(\theta(t_{k+1}))]$, we have

$$\begin{aligned} V(t,x) &\leq V(\theta(t_k+\tau), x(\theta(t_k+\tau))) - \gamma[t - \theta(t_k+\tau)] \\ &\leq v(\delta) - \gamma[t - \theta(t_k+\tau)]. \end{aligned}$$

If $\rho(\theta(t_{k+1}))$ is right-dense, then we have

$$\begin{aligned} V(\theta(t_{k+1}), x(\theta(t_{k+1}))) &\leq v(\delta) - \gamma[\theta(t_{k+1}) - \theta(t_k+\tau)] \\ &\leq v(\delta) - \gamma[t_0 + (k+1)(\tau + \frac{v(\delta)}{\gamma} + \bar{\mu}) \\ &\quad -t_0 - k(\tau + \frac{v(\delta)}{\gamma} + \bar{\mu}) - \tau - \bar{\mu}] \\ &\leq 0, \end{aligned}$$

which is a contradiction to (5.4).

If $\rho(\theta(t_{k+1}))$ is right-scattered, then

$$\begin{aligned} V(\tilde{t}, x(\tilde{t})) &= V(\rho(\tilde{t}), x(\rho(\tilde{t}))) + \mu(\rho(\tilde{t}))D^+V^{\Delta}(\rho(\tilde{t}), x(\rho(\tilde{t}))) \\ &\leq v(\delta) - \gamma[\rho(\tilde{t}) - \theta(t_k+\tau)] - \gamma\mu(\rho(\tilde{t})) \\ &= v(\delta) - \gamma[\tilde{t} - \theta(t_k+\tau)] \\ &\leq 0, \text{ where } \tilde{t} = \theta(t_{k+1}), \end{aligned}$$

which is a contradiction to (5.4).

Thus, there exists a $\bar{t}$ such that (5.3) holds.

Next, we claim that

$$V(t,x) \leq u(\eta) + (N-k-1)a, \text{ for all } t \geq \bar{t}. \tag{5.5}$$

Suppose this is not true, then there is a $t > \bar{t}$ such that

$$V(t,x) > u(\eta) + (N-k-1)a.$$

Let

$$t^* = \inf\{t \geq \bar{t} \mid V(t,x) > u(\eta) + (N-k-1)a\},$$

then $V(t^*, x(t^*)) \geq u(\eta) + (N-k-1)a$ and $V(t,x) \leq u(\eta)+(N-k-1)a$ for $\bar{t} \leq t < t^*$.

If t^* is left-dense, by the definition of t^*, we know that $V(t^*, x(t^*)) = u(\eta) + (N-k-1)a$ and t^* is right-dense. Then

$$\begin{aligned} V(\theta(t^*+s), x(\theta(t^*+s))) &\leq u(\eta) + (N-k)a \\ &= u(\eta) + (N-k-1)a + a \\ &= V(t^*, x(t^*)) + a \\ &< P(V(t^*, x(t^*))) \\ &= P(V(\sigma(t^*), x(\sigma(t^*)))), \text{ for } s \in [-\tau, 0]_{\mathbb{R}}. \end{aligned}$$

By condition (ii), we have $D^+V^\Delta(t^*, x(t^*)) \leq 0$, which is a contradiction to the definition of t^*.

If t^* is left-scattered, then $V(t^*, x(t^*)) \geq u(\eta) + (N-k-1)a$ and $V(t,x) \leq u(\eta) + (N-k-1)a$ for $\bar{t} \leq t \leq \rho(t^*)$. Then, by setting $\hat{t} = \rho(t^*)$, we have

$$\begin{aligned} V(\theta(\hat{t}+s), x(\theta(\hat{t}+s))) &\leq u(\eta) + (N-k)a \\ &= u(\eta) + (N-k-1)a + a \\ &\leq V(t^*, x(t^*)) + a \\ &< P(V(t^*, x(t^*))) \\ &= P(V(\sigma(\hat{t}), x(\sigma(\hat{t})))), \text{ for } s \in [-\tau, 0]_{\mathbb{R}}. \end{aligned}$$

By condition (ii), we have

$$D^+V^\Delta(\hat{t}, x(\hat{t})) = \frac{V(\sigma(\hat{(t)}), x(\sigma(\hat{t}))) - V(\hat{t}, x(\hat{t}))}{\mu(\hat{t})} \leq -w(\|x(t^*)\|) < 0,$$

i.e., $V(t^*, x(t^*)) < V(\hat{t}, x(\hat{t}))$, which is a contradiction to the definition of t^*.

Hence, (5.5) holds, so does (5.2). By a simple induction, we have

$$V(t,x) \leq u(\eta) + (N-k)a, \text{ for } t \geq \theta(t_k),$$

where $t_k = t_0 + k(\tau + \frac{v(\delta)}{\gamma} + \bar{\mu})$ and $k = 0, 1, \dots, N$. Therefore, choosing $k = N$, we obtain

$$V(t,x) \leq u(\eta) < u(\varepsilon), \text{ for } t \geq t_0 + T,$$

where $T = N(\tau + \frac{v(\delta)}{\gamma} + \bar{\mu}) + \bar{\mu} \geq \theta(t_0 + N(\tau + \frac{v(\delta)}{\gamma} + \bar{\mu})) - t_0$. Hence, $\|x(t)\| \leq \eta < \varepsilon$ for $t \geq t_0 + T$. This completes the proof. □

Remark 5.3. It is easy to see from the arguments in the proofs of Theorem 5.1 and Theorem 5.2 that the conclusions of these theorems remain true if the inequality $D^+V^\Delta(t,x) \leq -w(\|x(\sigma(t))\|)$ is replaced by $D^+V^\Delta(t,x) \leq -w(\|x(t)\|)$. If $\mathbb{T} = \mathbb{R}$, then the continuous versions of these results can be found in [5]; if $\mathbb{T} = \mathbb{Z}$, then the discrete versions of these results are contained in [10]. Since there are many other time scales than just the real numbers and the integers, our results are much more general.

