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NOTE ON MONOTONICITY IN SINGULAR VOLTERRA INTEGRAL EQUATIONS

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ABSTRACT. Observing the monotonic type for a class of singular Volterra integral equations we get a short proof of the singular Gronwall inequality in a completed setting with upper bounds as usual and additional lower bounds. Moreover, the solutions to linear singular Volterra integral equations admit norm bounds which (under an obvious restriction) depend in a monotone increasing way on the prescribed data. We use this observation to solve a nonlinear problem: In terms of linear singular Volterra equations we formulate an (seemingly new) iterative approximation scheme to mild Navier-Stokes solutions. The monotonicity of the bounds mentioned above leads to the proof of convergence and error estimates to our scheme inside a scale of Banach spaces locally in time.

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1. Introductory remarks

The integral representation of initial value problems in ordinary or partial differential equations leads to Volterra integral equations which, in important parabolic cases, are (weakly) singular, and any solution of the integral equation presents a mild solution to the related initial value problem [16].

The solutions to linear singular Volterra integral equations allow useful norm estimates like the singular Gronwall inequality [1, 2, 11]. In many cases of nonlinear parabolic problems their formulation by a (nonlinear) singular Volterra integral equation shows the inherent smoothing properties of the problem which opens the way to convergent iteration procedures and error estimates.

Famous examples are given by Fujita and Kato's Hilbert space solution to the 3-dimensional initial-boundary value problem of the Navier-Stokes equations [6, 13], and by Giga and Miyakawa's extension of their method to $n \geq 2$ dimensions and to solutions in Sobolev spaces, [10].

Having listed some basic notations in Section 2 we recall the classical formulas for the solutions to linear singular Volterra integral equations and for bounds of their

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norms. In particular we stress the monotone dependence of these bounds upon the prescribed data (under some obvious restriction). Then in Section 3 we will see the monotonic type of linear singular Volterra integral equations with positive real kernels, which implies the singular Gronwall inequality completed by additional lower bounds. Finally in Section 4, formulated in terms of linear singular Volterra integral equations we present a (seemingly new) iterative approximation scheme to mild Navier-Stokes solutions. Using the monotonicity of the bounds (stated in Section 2) we prove convergence and error estimates to our scheme inside a scale of Banach spaces locally in time.

2. Notations and some basic facts on linear Volterra integral equations

To any Banach space X with norm $\|\cdot\|_X$, let $B = B(X)$ denote the Banach space of bounded linear operators $S : X \to X$ with norm $\|S\|_B$. Moreover on the interval $J = [0, a]$, $0 < a < \infty$, we will use the Banach space $C^0(J, X)$ of all X-valued continuous functions $f : J \to X$ having the norm $\sup_{t \in J} \|f(t)\|_X$. On the triangle

$$\mathcal{T} = \{(t, s) \in \mathbb{R}^2 \mid 0 \le s \le t \le a\}$$

we consider any strongly continuous, uniformly bounded function

$$H : \mathcal{T} \to B(X), \quad \|H(t, s)\|_B \le N$$

with some constant $N > 0$. Then there hold the following two propositions:

Proposition 2.1. *To any given $g \in C^0(J, X)$, $\lambda \in \mathbb{R}$, $\alpha \in [0, 1)$, the singular Volterra integral equation*

$$u(t) = g(t) + \lambda \cdot \int_0^t \frac{H(t, s)}{(t - s)^\alpha} \cdot u(s) ds \tag{2.1}$$

has a unique solution $u \in C^0(J, X)$.

Proposition 2.2. *With the assumptions above, the solution u of equation* (2.1) *has the representation*

$$u(t) = g(t) + \lambda \cdot \int_0^t \frac{\tilde{H}(t, s, \lambda)}{(t - s)^\alpha} \cdot g(s) ds$$

by means of the strongly continuous kernel

$$\tilde{H}(t, s, \lambda) \in B(X), \quad \|\tilde{H}(t, s, \lambda)\|_B \le \tilde{N}(N, \lambda, a),$$

where

$$\frac{\tilde{H}(t, s, \lambda)}{(t - s)^\alpha} := K(t, s, \lambda)$$

is given by the strongly in $B(X)$, for all $b \in (0, \infty)$ uniformly on $\mathcal{T} \times \{\lambda \in \mathbb{R} \mid |\lambda| \le b\}$ convergent power expansion

$$K(t, s, \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} \cdot K_m(t, s), \text{ where}$$

$$K_1(t,s) := \frac{H(t,s)}{(t-s)^\alpha}, K_m(t,s) := \int_s^t K_1(t,\sigma)\cdot K_{m-1}(\sigma,s)d\sigma, \quad m \geq 2.$$

