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STRONG MAXIMUM PRINCIPLES FOR FRACTIONAL DIFFUSION DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, a strong maximum principle for the fractional diffusion differential equation is established. The differential equation being studied is defined in the sense of Riemann-Liouville fractional derivative, which is used to formulate the heat diffusion in material with subdiffusive properties.

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1. INTRODUCTION

Let α and T be positive real numbers with $0 < \alpha < 1$, and N be a natural number. Let $x = (x_1, x_2, \dots, x_N)$, Ω be a bounded open domain in $\mathbb{R}^N$, and $\partial\Omega$ be the smooth boundary of Ω. Let $a_i > 0$ for $i = 1, \dots, N$, $c(x,t)$ be a continuous function on $\bar{\Omega} \times [0,T]$, and

$$L_\alpha u = u_t - \sum_{i=1}^{N} a_i \frac{\partial^2}{\partial x_i^2}\left(D_t^{1-\alpha} u\right) - c(x,t)u,$$

where $D_t^{1-\alpha} u$ is the Riemann-Liouville fractional derivative defined as:

$$D_t^{1-\alpha} u(x,t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-1+\alpha} u(x,s)\,ds.$$

We consider the following fractional diffusion equation

$$L_\alpha u = F(x,t) \text{ in } \Omega \times (0,T], \tag{1.1}$$

subject to the initial and boundary conditions

$$\begin{cases} u(x,0) = \phi(x) \text{ on } \bar{\Omega}, \\ u(x,t) = \psi(t) \text{ on } \partial\Omega \times (0,T], \end{cases} \tag{1.2}$$

where F, ϕ, and ψ are continuous functions.

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It follows from a direct computation that for a positive integer k, if g is a non-constant function with $g \in C^k$, then the Riemann-Liouville fractional derivatives of order $k \geq 1$ coincide with the conventional derivatives of order k (cf. Podlubny [7, p. 69]), that is $D_t^k g(t) = d^k/dt^k g(t)$.

Problems involving partial differential equations of fractional orders have been used for modeling in engineering, science, economics and other fields (cf. Chechkin, Gorenflo and Sokolov [1], Gorenflo and Mainmardi [3], and Podlubny [7]). In particular, problems of thermal diffusion with subdiffusive and superdiffusive properties are formulated in terms of fractional diffusion equations (cf. Kirk and Olmstead [4], and Olmstead and Roberts [6]). To investigate this type of problems, techniques (such as fixed point theorems, and the method of lower and upper solutions) analogous to the classical diffusion equation are used.

Recently, Chan and Liu [2] obtained a weak maximum principle for fractional diffusion equations. In this paper, we establish a strong form of the maximum principle for the fractional diffusion equations, which is comparable to the maximum principle for the classical diffusion equation.

2. MAXIMUM PRINCIPLES

At the local extreme value of a function, Luchko [5] showed that the Riemann-Liouville derivative may not be zero. We give an estimation of the derivative at the local extrema as follow. The technique obtaining these results is similar to the proof of Theorem 2.4 of Al-Refai [8].

Theorem 2.1. *Let $g(t) \in C[0,T]$. Assume that $g'(t)$ exists and is continuous for $t \in (0,T]$, and $g(t)$ attains its minimum value over $[0,T]$ at the point $t_0 \in (0,T]$. Then for $0 < \alpha < 1$,*

$$D_t^{1-\alpha} g(t_0) \leq \frac{t_0^{\alpha-1}}{\Gamma(\alpha)} g(t_0).$$

Remark 2.2. By applying the above argument to $-g(t)$, we obtain the result that if $g(t)$ attains its maximum at $t_0 \in (0,T]$, then

$$D_t^{1-\alpha} g(t_0) \geq \frac{t_0^{\alpha-1}}{\Gamma(\alpha)} g(t_0).$$

The following result on positivity of the solution u is an extended result of Theorem 2.3 of Chan and Liu [2].

Lemma 2.3. *If u satisfies (1.1), $u(x,0)>0$ for $x\in\bar{\Omega}$, $u>0$ on $\partial\Omega\times(0,T]$, and $F>0$ and $c\geq 0$ for $(x,t)\in\Omega\times(0,T]$, then $u>0$ on $\bar{\Omega}\times[0,T]$.*

Proof. We multiply $(\tau-t)^{-\alpha}$ and integrate with respect to t from 0 to τ on both sides of (1.1). By using $F>0$, we obtain

$$\begin{aligned}\int_0^\tau(\tau-t)^{-\alpha}u_t(x,t)dt &-\int_0^\tau(\tau-t)^{-\alpha}\sum_{i=1}^N a_i\frac{\partial^2}{\partial x_i^2}\left(D_t^{1-\alpha}u(x,t)\right)dt\\ &-\int_0^\tau(\tau-t)^{-\alpha}c(x,t)u(x,t)dt>0.\end{aligned}\tag{2.1}$$