Next, we shall apply the previous theorems to the following linear delay differential equations on time scale $\mathbb{T}$.

$$\begin{cases} x^{\Delta}(t) &= Ax(t) + Bx(\theta(t-\tau)),\ t \in \mathbb{T}, \\ x_{t_0} &= \phi,\ t_0 \in \mathbb{T}, \end{cases} \tag{5.6}$$

where $x \in \mathbb{R}^n$, $A, B \in \mathbb{R}^{n\times n}$ and τ represents the time delay.

Corollary 5.4. *Let* $a_1 = \lambda_{\max}(A + A^T)$, $a_2 = \lambda_{\max}(AA^T)$ *and* $b = \lambda_{\max}(BB^T)$. *If*

$$a_1 + (a_2 + 2b)\mu(t) + 2\sqrt{b} < 0,\ t \in \mathbb{T},$$

then the trivial solution of system (5.6) *is uniformly asymptotically stable.*

Proof. It follows from Theorem 5.2 on choosing $V(x) = x^T x$. □

It can be seen that the stability conditions in Corollary 5.4 are independent of τ and very conservative since $\lambda_{\max}(A + A^T) < 0$. In order to get the less conservative and delay dependent stability criteria, we can proceed in the following manner.

Let $x(t) = x(t, t_0, \phi)$ be the solution of equation (5.6). Since $x^{\Delta}(t)$ is rd-continuous, we have

$$x(\theta(t-\tau)) = x(t) - \int_{\theta(t-\tau)}^{t} x^{\Delta}(s)\Delta s,\ t \geq \tau.$$

Then equation (5.6) can be generalised as the following form

$$x^{\Delta}(t) = (A + B)x(t) - B\int_{\theta(t-\tau)}^{t} [Ax(s) + Bx(\theta(s-\tau))]\Delta s \tag{5.7}$$

with initial data ψ on $[-2\tau, 0]_{\mathbb{R}}$ satisfying $\psi(s) = \phi(s)$ for $s \in [-\tau, 0]_{\mathbb{R}}$. Since equation (5.6) is a special case of equation (5.7), the trivial solution of equation (5.7) is asymptotically stable implies that the trivial solution of equation (5.6) is asymptotically stable.

As an example, consider the equation

$$x^{\Delta}(t) = -bx(\theta(t-\tau)) \tag{5.8}$$

on time scale $\mathbb{T}$. The generalized equation can be written as

$$x^{\Delta}(t) = -bx(t) - b^2\int_{\theta(t-\tau)}^{t} x(\theta(s-\tau))\Delta s \tag{5.9}$$

If $V(x) = x^2$, then, for any $q > 1$,

$$\begin{aligned} V^{\Delta}(x) &= 2xx^{\Delta} + \mu(x^{\Delta})^2 \\ &= -2bx^2 - 2b^2\int_{\theta(t-\tau)}^{t} x(\theta(s-\tau))\Delta s \end{aligned}$$

$$+\mu\left[b^2x^2+b^2\Big(\int_{\theta(t-\tau)}^{t}x(\theta(s-\tau))\Delta s\Big)^2+2b^3\int_{\theta(t-\tau)}^{t}x(\theta(s-\tau))\Delta s\right]$$
$$\leq\ [-2b+2b^2q\tau+\mu b^2(bq\tau+1)^2]V(x)$$

whenever $V(x(\theta(t+s)))\leq q^2V(x(t))$ for $s\in[-2\tau,0]_{\mathbb{R}}$. Therefore, if there exists a $q>1$ such that

$$-2b+2b^2q\tau+\mu b^2(bq\tau+1)^2<0,$$

then, by Theorem 5.2, the trivial solution of equation (5.8) is asymptotically stable.

6. EXPONENTIAL STABILITY RESULTS

In this section, the global exponential stability of system (3.1) is investigated base on the method of Lyapunov functions and Razumikhin technique. Two Razumikhin-type stability criteria are established.

Theorem 6.1. *Assume that there exist a Lyapunov function* $V\in C_{rd}(\mathbb{T}\times\mathbb{R}^n,\mathbb{R}^n)$ *and positive constants* p, c_1, c_2, λ, *such that the following conditions hold*

(i) $c_1\|x\|^p\leq V(t,x)\leq c_2\|x\|^p$ *and* $V(t,x)$ *is locally Lipschitz in* x *for each right-dense point* $t\in\mathbb{T}$;

(ii) *if* $V(\sigma(t)x(\sigma(t)))e^{\int_{\theta(t-\tau)}^{\sigma(t)}w(t)\Delta t}\geq V(\theta(t+s),x(\theta(t+s)))$ *for all* $s\in[-\tau,0]_{\mathbb{R}}$, *then*

$$D^+V^{\Delta}(t,x)\leq -w(t)V(t,x),$$

where $w(t)\in C_{rd}(\mathbb{T},\mathbb{R}^+)$, $-w(t)\in C_{rd}\mathcal{R}^+$ *and* $\inf_{t\geq\theta(t_0-\tau)}w(t)\geq\lambda$.

Then the trivial solution of system (3.1) *is globally exponentially stable.*

Proof. Let $x(t)=x(t,t_0,\phi)$ be a solution of system (3.1) and $V(t)=V(t,x)$. We shall show that

$$V(t)\leq c_2\|\phi\|_\tau^pe^{-\int_{t_0}^{t}w(t)\Delta t},\ \text{for } t\in\mathbb{T}_0.$$

Let

$$Q(t)=V(t)-c_2\|\phi\|_\tau^pe^{-\int_{t_0}^{t}w(t)\Delta t},\ t\geq\theta(t_0-\tau).$$

We shall show that $Q(t)\leq 0$ for $t\geq\theta(t_0-\tau)$. It is clear that $Q(t)\leq 0$ for $t\in[\theta(t_0-\tau),t_0]$, since $Q(t)\leq V(t)-c_2\|\phi\|_\tau^p\leq 0$ by condition (i).