With $\kappa_m := N^m \cdot \frac{(\Gamma(1-\alpha))^m}{\Gamma(m\cdot(1-\alpha))}$, *there holds*

$$(t-s)^\alpha \cdot \|K_m(t,s)\|_B \leq (t-s)^{(m-1)(1-\alpha)} \cdot \kappa_m, \quad m \geq 1,$$

Γ *denoting Legendre's Gamma function. For all* $\lambda \geq 0$, *the (absolutely convergent) bound*

$$\tilde{N}(N,\lambda,a) := \sum_{m=1}^{\infty} |\lambda|^{m-1} \cdot a^{(m-1)(1-\alpha)} \cdot \kappa_m \tag{2.2}$$

is monotone increasing in N, λ, *and* a.

Proof. The proofs of both propositions above given in [14, pp. 17–18] and [21, pp. 151–153], which are formulated there in the case of real valued functions, extend immediately to our abstract case, if we recall the abstract integration [12, pp. 59, 66], and the abstract Cauchy-Hadamard convergence theorem [12, p. 96]. □

Definition 2.3. To any function $f \in C^0(J,X)$, the defect Pf with respect to equation (2.1) is given by

$$(Pf)(t) := f(t) - \left[g(t) + \lambda \cdot \int_0^t \frac{H(t,s)}{(t-s)^\alpha} \cdot f(s)ds\right], \quad t \in J. \tag{2.3}$$

Note 2.1. In case of any given $Pf \in C^0(J,X)$, equation (2.3) is equivalent to the Volterra integral equation for f:

$$f(t) = (Pf)(t) + g(t) + \lambda \cdot \int^t \frac{H(t,s)}{(t-s)^\alpha} \cdot f(s)ds, \quad t \in J.$$

Since with our assumptions above $f \in C^0(J,X)$ implies $Pf \in C^0(J,X)$, by the two preceding Propositions we find that each function $f \in C^0(J,X)$ allows the representation

$$f(t) = (Pf)(t) + g(t) + \lambda \cdot \int_0^t \frac{\tilde{H}(t,s,\lambda)}{(t-s)^\alpha} \cdot [(Pf)(s) + g(s)]ds, \quad t \in J.$$

3. The monotonic type of linear Volterra integral equations with positive real kernels and the singular Gronwall inequality

Let $H : \mathcal{T} \to \mathbb{R}_+ := [0,\infty)$ denote a continuous function,

$$0 \leq H(t,s) \leq N \text{ with constant } N > 0 \text{ for all } (t,s) \in \mathcal{T}. \tag{3.1}$$

Due to the Propositions 2.1 and 2.2 above, to any given $g \in C^0(J,\mathbb{R})$, $\lambda > 0$, $0 \leq \alpha < 1$, the unique solution $u \in C^0(J,\mathbb{R})$ of the Volterra integral equation

$$u(t) = g(t) + \lambda \cdot \int_0^t \frac{H(t,s)}{(t-s)^\alpha} \cdot u(s)ds \tag{3.2}$$

is given by

$$u(t) = g(t) + \lambda \cdot \int_0^t \frac{\tilde{H}(t,s,\lambda)}{(t-s)^\alpha} \cdot g(s)ds.$$

There holds

Proposition 3.1. *For any two functions $v, w \in C^0(J,\mathbb{R})$, the condition*

(i) $(Pv)(t) \le 0 \le (Pw)(t)$ *for all* $t \in J$

implies the estimate

(ii) $v(t) \le u(t) \le w(t)$ *for all* $t \in J$.

Proposition 3.1 states the fact that, under the positivity assumptions above, the integral equation (3.2) presents a problem of monotonic type in the sense of L. Collatz [4, 5]. Proposition 3.1 is not implied in Walter's comparison theorem for monotone increasing kernels [20, II. Theorem, p. 14], since in requirement (i) we admit the equality sign at both places. Therefore a proof of Proposition 3.1 with the methods of [20] would require the additional arguments from [20, Section 1, IX].