Let us rewrite

$$\int_0^\tau(\tau-t)^{-\alpha}u_t(x,t)dt=\frac{1}{1-\alpha}\frac{d}{d\tau}\int_0^\tau(\tau-t)^{1-\alpha}u_t(x,t)dt.$$

Upon integration by parts the term on the right-hand side, we get

$$\begin{aligned}\int_0^\tau(\tau-t)^{-\alpha}u_t(x,t)dt &=-\tau^{-\alpha}u(x,0)+\frac{d}{d\tau}\int_0^\tau(\tau-t)^{-\alpha}u(x,t)dt\\ &=-\tau^{-\alpha}u(x,0)+\Gamma(1-\alpha)D_\tau^\alpha u.\end{aligned}$$

For the second term on the left-hand side of the inequality (2.1), we interchange the order of differentiation and integration, and get

$$\begin{aligned}\int_0^\tau(\tau-t)^{-\alpha}\sum_{i=1}^N a_i\frac{\partial^2}{\partial x_i^2}\left(D_t^{1-\alpha}u(x,t)\right)dt &=\sum_{i=1}^N a_i\frac{\partial^2}{\partial x_i^2}\int_0^\tau(\tau-t)^{-\alpha}\left(D_t^{1-\alpha}u(x,t)\right)dt\\ &=\sum_{i=1}^N a_i\frac{\partial^2}{\partial x_i^2}u(x,\tau).\end{aligned}$$

From (2.1), u satisfies

$$\Gamma(1-\alpha)D_\tau^\alpha u-\sum_{i=1}^N a_i\frac{\partial^2}{\partial x_i^2}u(x,\tau)>\tau^{-\alpha}u(x,0)+\int_0^\tau(\tau-t)^{-\alpha}c(x,t)u(x,t)dt.\tag{2.2}$$

Suppose that $u\leq 0$ somewhere on $\bar{\Omega}\times[0,T]$. Let $t_0=\sup\{s>0:u(x,t)>0$ for $(x,t)\in\bar{\Omega}\times[0,s)\ \}$. Since $u>0$ on $\partial\Omega\times[0,T]\cup\bar{\Omega}\times\{0\}$, we have $t_0>0$, and there is x_0 in Ω such that $u(x_0,t_0)=0$ and $u(x,t)>0$ in $\bar{\Omega}\times[0,t_0)$. Hence u attains its minimum value at (x_0,t_0) on $\bar{\Omega}\times[0,t_0]$. It follows from Lemma 2.1 of Chan and Liu [2] that $D_\tau^\alpha u(x_0,t_0)\leq t_0^{-\alpha}u(x_0,t_0)/\Gamma(1-\alpha)=0$. Since u attains its minimum at (x_0,t_0) on $\bar{\Omega}\times[0,t_0]$, we get $\sum_{i=1}^N a_i\partial^2/\partial x_i^2 u(x_0,t_0)\geq 0$. Hence

$\Gamma(1-\alpha)D_\tau^\alpha u(x_0,t_0)-\sum_{i=1}^N a_i\partial^2/\partial x_i^2 u(x_0,t_0)\le 0$. On the other hand, since $u(x_0,t)>0$ for $t\in[0,t_0)$, we have

$$\int_0^{t_0}(t_0-t)^{-\alpha}c(x_0,t)u(x_0,t)dt\ge 0.$$

Therefore, it follows from (2.2) that

$$\begin{aligned}0 &\ge \Gamma(1-\alpha)D_\tau^\alpha u(x_0,t_0)-\sum_{i=1}^N a_i\frac{\partial^2}{\partial x_i^2}u(x_0,t_0)\\ &> t_0^{-\alpha}u(x_0,0)+\int_0^{t_0}(t_0-t)^{-\alpha}c(x_0,t)u(x_0,t)dt>0.\end{aligned}$$

This contradiction shows that $u>0$ on $\bar\Omega\times[0,T]$. □

A more general case for non-strict inequality is given as follows.