Next, we shall show that $Q(t)\leq 0$ for $t\geq t_0$. In order to do this, let $\epsilon>0$ be arbitrary and we claim that $Q(t)\leq\epsilon$ for $t\geq t_0$. Suppose this is not true, then there exists some $t\geq t_0$ such that $Q(t)>\epsilon$. Let

$$t^*=\inf\{t\geq t_0\mid Q(t)>\epsilon\}.$$

By the definition of t^*, we have

$$Q(t^*)\geq\epsilon,$$
$$Q(t)\leq\epsilon,\ \text{for } t\in[\theta(t_0-\tau),t^*).$$

For the point of t^*, it is enough to consider the following two cases:

CASE 1. If t^* is left-dense, by the definition of t^*, we know that t^* is also right-dense, $Q(t^*) = \epsilon$ and $Q(t) \le \epsilon$ for $t \in [\theta(t_0 - \tau), t^*)$.

Notice $V(t^*) = Q(t^*) + c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} w(t)\Delta t}$; and for $s \in [-\tau, 0]_{\mathbb{R}}$, we have

$$\begin{aligned} V(\theta(t^*+s)) &= Q(\theta(t^*+s)) + c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{\theta(t^*+s)} w(t)\Delta t} \\ &\le \epsilon + c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{\theta(t^*+s)} w(t)\Delta t} \\ &\le \Big(\epsilon + c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} w(t)\Delta t}\Big) e^{-\int_{t^*}^{\theta(t^*+s)} w(t)\Delta t} \\ &= V(t^*) e^{\int_{\theta(t^*+s)}^{t^*} w(t)\Delta t} \\ &\le V(\sigma(t^*)) e^{\int_{\theta(t^*-\tau)}^{\sigma(t^*)} w(t)\Delta t}. \end{aligned}$$

So by condition (ii), we have

$$D^+V^\Delta(t^*) \le -w(t^*)V(t^*).$$

By the Chain Rule (Lemma 2.7), we have

$$\begin{aligned} \Big(e^{-\int_{t_0}^{t} w(t)\Delta t}\Big)^\Delta &= \left\{\int_0^1 e^{\left[-\int_{t_0}^{t} w(t)\Delta t + h\mu(t)\left(-\int_{t_0}^{t} w(t)\Delta t\right)^\Delta\right]} dh\right\}\Big(-\int_{t_0}^{t} w(t)\Delta t\Big)^\Delta \\ &= -w(t)\int_0^1 e^{\left[-\int_{t_0}^{t} w(t)\Delta t - h\mu(t)w(t)\right]} dh. \end{aligned}$$

Since t^* is right-dense, we have $\mu(t^*) = 0$ and

$$\Big(e^{-\int_{t_0}^{t} w(t)\Delta t}\Big)^\Delta\Big|_{t=t^*} = -w(t^*)e^{-\int_{t_0}^{t^*} w(t)\Delta t}.$$

Hence,

$$\begin{aligned} D^+Q^\Delta(t^*) &= D^+V^\Delta(t^*) + w(t^*)c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} w(t)\Delta t} \\ &\le -w(t^*)\Big(V(t^*) - c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} w(t)\Delta t}\Big) \\ &= -w(t^*)\epsilon \\ &< 0, \end{aligned}$$

which, by Lemma 2.6, leads to a contradiction to the definition of t^*.

CASE 2. If t^* is left-scattered, by the definition of t^*, we know that $Q(t^*) \ge \epsilon$ and $Q(t) \le \epsilon$ for $t \in [\theta(t_0 - \tau), \rho(t^*)]$.

Let $\bar{t} = \rho(t^*)$, then for $s \in [-\tau, 0]_{\mathbb{R}}$, we have

$$\begin{aligned} V(\theta(\bar{t}+s)) &= Q(\theta(\bar{t}+s)) + c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{\theta(\bar{t}+s)} w(t)\Delta t} \\ &\le \epsilon + c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{\theta(\bar{t}+s)} w(t)\Delta t} \\ &\le \Big(\epsilon + c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} w(t)\Delta t}\Big) e^{-\int_{t^*}^{\theta(\bar{t}+s)} w(t)\Delta t} \\ &\le \Big(Q(t^*) + c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} w(t)\Delta t}\Big) e^{\int_{\theta(\bar{t}+s)}^{t^*} w(t)\Delta t} \\ &\le V(t^*) e^{\int_{\theta(\bar{t}-\tau)}^{t^*} w(t)\Delta t} \end{aligned}$$

$$= \quad V(\sigma(\bar{t}))e^{\int_{\theta(\bar{t}-\tau)}^{\sigma(\bar{t})} w(t)\Delta t},$$

thus, by condition (ii), we have

$$D^+V^\Delta(\bar{t}) \leq -w(\bar{t})V(\bar{t}).$$

Since $\bar{t}$ is right-scattered, we can obtain

$$\begin{aligned}
& D^+Q^\Delta(\bar{t}) \\
= \ & \frac{Q(\sigma(\bar{t})) - Q(\bar{t})}{\mu(\bar{t})} \\
= \ & \frac{1}{\mu(\bar{t})}\Big[V(\sigma(\bar{t})) - c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{\sigma(\bar{t})} w(t)\Delta t} - V(\bar{t}) + c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{\bar{t}} w(t)\Delta t}\Big] \\
= \ & \frac{V(\sigma(\bar{t})) - V(\bar{t})}{\mu(\bar{t})} + \frac{1}{\mu(\bar{t})} c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{\bar{t}} w(t)\Delta t}\Big(1 - e^{-\int_{\bar{t}}^{\sigma(\bar{t})} w(t)\Delta t}\Big) \\
= \ & D^+V^\Delta(\bar{t}) + \frac{1}{\mu(\bar{t})} c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{\bar{t}} w(t)\Delta t}\Big(1 - e^{-\mu(\bar{t})w(\bar{t})}\Big) \\
\leq \ & -w(\bar{t})\Big(V(\bar{t}) - c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{\bar{t}} w(t)\Delta t}\Big) \\
= \ & -w(\bar{t})Q(\bar{t}),
\end{aligned}$$

which, coupled with $-w \in C_{rd}\mathcal{R}^+$, yields

$$\begin{aligned}
\epsilon \ & \leq \ Q(t^*) = Q(\sigma(\bar{t})) \\
& \leq \ [1 - \mu(\bar{t})w(\bar{t})]Q(\bar{t}) \\
& < \ Q(\bar{t}) \\
& = \ Q(\rho(t^*)),
\end{aligned}$$

i.e., $Q(\rho(t^*)) > \epsilon$, which is a contradiction to the definition of t^*.