Evidently the left-hand sides of inequalities (i) and (ii) in Proposition 3.1 yield

Corollary 3.2 (the singular Gronwall inequality)**.** *Assume* (3.1) *and* $\lambda > 0$. *Then each continuous function* $v \in C^0(J,\mathbb{R})$ *which fulfils the inequality*

$$v(t) \le g(t) + \lambda \cdot \int_0^t \frac{H(t,s)}{(t-s)^\alpha} v(s)ds, \quad t \in J, \quad 0 \le \alpha < 1, \quad 0 < \lambda,$$

is bounded from above by the solution u *to* (3.2).

Proof. As we have observed in the general Note 2.1., under the assumptions of Proposition 3.1 each $f \in C^0(J,\mathbb{R})$ has the representation

$$f(t) = (Pf)(t) + g(t) + \lambda \cdot \int_0^t \frac{\tilde{H}(t,s,\lambda)}{(t-s)^\alpha}[(Pf)(s) + g(s)]ds, \tag{3.3}$$

where

$$\frac{\tilde{H}(t,s,\lambda)}{(t-s)^\alpha} = \sum_{m=1}^{\infty} \lambda^{m-1} \cdot K_m(t,s),$$

$$K_1(t,s) = \frac{H(t,s)}{(t-s)^\alpha}, K_m(t,s) = \int_s^t K_1(t,\sigma) \cdot K_{m-1}(\sigma,s)d\sigma, \quad m \ge 2,$$

$t \in J$.

Therefore our assumption (3.1) and $\lambda > 0$ imply $\lambda \cdot \tilde{H}(t,s,\sigma) \ge 0$ for all $(t,s) \in \mathcal{T}$. Since the function $f = u$ obeys (3.3) with $Pf = 0$, from the non-negativity of the kernel $\lambda \cdot \frac{\tilde{H}(t,s,\sigma)}{(t-s)^\alpha}$ in (3.3) we conclude $v(t) \le u(t)$ for each $v \in C^0(J,\mathbb{R})$ fulfilling $Pv(t) \le 0$, and $u(t) \le w(t)$ for each $w \in C^0(J,\mathbb{R})$ with $Pw(t) \ge 0$ for all $t \in J$, respectively. □

Remark 3.3. In a more general setting with functions $v \in L^\infty_{loc}(J,\mathbb{R})$, the singular Gronwall inequality has been proved by Amann in [1, 2] and Henry in [11].

4. **Application to Navier-Stokes approximations**

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, $n \geq 2$. Besides the Lebesgue spaces $L^r = L^r(\Omega)$ of functions v, $v(x) \in \mathbb{R}^n$, $x \in \bar{\Omega}$, in the following we will use the fractional order spaces $H^{\alpha,r} = H^{\alpha,r}(\Omega)$, $0 \leq \alpha < \infty$, $1 < r < \infty$, and the subspace $L^r_\sigma \subset L^r(\Omega)$ of weakly divergence free functions on Ω, L^r_σ being the L^r-closure of the linear space $C^\infty_{c,\sigma}(\Omega)$ of C^∞-test functions, which are divergence free and have compact support in Ω.

By means of the Helmholtz-Weyl projection $P_r : L^r(\Omega) \to L^r_\sigma$ and the Laplacian operator $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ we define the Stokes operator $A := -P_r\Delta$ with its domain $D_A :\subset H^{2,r}(\Omega) \cap L^r_\sigma$. The operator A generates the holomorphic semigroup e^{-tA} on L^r_σ, $t \geq 0$. Therefore the fractional powers A^α are well defined for all $\alpha \in \mathbb{R}$, $A^\alpha \in B(L^r_\sigma)$ if $\alpha \leq 0$, $D_{A^\alpha} \subset H^{2\alpha,r} \cap L^r_\sigma$ if $\alpha \geq 0$, [7, 9, 10, 6, 13, 3, 17, 18, 19]. In addition there hold the relations

$$\|A^\alpha e^{-tA} w\|_{L^r} \leq C_\alpha \cdot t^{-\alpha} \cdot \|w\|_{L^r}, t > 0, \alpha \geq 0 \text{ with some constant } C_\alpha > 0, \tag{4.1}$$

$$\|(1 - e^{-tA})w\|_{L^r} \to 0 \quad \text{with} \quad t \to +0, \tag{4.2}$$

$$\|t^\alpha A^\alpha e^{-tA} w\|_{L^r} \to 0 \qquad \text{with} \quad t \to +0, \alpha > 0 \tag{4.3}$$

for $w \in L^r_\sigma$, [6, 7, 8, 16].