Theorem 2.4. *If u satisfies* (1.1)*, $u(x,0)\ge 0$ for $x\in\bar\Omega$, $u\ge 0$ on $\partial\Omega\times(0,T]$, and $F\ge 0$ and $c\ge 0$ in $\Omega\times(0,T]$, then $u(x,t)\ge 0$ on $\bar\Omega\times[0,T]$.*

Proof. Let R be a positive real number and $B=(-R,R)\times(-R,R)\cdots\times(-R,R)$ be a N-dimensional box with $\bar\Omega\subsetneq B$. Let $v(x,t)=\sum_{i=1}^N[(R^2-x_i^2)E_{1-\alpha,1}(\lambda t^{1-\alpha})]$, where $|x_i|<R$ for $i=1,2,\dots,N$, $E_{\mu,\nu}(z)$ is the Mittag-Leffler function which is a bounded, positive, and nondecreasing function in z, and λ is a positive number to be determined. Since $E_{1-\alpha,1}(z)>0$ for any $z\ge 0$, we have $v>0$ in $B\times[0,T]$. Since $\bar\Omega\subsetneq B$, we obtain $v>0$ on $\bar\Omega\times[0,T]$. The t derivative of Mittag-Leffler function can be obtained by a direct computation,

$$\begin{aligned}\frac{d}{dt}\left(E_{1-\alpha,1}(\lambda t^{1-\alpha})\right) &= \left(E_{1-\alpha,0}(\lambda t^{1-\alpha})-(1-1)E_{1-\alpha,1}(\lambda t^{1-\alpha})\right)\cdot\frac{\lambda(1-\alpha)t^{-\alpha}}{\lambda t^{1-\alpha}}\\ &=(1-\alpha)t^{-1}E_{1-\alpha,0}(\lambda t^{1-\alpha}).\end{aligned}$$

Furthermore its fractional derivative is given as

$$D_t^{1-\alpha}\left(E_{1-\alpha,1}(\lambda t^{1-\alpha})\right)(t)=t^{1-(1-\alpha)-1}E_{1-\alpha,\alpha}(\lambda t^{1-\alpha})=t^{-1+\alpha}E_{1-\alpha,\alpha}(\lambda t^{1-\alpha}).$$

By using the identity $t^{-1+\alpha}E_{1-\alpha,\alpha}(\lambda t^{1-\alpha})=t^{-1+\alpha}/\Gamma(\alpha)+\lambda E_{1-\alpha,1}(\lambda t^{1-\alpha})$, we get

$$v_t=\sum_{i=1}^N\left[(R^2-x_i^2)(1-\alpha)t^{-1}E_{1-\alpha,0}(\lambda t^{1-\alpha})\right]$$

and

$$D_t^{1-\alpha}v=\sum_{i=1}^N\left[(R^2-x_i^2)\left(\frac{t^{-1+\alpha}}{\Gamma(\alpha)}+\lambda E_{1-\alpha,1}(\lambda t^{1-\alpha})\right)\right].$$

Hence v satisfies

$$\begin{aligned} L_\alpha v = & \sum_{i=1}^N \left[(R^2 - x_i^2)(1-\alpha)t^{-1}E_{1-\alpha,0}(\lambda t^{1-\alpha}) \right] + 2N\frac{t^{-1+\alpha}}{\Gamma(\alpha)}\sum_{i=1}^N a_i \\ & + \left\{ 2N\lambda \sum_{i=1}^N a_i - c(x,t)\left[\sum_{i=1}^N (R^2 - x_i^2) \right] \right\} E_{1-\alpha,1}(\lambda t^{1-\alpha}). \end{aligned}$$

By taking $\lambda > \max_{(x,t)\in\bar{\Omega}\times[0,T]} c(x,t)R^2/2\sum_{i=1}^N a_i$, we have

$$\left\{ 2N\lambda \sum_{i=1}^N a_i - c(x,t)\left[\sum_{i=1}^N (R^2 - x_i^2) \right] \right\} E_{1-\alpha,1}(\lambda t^{1-\alpha}) > 0.$$

Thus $L_\alpha v > 0$ in $\Omega \times (0,T]$.

Now let $w = u + \epsilon v$ for any $\epsilon > 0$ on $\bar{\Omega} \times [0,T]$. Then $L_\alpha w = L_\alpha u + \epsilon L_\alpha v > 0$ in $\Omega \times (0,T]$, $w(x,0) = u(x,0) + \epsilon v(x,0) > 0$ for $x \in \bar{\Omega}$, and $w = u + \epsilon v > 0$ on $\partial\Omega \times [0,T]$. It follows from Lemma 2.3 that $w > 0$ on $\bar{\Omega} \times [0,T]$. This gives $u > -\epsilon v$ for any $\epsilon > 0$. Therefore $u \geq 0$ on $\bar{\Omega} \times [0,T]$. □

A similar result can be obtained for the negativity of the solution $u(x,t)$ by considering $-u(x,t)$ for $F \leq 0$, $c \geq 0$, and $u(x,t) \leq 0$ initially and on the boundary $\partial\Omega$.