Based on the above contradictions, we know that $Q(t) \leq \epsilon$ for all $t \geq t_0$. Let $\epsilon \to 0^+$, we have $Q(t) \leq 0$ for $t \geq t_0$. Thus, we get

$$V(t) \leq c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t} w(t)\Delta t}, \ t \geq t_0.$$

By condition (i) and (ii), we have

$$c_1\|x\|^p \leq V(t) \leq c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t} w(t)\Delta t} \leq c_2\|\phi\|_\tau^p e^{-\lambda(t-t_0)}, \ t \geq t_0.$$

i.e,

$$\|x\| \leq \Big(\frac{c_2}{c_1}\Big)^{\frac{1}{p}} \|\phi\|_\tau e^{-\frac{\lambda}{p}(t-t_0)}, \ t \geq t_0.$$

which completes the proof. □

If $w(t) \equiv \lambda$ in Theorem 6.1, we have the following result.

Corollary 6.2. *Assume that all conditions of Theorem* 6.1 *hold with the following change:*

(ii)* *if* $V(\sigma(t)x(\sigma(t)))e^{\lambda(\sigma(t)-\theta(t-\tau))} \geq V(\theta(t+s), x(\theta(t+s)))$ *for all* $s \in [-\tau, 0]_{\mathbb{R}}$*, then*

$$D^+V^{\Delta}(t,x) \leq -\lambda V(t,x),$$

where $-\lambda \in C_{rd}\mathcal{R}^+$.

Then the trivial solution of system (3.1) *is globally exponentially stable.*

Next, we further assume that the graininess function μ is bounded from above, i.e.,

$$\bar{\mu} = \sup_{t\in\mathbb{T}}\{\mu(t)\} < \infty.$$

Then we have the following conservative result, the conditions of which are easier to testify than that of Theorem 6.1 and Corollary 6.2. The proof of the following theorem is identical to that of Theorem 6.1 and thus omitted.

Theorem 6.3. *Assume that there exist a Lyapunov function* $V \in C_{rd}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ *and positive constants* p*,* c_1*,* c_2*,* λ*, such that the following conditions hold*

(i) $c_1\|x\|^p \leq V(t,x) \leq c_2\|x\|^p$ *and* $V(t,x)$ *is locally Lipschitz in* x *for each right-dense point* $t \in \mathbb{T}$*;*

(ii) *if* $qV(\sigma(t)x(\sigma(t))) \geq V(\theta(t+s), x(\theta(t+s)))$ *for all* $s \in [-\tau, 0]_{\mathbb{R}}$*, then*

$$D^+V^{\Delta}(t,x) \leq -\lambda V(t,x),$$

where $-\lambda \in C_{rd}\mathcal{R}^+$ *and* q *is a constant such that* $q \geq e^{\lambda(\bar{\mu}+\tau)}$.

Then the trivial solution of system (3.1) *is globally exponentially stable.*

Remark 6.4. By employing Lyapunov-Razumikhin method, we have established two exponential stability criteria, the conditions of which can be easily testified. Moreover, when the time scale $\mathbb{T}$ reduces to the real numbers $\mathbb{R}$ or the integers $\mathbb{Z}$, our results reduce to the results for functional differential equations contained in [11] or the results for delay difference equations. Since time scales contain not only $\mathbb{R}$ and $\mathbb{Z}$, but also some other hybrid time domains, our results are more general than the results in [11].

Remark 6.5. If the time scale $\mathbb{T}$ satisfies the following conditions:

(i): 0 and $\tau \in \mathbb{T}$;

(ii): $t+s \in \mathbb{T}$ for any $t \in \mathbb{T}$ and $s \in [-\tau, 0]$,

then the initial condition of system (3.1) can be given by $x_{t_0} = \phi$, $t_0 \geq 0$, where $\phi \in C([-\tau, 0], \mathbb{R}^n)$ and $x_{t_0}(s) = x(t_0+s)$ for $s \in [-\tau, 0]$. From the proof of Theorem 6.1, we can see that M, α and ϕ in Definition 3.3(S5) are all independent of t_0. This kind of stability is called uniform exponential stability in [12] for discrete delay systems.

Remark 6.6. If $\mathbb{T} = \mathbb{Z}$, then condition (ii) of Theorem 6.3 can be written in the form of

(ii) $qV(n+1, x(n+1)) \geq V(n+s, x(n+s))$ implies $V(n+1, x(n+1)) \leq \eta V(n, x(n))$, where $\eta = 1 - \lambda$.

We can see that, in [12], condition (ii) is changed into:

(ii)' $qV(n, x(n)) \geq V(n+s, x(n+s))$ implies $V(n+1, x(n+1)) \leq \eta V(n, x(n))$.