Any solution $u \in C^0(J, D_{A^\alpha})$ of the integral equation

$$A^\alpha u(t) = e^{-tA} A^\alpha u(0) - \int_0^t A^{\alpha+\delta} e^{-(t-s)A} A^{-\delta} P_r(u\nabla u)(s)ds, \tag{4.4}$$

with initial value $u(0) \in D_{A^\alpha}$ and $0 \leq \delta$, $0 \leq \alpha$, $\alpha + \delta < 1$, is called a mild Navier-Stokes solution on $J \times \Omega$. For short, we will assume exterior potential forces. For solving (4.4), Fujita-Kato [6, 13] and Giga-Miyakawa [8, 15, 10] have used the approximation scheme

$$A^\alpha u_0(t) := A^\alpha e^{-tA} u(0),$$

$$A^\alpha u_{m+1}(t) := A^\alpha u_0(t) - \int_0^t A^{\alpha+\delta} e^{-(t-s)A} A^{-\delta} P_r(u_m \nabla u_m)(s)ds,$$

$m \in \mathbb{N}$, with $u(0) \in D_{A^\beta}$.

Admitting $\beta < \alpha < 1 - \delta$ (Fujita-Kato: $r = 2$, $\beta = 1/4$, $\alpha \geq 1/2$) they proved convergence of the sequence $(A^\alpha u_m(t))_{m\in\mathbb{N}}$ with respect to the singular weighted norm $\sup_{0<\tau\leq t}\{\tau^{\beta-\alpha} \cdot \|A^\alpha u_m(\tau)\|_{L^r}\}$. In this way they could point out also the smoothing property of the integral equation (4.4).

The key for the construction of approximate solutions are estimates of the convective term:

Lemma 4.1 (Giga-Miyakawa [10])**.** *Let* $0 \le \delta < \frac{1}{2} + n \cdot \frac{1}{2} \cdot \left(1 - \frac{1}{r}\right)$*. Then there holds*

$$\|A^{-\delta} P_r(u\nabla v)\|_{L^r(\Omega)} \le M \cdot \|A^{\theta} u\|_{L^r(\Omega)} \cdot \|A^{\rho} v\|_{L^r(\Omega)}$$

with some constant $M = M(\delta, \theta, \rho, r)$*, provided that*

$$\frac{1}{2} + \frac{n}{2r} \le \delta + \theta + \rho, \quad 0 < \theta, \quad 0 < \rho, \quad \frac{1}{2} < \delta + \rho.$$

In the following we will always require $u(0) \in D_{A^\alpha}$, $\alpha = \theta = \rho$, $\alpha + \delta < 1$, thus the requirement of Lemma 4.1 reads

$$\text{(4.5)} \qquad \begin{cases} 0 \le \delta < \frac{1}{2} + n \cdot \frac{1}{2} \cdot \left(1 - \frac{1}{r}\right), & \frac{1}{2} + \frac{n}{2r} \le \delta + 2\alpha, \\ 0 < \alpha, \frac{1}{2} < \delta + \alpha < 1, & 1 < r < \infty. \end{cases}$$

Then from Lemma 4.1 evidently there results

Corollary 4.2. *Under the assumptions* (4.5) *there holds*

$$\text{(4.6)} \qquad \|A^{-\delta} P_r(u\nabla v)\|_{L^r} \le M \cdot \|A^{\alpha} u\|_{L^r} \cdot \|A^{\alpha} v\|_{L^r}, \ \textit{and}$$

$$\text{(4.7)} \qquad \begin{aligned} \|A^{-\delta} P_r[u\nabla v - \tilde{u}\nabla\tilde{v}]\|_{L^r} \le M \cdot \{ &\|A^{\alpha}(u - \tilde{u})\|_{L^r} \cdot \|A^{\alpha} v\|_{L^r} + \\ &+ \|A^{\alpha}\tilde{u}\|_{L^r} \cdot \|A^{\alpha}(v - \tilde{v})\|_{L^r} \}, \end{aligned}$$

for all $u, v, \tilde{u}, \tilde{v} \in D_{A^\alpha}$*, where* $M = M(\delta, \alpha, r)$*.*