Theorem 2.5. *If u satisfies* (1.1)*, $u(x,0) \leq 0$ for $x \in \bar{\Omega}$, $u \leq 0$ on $\partial\Omega \times (0,T]$, and $F \leq 0$ and $c \geq 0$ in $\Omega \times (0,T]$, then $u(x,t) \leq 0$ on $\bar{\Omega} \times [0,T]$.*

Remark 2.6. Suppose that u satisfies $L_\alpha u > 0$ in $\Omega \times (0,T]$ and other conditions in Theorem 2.4 hold, then $u(x,t) \geq 0$ on $\bar{\Omega} \times [0,T]$. If $u(x_0,t_0) = 0$ for some $(x_0,t_0) \in \Omega \times (0,T]$, then $u(x_0,t_0)$ is a minimum of $u(x,t)$ on $\bar{\Omega} \times [0,T]$. It follows from a similar argument as in the proof of Lemma 2.3 that $L_\alpha u \not> 0$, which leads to a contradiction. Hence $u > 0$ in $\Omega \times (0,T]$.

Similar to Theorem 2.4 of Chan and Liu [2], we can show that u attains its minimum value on the parabolic boundary of $\Omega \times (0,T]$.

Theorem 2.7. *Suppose u satisfies* (1.1)*, $u(x,0) = \phi(x) \geq 0$ on $\bar{\Omega}$, $u(x,t) = \psi(t) \geq 0$ on $\partial\Omega\times(0,T]$. If $F \geq 0$ and $c \geq 0$ in $\Omega\times(0,T]$, then $u \geq \min\{\min_{\bar{\Omega}}\phi(x), \min_{[0,T]}\psi(t)\}$ on $\bar{\Omega} \times [0,T]$.*

Proof. Let $m = \min\{\min_{\bar{\Omega}} \phi(x), \min_{[0,T]} \psi(t)\}$, and $\bar{u} = u - m$. Then, $\bar{u}(x,0) = \phi(x) - m \geq 0$ for $x \in \bar{\Omega}$, $\bar{u} = \psi(t) - m \geq 0$ on $\partial\Omega \times [0,T]$. Since

$$\frac{\partial}{\partial t}\bar{u} = \frac{\partial}{\partial t}u, \sum_{i=1}^{N} a_i \frac{\partial^2}{\partial x_i^2} D_t^{1-\alpha}\bar{u} = \sum_{i=1}^{N} a_i \frac{\partial^2}{\partial x_i^2} D_t^{1-\alpha}u,$$

and $m \geq 0$, $\bar{u}$ satisfies

$$L_\alpha \bar{u} \geq cm \geq 0. \tag{2.3}$$

It follows from Theorem 2.4 that $\bar{u} \geq 0$ on $\bar{\Omega} \times [0,T]$. That is,

$$u \geq \min\{\min_{\bar{\Omega}} \phi(x), \min_{[0,T]} \psi(t)\} \text{ on } \bar{\Omega} \times [0,T].$$

□

We note that in the proof of Theorem 2.7, (2.3) with $c \equiv 0$ gives $L_\alpha \bar{u} \geq 0$ for any m. The following result holds.

Theorem 2.8. *Suppose u satisfies* (1.1) *with $c \equiv 0$, $u(x,0) = \phi(x)$ on $\bar{\Omega}$, $u(x,t) = \psi(t)$ on $\partial\Omega \times (0,T]$. If $F \geq 0$, then $u \geq \min\{\min_{\bar{\Omega}} \phi(x), \min_{[0,T]} \psi(t)\}$ on $\bar{\Omega} \times [0,T]$.*

Similar to Theorems 2.7 and 2.8, we have the following results.

Theorem 2.9. *Suppose u satisfies* (1.1), *$u(x,0) = \phi(x) \leq 0$ on $\bar{\Omega}$, $u(x,t) = \psi(t) \leq 0$ on $\partial\Omega \times (0,T]$. If $F \leq 0$ and $c \geq 0$ in $\Omega \times (0,T]$, then $u \leq \max\{\max_{\bar{\Omega}} \phi(x), \max_{[0,T]} \psi(t)\}$ on $\bar{\Omega} \times [0,T]$.*

Theorem 2.10. *Suppose u satisfies* (1.1) *with $c \equiv 0$, $u(x,0) = \phi(x)$ on $\bar{\Omega}$, $u(x,t) = \psi(t)$ on $\partial\Omega \times (0,T]$. If $F \leq 0$, then $u \leq \max\{\max_{\bar{\Omega}} \phi(x), \max_{[0,T]} \psi(t)\}$ on $\bar{\Omega} \times [0,T]$.*