It is worth noting that condition (ii) and condition $(ii)'$ serve similarly, that is, comparing $V(n+1, x(n+1))$ with $V(n, x(n))$ after having compared $V(n+1, x(n+1))$ or $V(n, x(n))$ with its τ backward items. Since, in [12], an additional condition is needed:

(iii)' for some $s \in N_{-\tau} - \{0\}$, $V(n+s, x(n+s)) \geq e^{\alpha} V(n, x(n))$ implies $V(n+1, x(n+1)) \leq \frac{1}{q} \max_{s \in N_{-\tau}} \{V(n+s, x(n+s))\}$, where $\alpha = \min\{\ln(\frac{1}{\eta}), \frac{\ln q}{\tau+1}\}$, and $N_{-\tau} = \{-\tau, -\tau+1, \ldots, -1, 0\}$,

we know that Theorem 3.3 in [12] is more complicated than the discrete version of our results. Moreover, Theorem 6.3 will be used to analyze the stability of a class of delay systems in the following section. Some exponential stability criteria which contain Theorem 4.1 in [12] will be established. Hence, the discrete version of Theorem 6.3 is not only simpler but also more efficient than the results in [12] to analyze stability of delay discrete systems.

Next, we shall apply these Razumikhin-type results to some special cases of system (3.1). Consider the nonlinear delay systems on time scales of the form

$$\text{(6.1)} \quad \begin{cases} x^{\Delta}(t) &= F(t, x(t), x(\theta(t+h_1(t))), \ldots, x(\theta(t+h_m(t))), \ t \in \mathbb{T}_0 \\ x_{t_0} &= \phi, \ t_0 \geq 0. \end{cases}$$

where $F \in C_{rd}(\mathbb{T} \times \mathbb{R}^{n \times (m+1)}, \mathbb{R}^n)$ and $h_j : \mathbb{T}_0 \to [-\tau, 0]_{\mathbb{R}}$ for $j = 1, 2, \ldots, m$. We assume here that $F(t, 0, 0, \ldots, 0) \equiv 0$ for any $t \in \mathbb{T}$.

Theorem 6.7. *Assume that condition* (i) *of Theorem* 6.3 *holds, while condition* (ii) *of Theorem* 6.3 *is replaced by the following condition:*

(ii)* *there exist positive constants* λ, λ_i, $i = 1, 2, \ldots, m$, *such that*

$$D^+ V^{\Delta}(t, x) \leq -\lambda V(t, x) + \sum_{i=1}^{m} \lambda_i V(\theta(t+h_i(t)), x(\theta(t+h_i(t)))).$$

If $\lambda > \sum_{i=1}^{m} \lambda_i$ *and* $1 > \bar{\mu}\lambda$, *then the trivial solution of system* (6.1) *is globally exponentially stable.*

Proof. If $\lambda > \sum_{i=1}^{m} \lambda_i$ and $1 > \bar{\mu}\lambda$,, then we see that equation

$$\text{(6.2)} \qquad \lambda - q \sum_{i=1}^{m} \lambda_i = \frac{\ln q}{\bar{\mu} + \tau}$$

has a unique solution q satisfying

$$1 < q < \frac{\lambda}{\sum_{i=1}^m \lambda_i} \text{ and } q\bar{\mu} < \frac{1}{\sum_{i=1}^m \lambda_i}.$$

Thus, for any $s \in [-\tau, 0]_{\mathbb{R}}$, if $V(\theta(t+s), x(\theta(t+s))) \le qV(\sigma(t), x(\sigma(t)))$, then, by condition (ii)*, we have

$$D^+V^\Delta(t,x) \le -\lambda V(t,x) + q\sum_{i=1}^m \lambda_i V(\sigma(t), x(\sigma(t))).$$

If t is right-dense, i.e., $\sigma(t) = t$, we have

$$D^+V^\Delta(t,x) \le -\Big(\lambda - q\sum_{i=1}^m \lambda_i\Big)V(t,x). \tag{6.3}$$

If t is right-scattered, i.e., $\sigma(t) > t$, we have

$$\begin{aligned} D^+V^\Delta(t,x) &= \frac{V(\sigma(t), x(\sigma(t))) - V(t,x)}{\mu(t)} \\ &\le -\lambda V(t,x) + q\sum_{i=1}^m \lambda_i V(\sigma(t), x(\sigma(t))) \\ &= -\lambda V(t,x) + \mu(t) q\sum_{i=1}^m \lambda_i \frac{V(\sigma(t), x(\sigma(t))) - V(t,x)}{\mu(t)} \\ &\quad + q\sum_{i=1}^m \lambda_i V(t,x) \\ &= -\Big(\lambda - q\sum_{i=1}^m \lambda_i\Big)V(t,x) + \mu(t) q \sum_{i=1}^m \lambda_i D^+V^\Delta(t,x), \end{aligned}$$

i.e.,

$$\Big(1 - \mu(t) q\sum_{i=1}^m \lambda_i\Big) D^+V^\Delta(t,x) \le -\Big(\lambda - q\sum_{i=1}^m \lambda_i\Big)V(t,x). \tag{6.4}$$

Since the unique root of equation (6.2) satisfies $q\bar{\mu} < (\sum_{i=1}^m \lambda_i)^{-1}$, we have

$$1 \ge 1 - \mu(t) q\sum_{i=1}^m \lambda_i \ge 1 - q\bar{\mu}\sum_{i=1}^m \lambda_i > 0. \tag{6.5}$$

Then, by (6.4) and (6.5), we have

$$\begin{aligned} D^+V^\Delta(t,x) &\le -\frac{\lambda - q\sum_{i=1}^m \lambda_i}{1 - \mu(t) q\sum_{i=1}^m \lambda_i} V(t,x) \\ &\le -\Big(\lambda - q\sum_{i=1}^m \lambda_i\Big)V(t,x). \end{aligned} \tag{6.6}$$

It follows from (6.3) and (6.6) that

$$D^+V^\Delta(t,x) \le -\Big(\lambda - q\sum_{i=1}^m \lambda_i\Big)V(t,x)$$

$$= -\frac{\ln q}{\bar{\mu}+\tau} V(t,x).$$

By $1 > \bar{\mu}\lambda$, we have

$$\begin{aligned} 1-\mu(t)\frac{\ln q}{\bar{\mu}+\tau} &= 1-\mu(t)\Big(\lambda - q\sum_{i=1}^{m}\lambda_i\Big) \\ &\geq 1-\bar{\mu}\lambda+\mu(t)q\sum_{i=1}^{m}\lambda_i \\ &> 0, \end{aligned}$$

which implies that $-\frac{\ln q}{\bar{\mu}+\tau} \in C_{rd}\mathcal{R}^{+}$. Therefore, by Theorem 6.3, the trivial solution of system (6.1) is globally exponentially stable. □