For solving (4.4) with any prescribed $u(0) \in D_{A^\alpha}$ we consider the approximation scheme

$$\text{(4.8)} \qquad A^{\alpha} u_0(t) := e^{-tA} A^{\alpha} u(0),$$

$$\text{(4.9)} \qquad A^{\alpha} u_{m+1}(t) := A^{\alpha} u_0(t) - \int_0^t A^{\alpha+\delta} e^{-(t-s)A} A^{-\delta} P_r(u_m \cdot \nabla u_{m+1})(s) ds,$$

$t \in J$, $m \in \mathbb{N}$, α, δ, r from (4.5).

Let $u \in C^0(J, D_{A^\alpha})$. With any $w \in L^r_\sigma$, we introduce the kernel

$$H(t, s, u)w := H(t, s) \cdot (Fu(s))w,$$

where

$$\text{(4.10)} \qquad H(t, s) := -A^{\alpha+\delta} e^{-(t-s)A} \cdot (t - s)^{\alpha+\delta} \ \text{and}$$

$$\text{(4.11)} \qquad (Fu(s))w := A^{-\delta} P_r(u(s) \cdot \nabla A^{-\alpha} w), \quad (t, s) \in \mathcal{T}.$$

Using these notations we get from (4.9) the linear singular Volterra integral equation

$$\text{(4.12)} \qquad A^{\alpha} u_{m+1}(t) = A^{\alpha} u_0(t) + \int_0^t \frac{H(t, s, u_m)}{(t - s)^{\alpha+\delta}} A^{\alpha} u_{m+1}(s) ds, \quad t \in J,$$

with prescribed $A^{\alpha} u_0(t) = e^{-tA} A^{\alpha} u(0), u(0) \in D_{A^\alpha}, u_m \in C^0(J, D_{A^\alpha})$.

In the following, with any fixed value $r \in (1, \infty)$, we will always write $\|\cdot\| = \|\cdot\|_{L^r}$, omitting the norm index L^r.

Proposition 4.3. *Let* $u \in C^0(J, D_{A^\alpha})$, $\|A^\alpha u(t)\| \le c$, $(t,s) \in \mathcal{T}$, *with some constant* $c > 0$. *Then*

(i) $\|H(t,s,u)\|_{B(L^r_\sigma)} \le N_c := C_{\alpha+\delta} \cdot M \cdot c$, *and*
(ii) $H(t,s,u)$ *is strongly* L^r_σ*-continuous.*

Note 4.1. The bound N_c is monotone increasing in the norm bound $c \ge \|A^\alpha u\|$.

Proof. Inequality (i) results from (4.6) in Corollary 4.2 because of the semigroup estimate (4.1). To show (ii), we prove that the difference $D := \|H(t,s,u)w - H(\tau,\sigma,u)w\|$ tends to zero with $(t,s) \to (\tau,\sigma)$ inside of $\mathcal{T}$. Without any restriction we may assume

$$0 \le s \le t \le \tau \le a, \quad 0 \le \sigma \le \tau, \tag{4.13}$$

with any fixed $(\tau,\sigma) \in \mathcal{T}$. Let $u \in C^0(J, D_{A^\alpha})$, $w \in L^r_\sigma$. Then in the inequality

$$D \le \|H(t,s)[(Fu(s)) - (Fu(\sigma))]w\| + \|[H(t,s) - H(\tau,\sigma)](Fu(\sigma))w\| := D_1 + D_2,$$

the first term D_1 on the right-hand side tends to zero with $|s - \sigma| \to 0$ because of (4.1) and (4.7) with $v = \tilde{v} = A^{-\alpha}w$.