Suppose that the conditions in Theorem 2.4 hold. Then $u \geq 0$ on $\bar{\Omega} \times [0,T]$. For $u \not\equiv 0$, assume that there is $t_1 > 0$ such that $u \equiv 0$ on $\bar{\Omega} \times [0,t_1)$, and $u > 0$ somewhere on $\bar{\Omega} \times \{t_1\}$. Then we have $D_t^{1-\alpha}u = 0$ for t in $(0,t_1)$ and $x \in \bar{\Omega}$. Hence, for $t_1 \leq t \leq T$,

$$D_t^{1-\alpha}u = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-1+\alpha} u(x,s)ds = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_{t_1}^t (t-s)^{-1+\alpha} u(x,s)ds.$$

Let us denote

$${}_{t_1}D_t^{1-\alpha}u = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_{t_1}^t (t-s)^{-1+\alpha} u(x,s)ds.$$

Then u satisfies

$$u_t - \sum_{i=1}^{N} a_i \frac{\partial^2}{\partial x_i^2} \left({}_{t_1}D_t^{1-\alpha} u\right) - cu \geq 0$$

in $\Omega \times (t_1, T)$ with $u(x, t_1) \geq 0$ for $x \in \bar{\Omega}$ and $u(x,t) \geq 0$ on $\partial\Omega \times [t_1, T]$. By using the change of variables $\tau = t - t_1$, we get

$$u_\tau - \sum_{i=1}^{N} a_i \frac{\partial^2}{\partial x_i^2} \left(D_\tau^{1-\alpha} u\right) - cu \geq 0$$

in $\Omega \times (0, T - t_1)$ with $u(x, 0) \geq 0$ for $x \in \bar{\Omega}$, $u(\tilde{x}, 0) > 0$ for some $\tilde{x} \in \Omega$ and $u(x,t) \geq 0$ on $\partial\Omega \times [0, T - t_1]$. Thus, without loss of generality, we can assume that $t_1 = 0$, that is, there is $x \in \bar{\Omega}$ such that $u(x,t) > 0$ for $t > 0$ if $u \not\equiv 0$.

The following theorem gives a strong form of the maximum principle similar to the classical cases.

Theorem 2.11. *Suppose u satisfies* (1.1), *$u(x, 0) \geq 0$ on $\bar{\Omega}$, $u(x,t) \geq 0$ on $\partial\Omega \times (0, T]$ with $F \geq 0$ and $c \geq 0$ in $\Omega \times (0, T]$. If $u(x,t) = 0$ for some $(x,t) \in \Omega \times (0, T)$, then $u \equiv 0$ on $\bar{\Omega} \times [0, T]$.*

Proof. Since u satisfies the conditions of Theorem 2.4, we have $u \geq 0$ on $\bar{\Omega} \times [0, T]$. Firstly, we assume that there is $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_i, \ldots, \tilde{x}_N) \in \Omega$ such that $u(\tilde{x}, t) > 0$ for $0 < t \leq \tilde{t}$, and we claim that $u(X_i, t) > 0$ for any $X_i = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{i-1}, x_i, \tilde{x}_{i+1}, \ldots, \tilde{x}_N)$ on line segment along the i-th component direction in Ω passing through the point $\tilde{x}$, and $0 < t \leq \tilde{t}$. Suppose the claim is false. Let $Y_i = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{i-1}, y_i, \tilde{x}_{i+1}, \ldots, \tilde{x}_N) \in \Omega$ and $0 < \hat{t} \leq \tilde{t}$ such that $u(Y_i, \hat{t}) = 0$. Then u attains its minimum value zero at $(Y_i, \hat{t})$ in $\bar{\Omega} \times [0, \tilde{t}]$. Without loss of generality, we assume that $y_i < \tilde{x}_i$. We take $\delta > 0$ such that $X_i = (\tilde{x}_1, \ldots, \tilde{x}_{i-1}, x_i, \tilde{x}_{i+1}, \ldots, \tilde{x}_N) \in \Omega$ for any $y_i - \delta \leq x_i \leq \tilde{x}_i$. We now consider the one-dimensional spatial case as $D_i = (y_i - \delta, \tilde{x}_i) \times (0, \hat{t}]$, and $\partial D_i = ([y_i - \delta, \tilde{x}_i] \times \{0\}) \cup \left(\{y_i - \delta, \tilde{x}_i\} \times [0, \hat{t}]\right)$. Then $u \geq 0$ on ∂D_i. Let $v(x_i, t) = E_{1,3}(\lambda t) \cdot \left(e^{\beta y_i} - \frac{t^2}{\hat{t}^2} e^{\beta x_i}\right)$ where λ, β are positive numbers to be determined. Then $v(x_i, 0) = e^{\beta y_i} > 0$, $v(y_i - \delta, t) = E_{1,3}(\lambda t) e^{\beta y_i} \left(1 - \frac{t^2}{\hat{t}^2} e^{-\beta\delta}\right) > 0$, $v(\tilde{x}_i, t) = E_{1,3}(\lambda t) \left(e^{\beta y_i} - \frac{t^2}{\hat{t}^2} e^{\beta \tilde{x}_i}\right)$ for $0 < t \leq \hat{t}$, and $v(y_i, \hat{t}) = 0$. By a direct computation, we get