Remark 6.8. If $\mathbb{T} = \mathbb{Z}$ and denote $\lambda_0 = 1 - \lambda$, then condition $(ii)^{*}$ of Theorem 6.7 can be rewritten in the following form:

$$V(n+1, x(n+1)) \leq \lambda_0 V(n, x(n)) + \sum_{i=1}^{m} \lambda_i V(n+h_i(n), x(n+h_i(n))).$$

Hence, Theorem 6.7 reduces to Theorem 4.1 in [12]. Since the discrete version of Theorem 6.3 is easier than Theorem 3.3 in [12] to apply, our proof for the discrete version of Theorem 6.7 is simpler than the proof of Theorem 4.1 in [12].

Since it may not be easy to find a suitable Lyapunov function satisfying condition (ii)* of Theorem 6.7, we shall introduce the following corollary which makes our results more applicable.

Corollary 6.9. *Assume that there exist constants $L > 0$, $\lambda > 0$ and $\alpha_i \geq 0$, $i = 1, 2, \ldots, m$, such that*

$$\|F(t,x,0,\ldots,0)\|^2 \leq L\|x\|^2 \tag{6.7}$$

$$x^T F(t,x,0,\ldots,0) \leq -\lambda\|x\|^2 \tag{6.8}$$

$$\|F(t,x,y_1,\ldots,y_m) - F(t,x,0,\ldots,0)\| \leq \sum_{i=1}^{m} \alpha_i \|y_i\| \tag{6.9}$$

for all $t \in \mathbb{T}$ and $x, y_1, y_2, \ldots, y_m \in \mathbb{R}^m$. If $1 > \lambda_0\bar{\mu}$ and

$$\bar{\mu}L + \sum_{i=1}^{m}(1+\bar{\mu}m\alpha_i)\alpha_i < \lambda,$$

where $\lambda_0 = 2\lambda - 2\bar{\mu}L - \sum_{i=1}^{m}\alpha_i$, then the trivial solution of system (6.1) *is globally exponentially stable.*

Proof. Let $V(t,x)=x^Tx$ and $y_i(t)=x(\theta(t+h_i(t)))$, $i=1,2,\ldots,m$, then it suffices to show that (6.7), (6.8) and (6.9) imply condition (ii)* of Theorem 6.7. For, all $(t,x)\in\mathbb{T}_0\times\mathbb{R}^n$, we have

$$\begin{aligned} D^+V^\Delta(t,x) &= (x^Tx)^\Delta \\ &= x^Tx^\Delta+(x^\Delta)^Tx^\sigma \\ &= x^Tx^\Delta+(x^\Delta)^Tx+\mu(t)(x^\Delta)^Tx^\Delta \\ &= x^TF(t,x,y_1,\ldots,y_m)+F^T(t,x,y_1,\ldots,y_m)x \\ &\quad +\mu F^T(t,x,y_1,\ldots,y_m)F(t,x,y_1,\ldots,y_m), \end{aligned} \tag{6.10}$$

where $x^\sigma=x(\sigma(t))$. If we denote $G=F(t,x,y_1,\ldots,y_m)-F(t,x,0,\ldots,0)$ and $F=F(t,x,0,\ldots,0)$, then (6.10) implies that

$$\begin{aligned} D^+V^\Delta(t,x) &= 2x^TG+2x^TF+\mu G^TG+\mu F^TF+2\mu G^TF \\ &\le 2\|x\|\cdot\|G\|+2\mu\|G\|^2+2\mu\|F\|^2+2x^TF. \end{aligned}$$

Furthermore, by (6.7), (6.8) and (6.9), we have

$$\begin{aligned} D^+V^\Delta(t,x) &\le 2\sum_{i=1}^m\alpha_i\|x\|\cdot\|y_i\|-2\lambda x^Tx+2\mu(\sum_{i=1}^m\alpha_i\|y_i\|)^2+2\mu Lx^Tx \\ &\le (-2\lambda+2\mu L+\sum_{i=1}^m\alpha_i)x^Tx+\sum_{i=1}^m(\alpha_i+2\mu m\alpha_i^2)\|y_i\|^2 \\ &\le -(2\lambda-2\bar\mu L-\sum_{i=1}^m\alpha_i)V(t,x) \\ &\quad +\sum_{i=1}^m(\alpha_i+2\bar\mu m\alpha_i^2)V(\theta(t+h_i(t)),x(\theta(t+h_i(t)))), \end{aligned}$$

which implies that condition $(ii)^*$ of Theorem 6.7 holds. Hence, applying Theorem 6.7 yields the desired conclusion. □

When the function F is linear, then, for $m=1$, system (6.1) reduces to the following linear delay system on time scales

$$\begin{cases} x^\Delta(t) = Ax(t)+Bx(\theta(t+h(t))),\ t\in\mathbb{T}_0, \\ x_{t_0} = \phi,\ t_0\ge 0, \end{cases} \tag{6.11}$$

where $A,B\in\mathbb{R}^{n\times n}$ and $h:\mathbb{T}_0\to[-\tau,0]_\mathbb{R}$.