For estimating the second term D_2, firstly in addition to (4.13) we require

$$0 \le t - s \le \tau - \sigma. \tag{4.14}$$

Then $\sigma = \tau$ gives $s = t$, therefore $D_2 = 0$. Thus we have to consider only the case

$$0 \le \sigma < \tau.$$

Writing $w_u := (Fu(\sigma))w, \beta := \alpha + \delta$, we find

$$\begin{aligned} D_2 &= \|A^\beta[e^{-(t-s)A} \cdot (t-s)^\beta - e^{-(\tau-\sigma)A} \cdot (\tau-\sigma)^\beta]w_u\| \\ &\le \left\{ \begin{array}{l} \|A^\beta e^{-(t-s)A} \cdot (t-s)^\beta \cdot [1 - e^{-[(\tau-t)-(\sigma-s)]A}]w_u\| \\ +\|A^\beta e^{-(\tau-\sigma)A}[(t-s)^\beta - (\tau-\sigma)^\beta]w_u\| \end{array} \right\} := D_{21} + D_{22}. \end{aligned}$$

Since (4.14) means $0 \le \eta := \tau - t - [\sigma - s]$, from (4.1) and (4.2) we find $D_{21} \to 0$ if $(t,s) \to (\tau,\sigma)$. Moreover, because of $0 \le \sigma < \tau$ with fixed $(\tau,\sigma) \in \mathcal{T}$, the convergence $(t,s) \to (\tau,\sigma)$ implies $(t-s)^\beta \equiv (\tau-\sigma)^\beta \cdot (1+\epsilon)$ with values $|\epsilon| \to 0$. Therefore again using (4.1), we see

$$D_{22} = \|A^\beta e^{-(\tau-\sigma)A}(\tau-\sigma)^\beta w_u\| \cdot |\epsilon| \to 0$$

with $|\epsilon| \to 0$. In the remaining case (4.13) and $0 \le \tau - \sigma \le t - s$ we conclude similarly. ☐

From Proposition 4.3 we see that the kernel $H(t,s,u_m)$ in (4.12) satisfies the requirement of Propositions 2.1 and 2.2. Consequently there holds

Proposition 4.4. *Let* $u(0) \in D_{A^\alpha}$, $u_m \in C^0(J, D_{A^\alpha})$, $\|A^\alpha u_m(t)\| \le c_m$ *with constant* $c_m > 0$, $t \in J$, *for* $m = 0$ *and some fixed* $m \in \mathbb{N}$. *Then*

(i) *the singular Volterra integral equation* (4.12) *has a unique solution* $A^\alpha u_{m+1} \in C^0(J, L^r_\sigma)$, *and there holds with* $(t,s) \in \mathcal{T}$

(ii) $A^\alpha u_{m+1}(t) = A^\alpha u_0(t) + \int_0^t \frac{\tilde{H}(t,s,u_m)}{(t-s)^{\alpha+\delta}} A^\alpha u_0(s)ds$ *with* $A^\alpha u_0(t) = e^{-tA} A^\alpha u(0)$, $\tilde{H}(t,s,u_m) \in B(L^r_\sigma)$,

(iii) $\|\tilde{H}(t,s,u_m)\|_{B(L^r_\sigma)} \le \tilde{N}(N_{c_m})$, *the function* $\tilde{N}(N_{c_m}) := \tilde{N}(N_{c_m}, 1, a)$ *from* (2.2) *being monotone increasing in dependence on the bound* $N_{c_m} \ge \|H(t,s,u_m)\|$,

(iv) $\|A^\alpha u_{m+1}(t)\| \le c_0 \cdot [1 + \tilde{N}(N_{c_m}) \cdot T]$, *where* $T = \frac{t^{1-(\alpha+\delta)}}{1-(\alpha+\delta)}$.

Proof. The proof of (i)–(iii) results from the Propositions 2.1 and 2.2 because of Proposition 4.3. Then using the bounds c_0 and $\tilde{N}(N_{c_m})$ in (iii) we find (iv) by straightforward integration from (ii). □

According to Proposition 4.4, the sequence of functions $A^\alpha u_m \in C^0(J, L^r_\sigma)$ is well defined, $m \in \mathbb{N}$. Evidently we have $u_m = A^{-\alpha}(A^\alpha u_m) \in C^0(J, L^r_\sigma)$ by the boundedness of $A^{-\alpha}$, $\alpha \ge 0$.