$$v_t(x_i, t) = \left(\frac{E_{1,2}(\lambda t) - 2E_{1,3}(\lambda t)}{t}\right) \cdot \left(e^{\beta y_i} - \frac{t^2}{\hat{t}^2} e^{\beta x_i}\right) - E_{1,3}(\lambda t) \frac{2t}{\hat{t}^2} e^{\beta x_i},$$

and

$$\left(D_t^{1-\alpha}v(x_i,t)\right)_{x_ix_i} = -\frac{\beta^2 e^{\beta x_i}}{\hat{t}^2}\left(t^{1+\alpha}E_{1,2+\alpha}(\lambda t)\right).$$

Then,

$$\begin{aligned} L_\alpha v(x_i,t) = &\left(\frac{E_{1,2}(\lambda t)-2E_{1,3}(\lambda t)}{t}\right)\cdot\left(e^{\beta y_i}-\frac{t^2}{\hat{t}^2}e^{\beta x_i}\right)-E_{1,3}(\lambda t)\frac{2t}{\hat{t}^2}e^{\beta x_i} \\ &+\frac{a_i\beta^2 e^{\beta x_i}}{\hat{t}^2}\left(t^{1+\alpha}E_{1,2+\alpha}(\lambda t)\right)-c(X_i,t)E_{1,3}(\lambda t)\cdot\left(e^{\beta y_i}-\frac{t^2}{\hat{t}^2}e^{\beta x_i}\right). \end{aligned}$$

This gives

$$\begin{aligned} L_\alpha v(x_i,t) \geq &\left(\frac{E_{1,2}(\lambda t)-2E_{1,3}(\lambda t)}{t}\right)e^{\beta y_i}-c(X_i,t)E_{1,3}(\lambda t)e^{\beta y_i} \\ &-\left(\frac{E_{1,2}(\lambda t)-2E_{1,3}(\lambda t)}{t}\right)\frac{t^2}{\hat{t}^2}e^{\beta x_i}-E_{1,3}(\lambda t)\frac{2t}{\hat{t}^2}e^{\beta x_i} \\ &+\frac{\beta^2 e^{\beta x_i}}{\hat{t}^2}\left(t^{1+\alpha}E_{1,2+\alpha}(\lambda t)\right). \end{aligned} \tag{2.4}$$

It follows from the series representation form of Mittag-Leffler function $E_{\mu,\nu}$ that

$$\frac{E_{1,2}(\lambda t)-2E_{1,3}(\lambda t)}{t}=\sum_{n=1}^{\infty}\frac{n\lambda^n t^{n-1}}{(n+1)\Gamma(n+2)}.$$

The first two terms on the right-hand side of (2.4) can be estimated as follows:

$$\begin{aligned} &\left(\frac{E_{1,2}(\lambda t)-2E_{1,3}(\lambda t)}{t}\right) e^{\beta y_i}-c(X_i,t)E_{1,3}(\lambda t)e^{\beta y_i} \\ &=\left(\sum_{n=1}^{\infty}\frac{n\lambda^n t^{n-1}}{(n+1)\Gamma(n+2)}-c(X_i,t)\sum_{n=0}^{\infty}\frac{\lambda^n t^n}{\Gamma(n+3)}\right)e^{\beta y_i} \\ &\geq\left(\sum_{n=1}^{\infty}\frac{n\lambda^n t^{n-1}}{(n+1)\Gamma(n+2)}-\max c(X_i,t)\sum_{n=1}^{\infty}\frac{\lambda^{n-1}t^{n-1}}{\Gamma(n+2)}\right)e^{\beta y_i} \\ &=\left(\sum_{n=1}^{\infty}\frac{\lambda^{n-1}t^{n-1}}{\Gamma(n+2)}\left(\lambda\frac{n}{n+1}-\max c(X_i,t)\right)\right)e^{\beta y_i}. \end{aligned}$$