Corollary 6.10. *Let* $a=\sup_{t\in\mathbb{T}}\left\{\lambda_{\max}(A+A^T+2\mu A^TA+I)\right\}$ *and* $b=(1+2\bar\mu)\lambda_{\max}(B^TB)$. *In addition, we assume that* $a+b<0$ *and* $1+a\bar\mu>0$, *then the trivial solution of system* (6.11) *is globally exponentially stable.*

Proof. Let $V(t,x)=x^Tx$ and $x_h=x(\theta(t+h(t)))$, then we have

$$\begin{aligned} D^+V^\Delta(t,x) &= x^Tx^\Delta+(x^\Delta)^Tx+\mu(x^\Delta)^Tx^\Delta \\ &= x^T(A+A^T+\mu A^TA)x+2x^TBx_h+2\mu x^TA^TBx_h \\ &\quad +\mu x_h^TB^TBx_h \end{aligned}$$

$$\begin{aligned} &\leq x^T(A+A^T+2\mu A^TA+I)x+(1+2\mu)x_h^TB^TBx_h \\ &\leq aV(t,x)+bV(\theta(t+h(t)),x(\theta(t+h(t)))), \end{aligned}$$

which implies that condition (ii)* of Theorem 6.7 holds. Therefore, the trivial solution of system (6.11) is globally exponentially stable. □

7. NUMERICAL EXAMPLES

In this section, we shall apply the Razumikhin-type criteria established in previous sections to analyze stability of delay differential equations on time scales.

Example 7.1. Consider the following linear delay differential equation on time scale $\mathbb{T}$

$$\begin{cases} x^\Delta(t) &= cx(t)+dx(\theta(t-\tau)), \\ x_{t_0} &= \phi, \end{cases} \tag{7.1}$$

where $x \in \mathbb{R}$, $t_0 = 0 \in \mathbb{T}$, $c = -1.7$, $d = -0.2$, and $\tau = 2$.

If the graininess function $\mu(t)$ of time scale $\mathbb{T}$ satisfies $\mu(t) \leq 1$ for $t \in \mathbb{T}$, then all the conditions of Corollary 5.4 are satisfied. Hence, the trivial solution of equation (7.1) is uniformly asymptotically stable.

In the following simulations, we consider three types of time scales: $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = \mathbb{R}$, and time scale $\mathbb{T}$ chosen randomly with $\mu \leq 1$. For the two special cases $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{R}$, simulation results are shown in Figure 1(a) and Figure 1(b), respectively. Two time scales $\mathbb{T}_1$ and $\mathbb{T}_2$ are chosen randomly with $\mu \leq 1$, and the corresponding simulation results are shown in Figure 1(c) and Figure 1(d), respectively.

Next, we shall study the exponential stability of (7.1) with $c = -4$, $|d| = 1.75$ and $\tau > 0$ by using Corollary 6.10. Then, we have

$$\begin{aligned} a &= \sup_{t\in\mathbb{T}}\{1+2c+2c^2\mu(t)\}, \\ b &= (1+2\bar{\mu})d^2. \end{aligned}$$

(1) If $\mathbb{T} = \mathbb{T}_3 = \frac{1}{10}\mathbb{Z} = \{\dots, 0, \frac{1}{10}, \frac{2}{10}, \dots\}$, then $\mu(t) = \bar{\mu} = \frac{1}{10}$ and

$$\begin{aligned} a &= 1+2c+\frac{1}{5}c^2, \\ b &= \frac{6}{5}d^2. \end{aligned}$$

Obviously, $a + b < 0$ and $1 + a\bar{\mu} > 0$, which imply that all the conditions of Corollary 6.10 hold. Hence, the trivial solution of equation (7.1) is globally exponentially stable. Numerical simulation is shown in Figure 1(e) for equation (7.1) on time scale $\mathbb{T}_3$ with $d = -1.75$, $\tau = 0.5$ and $\phi(s) = -0.8$ for $s \in \{-0.5, -0.4, \dots, -0.1, 0\}$.

