A HOMOTOPY APPROACH TO COINCIDENCE THEORY

MOHAMED JLELI, DONAL O’REGAN, AND BESSEM SAMET

Department of Mathematics, College of Science,
King Saud University, P.O. Box 2455, Riyadh, 11451, Saudi Arabia
jleli@ksu.edu.sa; bsamet@ksu.edu.sa
School of Mathematics, Statistics and Applied Mathematics
National University of Ireland, Galway, Ireland
donal.oregan@nuigalway.ie

ABSTRACT. We use homotopy type arguments to establish new coincidence theory for general classes of maps. Our theory is based on the notions of Φ-essential or d-Φ-essential maps.

AMS (MOS) Subject Classification. 47H04

Keywords: essential maps, coincidence points, homotopy.

1. INTRODUCTION

The notion of essential maps was introduced by Granas [3] and extended in the literature by many authors (see [2, 4, 5, 6, 7, 9] and the references therein). In Section 2, using the notions of homotopy and Φ-essential maps we establish a variety of coincidence theorems (in particular we show if G is Φ-essential and if natural conditions are assumed so F ∼ G then F is Φ-essential). In Section 3 we discuss d-Φ-essential maps.

2. Φ-ESSENTIAL MAPS

Let E be a completely regular topological space and U an open subset of E.

We consider classes A and B of maps.

Definition 2.1. We say $F \in A(U, E)$ (respectively $F \in B(U, E)$) if $F : U \to 2^E$ and $F \in A(U, E)$ (respectively $F \in B(U, E)$); here $2^E$ denotes the family of nonempty subsets of E.

In this section we fix a $\Phi \in B(U, E)$.

Definition 2.2. We say $F \in A_{\partial U}(U, E)$ if $F \in A(U, E)$ with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$; here $\partial U$ denotes the boundary of $U$ in $E$.  

Received July 5, 2016
Definition 2.3. Let $F \in A_{\partial U}(\overline{U}, E)$. We say $F : \overline{U} \to 2^E$ is $\Phi$-essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$.

Theorem 2.4. Let $E$ be a completely regular (respectively normal) topological space, $U$ an open subset of $E$, $F \in A_{\partial U}(\overline{U}, E)$ and let $G \in A_{\partial U}(\overline{U}, E)$ be $\Phi$-essential in $A_{\partial U}(\overline{U}, E)$. For any map $R \in A_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = F|_{\partial U}$ assume there exists a map $H^R : \overline{U} \times [0, 1] \to 2^E$ with $H^R(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H^R_I(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0, 1)$ and \{ $x \in \overline{U} : \Phi(x) \cap H^R_I(x, t) \neq \emptyset$ for some $t \in [0, 1]$ \} is compact (respectively closed) and $H^R_0 = G, H^R_1 = R$; here $H^R_I(x) = H^R(x, t)$. Then $F$ is $\Phi$-essential in $A_{\partial U}(\overline{U}, E)$.

Proof. Let $R \in A_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = F|_{\partial U}$. We must show there exists a $x \in U$ with $R(x) \cap \Phi(x) \neq \emptyset$. Choose the map $H^R : \overline{U} \times [0, 1] \to 2^E$ as in the statement of Theorem 2.4. Let

\[ D = \{ x \in \overline{U} : \Phi(x) \cap H^R_I(x, t) \neq \emptyset \text{ for some } t \in [0, 1] \}. \]

Note $D \neq \emptyset$ since $H^R_0(= G)$ is $\Phi$-essential in $A_{\partial U}(\overline{U}, E)$. Also $D$ is compact (respectively closed) if $E$ is a completely regular (respectively normal) topological space. Next note $D \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $J_\mu : \overline{U} \to 2^E$ by $J_\mu(x) = H^R_I(x, \mu(x)) = H^R_{\mu(x)}(x)$. Note $J_\mu \in A_{\partial U}(\overline{U}, E)$ with $J_\mu|_{\partial U} = H^R_0|_{\partial U}$. Now since $H^R_0$ is $\Phi$-essential in $A_{\partial U}(\overline{U}, E)$ then there exists a $x \in U$ with $J_\mu(x) \cap \Phi(x) \neq \emptyset$ (i.e. $H^R_{\mu(x)}(x) \cap \Phi(x) \neq \emptyset$), and thus $x \in D$ so $\mu(x) = 1$ and as a result $\emptyset \neq H^R_1(x) \cap \Phi(x) = R(x) \cap \Phi(x)$. \qed

Remark 2.5. (i). In applications usually one puts conditions on the maps so that $D$ in the proof of Theorem 2.4 is closed and $\overline{D}$ is compact (so as a result $D$ is compact). However in the weak topology case one might need to work a little differently (the case we describe below occurs in applications [8]). Suppose $E$ is a metrizable locally convex linear topological space. Note $E = (E, w)$, the space $E$ endowed with the weak topology, is completely regular. Let $D \subseteq E$ and suppose $D$ is weakly sequentially closed and $\overline{D}^w$ is weakly compact. Then $D$ is weakly compact. To see this let $x \in \overline{D}^w$. Then the Eberlein–Smulian theorem [1 pg. 549] guarantees that that there is a sequence $(x_n)$ in $D$ with $x_n \to x$ (here $\to$ denotes weak convergence). Now since $D$ is weakly sequentially closed then $x \in D$, so $\overline{D}^w = D$, and $D$ is weakly compact.