By taking $\lambda > 2\max_{y_i-\delta\leq x_i\leq\tilde{x}_i, 0\leq t\leq\hat{t}} c(X_i,t)$, we have

$$\left(\frac{E_{1,2}(\lambda t)-2E_{1,3}(\lambda t)}{t}\right)e^{\beta y_i}-c(X_i,t)E_{1,3}(\lambda t)e^{\beta y_i}>0 \text{ on } \bar{D}_i.$$

Since the last three terms on the right-hand side of (2.4) approach zero as t tends to zero, there is $\bar{t}>0$ such that $L_\alpha v(x_i,t)>0$ for $x_i\in(y_i-\delta,\tilde{x}_i)$, and $t\in(0,\bar{t})$. If $\bar{t}\geq\hat{t}$, then the proof is complete. Otherwise, if $\hat{t}>\bar{t}$, we choose $\beta>0$ sufficiently large such that

$$e^{\beta x_i}\left(\frac{a_i\beta^2}{\hat{t}^2}\left(\bar{t}^{1+\alpha}E_{1,2+\alpha}(\lambda\bar{t})\right)-\left(\frac{E_{1,2}(\lambda\hat{t})-2E_{1,3}(\lambda\bar{t})}{\bar{t}}\right)\frac{\hat{t}^2}{\hat{t}^2}-E_{1,3}(\lambda\hat{t})\frac{2\hat{t}}{\hat{t}^2}\right)>0.$$

This gives that $L_\alpha v(x_i,t) > 0$ for $x_i \in (y_i - \delta, \tilde{x}_i)$ and $t \in [\bar{t}, \hat{t})$. Since $L_\alpha v(x_i,t) > 0$ for $x_i \in (y_i - \delta, \tilde{x}_i)$ and $t \in (0, \bar{t})$, we get $L_\alpha v(x_i,t) > 0$ in D_i. Next we let $\tilde{v}(X_i,t) = v(x_i,t)$ for $X_i = (\tilde{x}_1, \ldots, \tilde{x}_{i-1}, x_i, \tilde{x}_{i+1}, \ldots, \tilde{x}_N)$. Then $\tilde{v}(Y_i, \hat{t}) = 0$, and $L_\alpha \tilde{v}(X_i,t) > 0$ for $x_i \in (y_i - \delta, \tilde{x}_i)$ and $t \in (0, \hat{t})$. At $X_i = \tilde{x}$, we get $\tilde{v}(\tilde{x}, t) > 0$ for $t < \hat{t}e^{\beta(y_i - \tilde{x}_i)/2}$. For $t \in [\hat{t}e^{\beta(y_i - \tilde{x}_i)/2}, \hat{t}]$, since $u(\tilde{x}, t) > 0$ for $t > 0$, let $m = \min_{\hat{t}e^{\beta(y_i - \tilde{x}_i)/2} \le t \le \hat{t}} u(\tilde{x}, t)$, we have $m > 0$. Let us pick $\epsilon > 0$ such that $m + \epsilon E_{1,3}(\lambda \hat{t}) e^{\beta y_i} \left(1 - e^{\beta(y_i - \tilde{x}_i)}\right) > 0$. Then $u(\tilde{x}, t) + \epsilon \tilde{v}(\tilde{x}, t) > 0$ for $t \in [\hat{t}e^{\beta(y_i - \tilde{x}_i)/2}, \hat{t}]$. Let $w(X_i, t) = u(X_i, t) + \epsilon \tilde{v}(X_i, t)$. Then $w(X_i, t) > 0$ for $(x_i, t) \in \partial D_i$. Also $L_\alpha w(X_i, t) = L_\alpha \left(u(X_i, t) + \epsilon \tilde{v}(X_i, t)\right) > 0$. An argument similar to the proof of Theorem 2.4 shows that $w(X_i, t) > 0$ for $(x_i, t) \in \bar{D}_i$. But $w(Y_i, \hat{t}) = u(Y_i, \hat{t}) + \epsilon \tilde{v}(Y_i, \hat{t}) = 0$; this leads to a contradiction. This shows that $u(X_i, t) > 0$ for any X_i on line segment parallel to the spatial axis in Ω passing through the point $\tilde{x}$, and $0 < t \le \tilde{t}$. Then for any $(x, t) \in \Omega \times (0, T]$, the point x is joined to $\tilde{x}$ with line segment parallel to the spatial axis. Then $u(x, t) > 0$ for $0 < t \le \tilde{t}$. This proves our claim.