Proposition 4.5. *For* $t \in J$ *we assume* $c > c_0 \ge \|A^\alpha u_0(t)\|$ *with constants* c, c_0. *Then the sequence* $A^\alpha u_m(t)$, $m \in \mathbb{N}$, *is uniformly* L^r*-bounded by* c *on the interval*

$$J_1 = \left\{ t \in J \mid t \le t_1 := \left[\frac{(c-c_0)(1-(\alpha+\delta))}{c_0 \cdot \tilde{N}(N_c)} \right]^{\frac{1}{1-(\alpha+\delta)}} \right\}. \tag{4.15}$$

Note 4.2. By a short calculation we see that $t_1 \ge a$ is admitted if

$$\frac{a^{1-(\alpha+\delta)}}{1-(\alpha+\delta)} \cdot \tilde{N}(N_c) \le \frac{c-c_0}{c_0}.$$

Proof. To any $c > c_0$, the requirement $0 \le t \le t_1$, t_1 from (4.15), is equivalent with

$$c_0 \cdot [1 + \tilde{N}(N_c) \cdot T] \le c, \quad T = \frac{t^{1-(\alpha+\delta)}}{1-(\alpha+\delta)}. \tag{4.16}$$

Using inequality (iv) in Proposition 4.4, by induction we get from (4.16):

In case $m = 0$: (iv) implies $\|A^\alpha u_1(t)\| \le c_0 \cdot [1 + \tilde{N}(N_{c_0}) \cdot T] \le c$ because of (4.16) and $\tilde{N}(N_{c_0}) \le \tilde{N}(N_c)$ by the monotonicity of the function $\tilde{N}$ in N_{c_0}, and of the function N_c in c, respectively, where $c_0 < c$ (N_c from (i) in Proposition 4.3).

Now let $\|A^\alpha u_k(t)\| \le c_k \le c$ for all $k = 0, 1, \ldots, m$ and $t \in J_1$. Then again from (iv) in Proposition 4.4, using (4.16) and recalling the monotonicity as above we find

$$\|A^\alpha u_{m+1}(t)\| \le c_0 \cdot [1 + \tilde{N}(N_{c_m}) \cdot T] \le c,$$

which proves our claim for all $m \in \mathbb{N}$. □

Proposition 4.6. *Let* $u(0) \in D_{A^\alpha}$, $u_0(t) = e^{-tA}u(0)$. *Then with any constant* $c > c_0 \ge \|A^\alpha u_0(t)\|$, $t \in J$, *the sequence* $A^\alpha u_m(t)$, $m \in \mathbb{N}$, *is uniformly* L^r*-convergent for* *all*

$$t \in J_2 := \left\{ \tau \in J_1 \mid \tau \leq t_2 := \left[\frac{q \cdot (1-(\alpha+\delta))}{(1+q) \cdot \tilde{C} \cdot c} \right]^{\frac{1}{1-(\alpha+\delta)}} \right\} \tag{4.17}$$

with $\tilde{C} = C_{\alpha+\delta} \cdot M$ *and any constant* $q \in (0,1)$.

Note 4.3. A short calculation shows that $t_2 \geq a$ is admitted if $t_1 \geq a$ and additionally

$$\frac{a^{1-(\alpha+\delta)}}{1-(\alpha+\delta)} \cdot \tilde{C} \cdot \frac{1+q}{q} \leq \frac{1}{c}$$

holds.

Proof. As we have seen in Proposition 4.5, the functions $A^\alpha u_m \in C^0(J_1, L^r_\sigma)$ in (4.9) fulfill $\|A^\alpha u_m(t)\| \leq c$ with some constant $c > c_0$. Introducing the bounds

$$d_m(t) := \sup_{0 \leq \tau \leq t} \|A^\alpha(u_{m+1}(\tau) - u_m(\tau))\|,$$

from (4.7) we find

$$\|(Fu_{m+1}(t))A^\alpha u_{m+2}(t) - (Fu_m(t))A^\alpha u_{m+1}(t)\| \leq M \cdot c \cdot [d_m(t) + d_{m+1}(t)]. \tag{4.18}$$