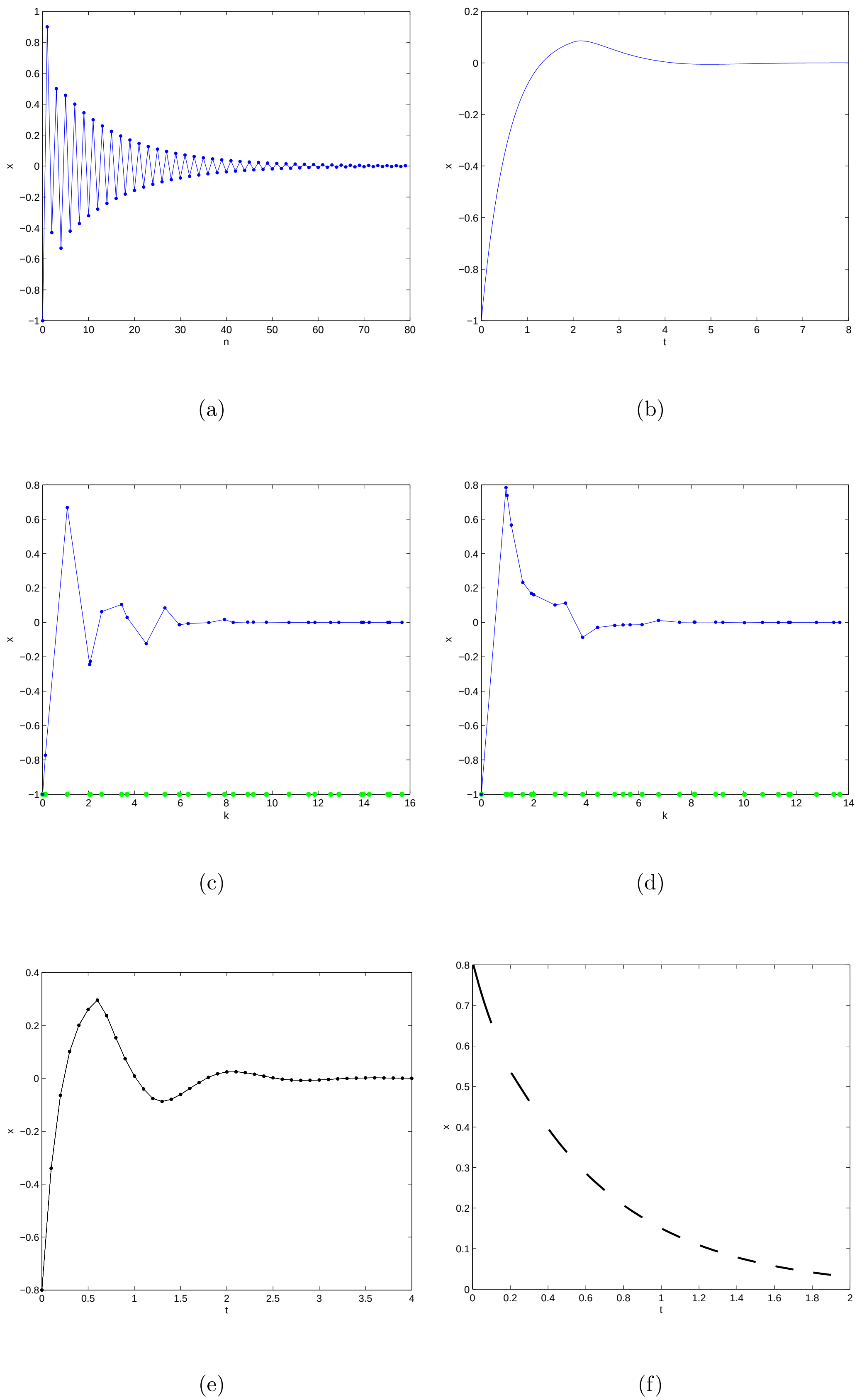

FIGURE 1. Numerical simulations of (7.1) on time scales: (a) $\mathbb{Z}$, (b) $\mathbb{R}$, (c) $\mathbb{T}_1$, (d) $\mathbb{T}_2$, (e) $\mathbb{T}_3$, (f) $\mathbb{T}_4$. In (c) and (d), the green dots represent the time scales randomly generated with $\mu \leq 1$.

(2) If $\mathbb{T} = \mathbb{T}_4 = \bigcup_{k=-1}^{\infty}[\frac{1}{5}k, \frac{1}{5}k + \frac{1}{10}]$, then we have

$$\mu(t) = \begin{cases} 0, & t \in \bigcup_{k=-1}^{\infty}[\frac{1}{5}k, \frac{1}{5}k + \frac{1}{10}), \\ \frac{1}{10}, & t = \frac{1}{5}k + \frac{1}{10},\ k = -1, 0, 1, 2, \ldots \end{cases}$$

and $\bar{\mu} = \frac{1}{10}$, $a = 1 + 2c + \frac{1}{5}c^2$, $b = \frac{6}{5}d^2$. Therefore, all the conditions of Corollary 6.10 are satisfied. We choose $d = 1.75$, $\tau = 0.2$ and $\phi(s) = 0.8$ for $s \in [-\tau, 0]_{\mathbb{R}}$, then, from Figure 1(f), one can see that the trivial solution of equation (7.1) is globally exponentially stable.

Example 7.2. Consider the nonlinear continuous systems with time delay

$$\begin{cases} \dot{x}(t) &= Ax(t) + g(t, x(t), x(t-\tau)),\ t \geq t_0 = 0, \\ x_{t_0} &= \phi, \end{cases} \tag{7.2}$$

where $x = (x_1, x_2, x_3)^T$,

$$A = \begin{bmatrix} -1 & 0 & -2 \\ 0 & -5 & -2 \\ 1 & 0 & -4 \end{bmatrix},$$

and $g(t, x(t), x(t-\tau)) = \frac{3}{2}\Big(x_1(t-\tau)\sin(x_3(t)),\ x_2(t-\tau)\cos(x_2(t)),\ \frac{x_3(t-\tau)}{1+\sin^2 t}\Big)^T$, τ is a positive constant.

Let $F(t, x(t), x(t-\tau)) = Ax(t) + g(t, x(t), x(t-\tau))$, then $F(t, x(t), 0) = Ax(t)$. It is easy to see that

$$x^T F(t, x, 0) = x^T A x \leq \lambda_{\max}(A) x^T x,$$

$$\|F(t, x, x(t-\tau)) - F(t, x, 0)\| = \|g(t, x, x(t-\tau))\| \leq \frac{3}{2}\|x(t-\tau)\|.$$

Since time scale $\mathbb{T} = \mathbb{R}$, i.e., $\mu(t) \equiv 0$ and $\bar{\mu} = 0$, all the conditions of Corollary 6.9 are satisfied with $\lambda = -\lambda_{\max}(A) = 2$, $m = 1$ and $\alpha_1 = \frac{3}{2}$. Hence, the trivial solution of system (7.2) is globally exponentially stable. The numerical simulation result is shown in Figure 2 with $\tau = 0.2$ and $\phi(s) = [0.75, 0.25, -0.5]^T$ for $s \in [-\tau, 0]$.

8. CONCLUSIONS

In this paper, the fundamental theory of general functional differential equations on time scales have been studied. Some results of local and global existence, uniqueness, and extended existence of solutions have been introduced and proved. Several Razumikhin-type stability criteria have also been presented, and then applied to discuss stability property of various linear and nonlinear delay systems on time scales.

One possible future direction is to investigate the stability property by using the Lyapunov-Krasovskii functionals; another research direction is to extend the work to impulsive functional differential equations on time scales.

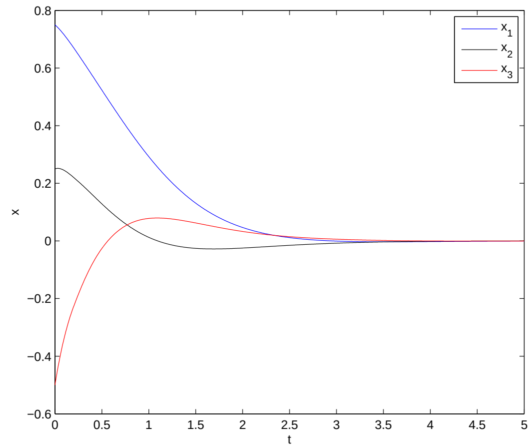

FIGURE 2. Numerical simulations of continuous-time system (7.2).

9. ACKNOWLEDGEMENT

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