Next, we assume that there is $(x_1, t_1) \in \Omega \times (0, T)$ such that $u(x_1, t_1) = 0$ and $u > 0$ somewhere in $\Omega \times (0, T)$. Using the above argument, we can assume that $u(x, t_1) = 0$ for $x \in \Omega$ and $u(x, t) > 0$ in $\Omega \times (0, t_1)$. For $F \ge 0$ and $c \ge 0$, $u(x, t)$ satisfies

$$u_t - \sum_{i=1}^{N} a_i \frac{\partial^2}{\partial x_i^2} \left(D_t^{1-\alpha} u\right) \ge 0 \text{ in } \Omega \times (0, t_1). \tag{2.5}$$

For any $0 < \eta < t_1$, we recall the operator

$${}_\eta D_t^\alpha u(x, t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_\eta^t (t-s)^{-\alpha} u(x, s) ds.$$

Then, a similar argument as in the proof of Lemma 2.3 on (2.2) shows that (2.5) becomes

$$\Gamma(1-\alpha)\, {}_\eta D_t^\alpha u - \sum_{i=1}^{N} a_i \frac{\partial^2}{\partial x_i^2} u \ge (t-\eta)^{-\alpha} u(x, \eta) > 0 \text{ in } \Omega \times (\eta, t_1).$$

Hence from Remark 2.6, we have $u(x, t) > 0$ in $\Omega \times (\eta, t_1]$, which leads to a contradiction. In other word, if $u = 0$ somewhere in $\Omega \times (0, T]$, then $u(x, t)$ cannot be positive anywhere inside. Therefore $u \equiv 0$ in $\Omega \times (0, T]$. Since u is continuous, we obtain $u \equiv 0$ on $\bar{\Omega} \times [0, T]$. □

It follows from a modification of Theorem 2.11, a general result holds.

Theorem 2.12. *Suppose* $u(x,t)$ *satisfies* (1.1)*,* $u(x,0) \geq 0$ *on* $\bar{\Omega}$*,* $u(x,t) \geq 0$ *on* $\partial\Omega \times (0,T]$*. If* $F(x,t) \geq 0$ *and* $c(x,t) \geq 0$ *for* $(x,t) \in \Omega \times (0,T]$*, and there is* $(x_0,t_0) \in \Omega \times (0,T]$ *such that* $u(x_0,t_0) = m = \min_{\bar{\Omega}\times[0,T]} u(x,t)$*, then* $u(x,t) \equiv m$ *for* $(x,t) \in \bar{\Omega} \times [0,T]$*.*

Proof. Let $w(x,t) = u(x,t) - m$. Then $w(x,t)$ satisfies the conditions in Theorem 2.11 and $w(x_0,t_0) = 0$. Therefore $w(x,t) \equiv 0$ on $\bar{\Omega} \times [0,T]$, and hence $u(x,t) \equiv m$. □

Similar to Theorem 2.8 for the case $c \equiv 0$, if u attains its minimum value in $\Omega \times (0,T)$, then u is a constant.

Theorem 2.13. *Suppose* $u(x,t)$ *satisfies* (1.1) *with* $c \equiv 0$*. If* $F(x,t) \geq 0$ *and there is* $(x_0,t_0) \in \Omega \times (0,T]$ *such that* $u(x_0,t_0) = m = \min_{\bar{\Omega}\times[0,T]} u(x,t)$*, then* $u(x,t) \equiv m$ *for* $(x,t) \in \bar{\Omega} \times [0,T]$*.*

The following results about the maximum values of the solution u can be proved similarly.

Theorem 2.14. *Suppose* $u(x,t)$ *satisfies* (1.1)*,* $u(x,0) \leq 0$ *on* $\bar{\Omega}$*,* $u(x,t) \leq 0$ *on* $\partial\Omega \times (0,T]$*. If* $F(x,t) \leq 0$ *and* $c(x,t) \geq 0$ *for* $(x,t) \in \Omega \times (0,T]$*, and there is* $(x_0,t_0) \in \Omega \times (0,T]$ *such that* $u(x_0,t_0) = M = \max_{\bar{\Omega}\times[0,T]} u(x,t)$*, then* $u(x,t) \equiv M$ *for* $(x,t) \in \bar{\Omega} \times [0,T]$*.*

Theorem 2.15. *Suppose* $u(x,t)$ *satisfies* (1.1) *with* $c \equiv 0$*. If* $F(x,t) \leq 0$ *and there is* $(x_0,t_0) \in \Omega \times (0,T]$ *such that* $u(x_0,t_0) = M = \max_{\bar{\Omega}\times[0,T]} u(x,t)$*, then* $u(x,t) \equiv M$ *for* $(x,t) \in \bar{\Omega} \times [0,T]$*.*

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