Equation (4.12) implies

$$\begin{aligned} &\|A^\alpha(u_{m+2}(t) - u_{m+1}(t))\| \leq \\ &\leq \int_0^t \|A^{\alpha+\delta} e^{-(t-s)A}[(Fu_{m+1}(s))A^\alpha u_{m+2}(s) - (Fu_m(s))A^\alpha u_{m+1}(s)]\| ds. \end{aligned}$$

Inserting the bounds from (4.1) and (4.18), by straightforward integration we get

$$d_{m+1}(t) \leq C \cdot T \cdot [d_m(t) + d_{m+1}(t)], \quad t \in J_1, \tag{4.19}$$

where $T = \frac{t^{1-(\alpha+\delta)}}{1-(\alpha+\delta)}$, $C = C_{\alpha+\delta} \cdot c \cdot M$. On the left-hand side we have used the monotonicity of $d_{m+1}(t)$ in t. Since in case $C \cdot T < 1$, inequality (4.19) is equivalent with

$$d_{m+1}(t) \leq \frac{CT}{1-CT} \cdot d_m(t), \tag{4.20}$$

the uniform convergence of the sequence $A^\alpha u_m(t)$ in L^r_σ results from the requirement

$$\frac{CT}{1-CT} \leq q \in (0,1), \tag{4.21}$$

or equivalently, $T \leq \frac{q}{C \cdot (1+q)}$, which implies $C \cdot T < 1$ and gives the bound t_2 in (4.17). □

Proposition 4.7. *Let* $u(0) \in D_{A^\alpha}, u_0(t) = e^{-tA}u(0)$, J_2 *from* (4.17). *Then there holds*

(i) *The Cauchy sequence* $A^\alpha u_m \in C^0(J_2, L^r_\sigma)$, $m \in \mathbb{N}$, *defines the Cauchy sequence* $u_m \in C^0(J_2, L^r_\sigma)$, *and*

(ii) $\lim_{m\to\infty} u_m = u \in C^0(J_2, L^r_\sigma)$, $\lim_{m\to\infty} A^\alpha u_m = v \in C^0(J_2, L^r_\sigma)$,

(iii) $u(t) \in D_{A^\alpha}, v(t) = A^\alpha u(t)$, $t \in J_2$,

(iv) $A^\alpha u(t)$ *represents a solution of the integral equation* (4.4),

(v) *the error estimate*

$$\|A^\alpha(u(t) - u_m(t))\| \leq \frac{q^m}{1-q} \cdot \|A^\alpha(u_1(t) - u_0(t))\|, \quad t \in J_2, \quad m \in \mathbb{N}.$$

Proof. Part (i) and (ii) clearly hold because of Proposition 4.6 and the boundedness of the linear operator $A^{-\alpha}$ on L^r_σ. Therefore (iii) results from the closedness of the linear operator A^α for each $t \in J_2$. Consequently also the right hand side of (4.4) is well defined. Using the notation (4.10), (4.11), we conclude the statement (iv) from Proposition 4.6 by means of the estimate

$$\left\| \int_0^t \left[\frac{H(t,s)}{(t-s)^{\alpha+\delta}} (Fu(s)) A^\alpha u(s) - \frac{H(t,s)}{(t-s)^{\alpha+\delta}} (Fu_m(s)) A^\alpha u_{m+1}(s) \right] ds \right\| \leq$$
$$\leq \frac{t^{1-(\alpha+\delta)}}{1-(\alpha+\delta)} \cdot C_{\alpha+\delta} \cdot M \cdot c \cdot \left[\sup_{s \in J_2} \|A^\alpha(u - u_m)(s)\| + \sup_{s \in J_2} \|A^\alpha(u - u_{m+1})(s)\| \right] \to 0$$

with $m \to \infty$, which because of Proposition 4.5 follows from (4.7) and (4.1) by straightforward integration. Finally, to prove (v), by definition of $d_m(t)$ from (4.20) and (4.21) we find

$$\|A^\alpha(u_{m+k}(t) - u_m(t))\| \leq \frac{q^m}{1-q} \|A^\alpha(u_1(t) - u_0(t))\|, \quad t \in J_2, \quad m, k \in \mathbb{N},$$

which in the limit $k \to \infty$ gives the error estimate (v). □

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