(ii). Suppose in the statement of Theorem 2.4 we have that $E$ is a topological vector space and $F$ and $G$ are as in the statement of Theorem 2.4. Assume there exists a map $H : \overline{U} \times [0, 1] \to 2^E$ with $H_0 = G$ and $H_1 = F$. Now for any map $R$ with $R|_{\partial U} = F|_{\partial U}$ note

\[
H^R(x, t) = \begin{cases} 
H(x, 2t), & t \in [0, \frac{1}{2}] \\
2(1 - t)F(x) + 2(t - \frac{1}{2})R(x), & t \in [\frac{1}{2}, 1]
\end{cases}
\]
connects $G$ to $R$ (note as well for $x \in \partial U$ and $t \in \left[\frac{1}{2}, 1\right]$ that $H^R(x, t) = 2(1 - t)F(x) + 2\left(t - \frac{1}{2}\right)F(x) = F(x)$).

It is possible to obtain an analogue result if we change Definition 2.3 as follows.

**Definition 2.6.** Let $E$ be a completely regular (respectively normal) topological space, and $U$ an open subset of $E$. Let $F, G \in A_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if there exists a map $H : \overline{U} \times [0, 1] \rightarrow 2^E$ with $H(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \rightarrow [0, 1]$ with $\eta(\partial U) = 0$, $H_t(x) \cap \Phi(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0, 1]$, $H_1 = F$, $H_0 = G$ and \{ $x \in \overline{U} : \Phi(x) \cap H(x, t) \neq \emptyset$ for some $t \in [0, 1]$\} is compact (respectively closed); here $H_t(x) = H(x, t)$.

**Definition 2.7.** Let $F \in A_{\partial U}(\overline{U}, E)$. We say $F : \overline{U} \rightarrow 2^E$ is $\Phi$-essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J : A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$.

**Theorem 2.8.** Let $E$ be a completely regular (respectively normal) topological space, $U$ an open subset of $E$, $F \in A_{\partial U}(\overline{U}, E)$ and let $G \in A_{\partial U}(\overline{U}, E)$ be $\Phi$-essential in $A_{\partial U}(\overline{U}, E)$. For any map $R \in A_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = F|_{\partial U}$ and $R \cong F$ in $A_{\partial U}(\overline{U}, E)$ assume there exists a map $H : \overline{U} \times [0, 1] \rightarrow 2^E$ with $H(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \rightarrow [0, 1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0, 1)$ and

\[
\{ x \in \overline{U} : \Phi(x) \cap H(x, t) \neq \emptyset \text{ for some } t \in [0, 1] \}
\]

is compact (respectively closed), $H_0 = G$, $H_1 = R$ (here $H_t(x) = H(x, t)$) and for any continuous map $\mu : \overline{U} \rightarrow [0, 1]$ with $\mu(\partial U) = 0$ assume

\[
\{ x \in \overline{U} : \emptyset \neq \Phi(x) \cap H(x, t\mu(x)) \text{ for some } t \in [0, 1] \}
\]

is closed. Then $F$ is $\Phi$-essential in $A_{\partial U}(\overline{U}, E)$.

**Proof.** Let $R \in A_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = F|_{\partial U}$ and $R \cong F$ in $A_{\partial U}(\overline{U}, E)$. We must show there exists a $x \in U$ with $R(x) \cap \Phi(x) \neq \emptyset$. Choose the map $H : \overline{U} \times [0, 1] \rightarrow 2^E$ as in the statement of Theorem 2.8. Let

\[
D = \{ x \in \overline{U} : \Phi(x) \cap H(x, t) \neq \emptyset \text{ for some } t \in [0, 1] \}.
\]

Note $D \neq \emptyset$. Also $D$ is compact (respectively closed) if $E$ is a completely regular (respectively normal) topological space. Next note $D \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \rightarrow [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $J_\mu : \overline{U} \rightarrow 2^E$ by $J_\mu(x) = H(x, \mu(x)) = H_\mu(x)(x)$. Note $J_\mu \in A_{\partial U}(\overline{U}, E)$ with $J_\mu|_{\partial U} = H_0|_{\partial U}$. We now claim

\[(2.1) \quad J_\mu \cong H_0 \text{ in } A_{\partial U}(\overline{U}, E).\]
If the claim is true then since $H_0$ is $\Phi$-essential in $A_{\partial U}(\overline{U}, E)$ then there exists a $x \in U$ with $J_\mu(x) \cap \Phi(x) \neq \emptyset$ (i.e. $H_\mu(x) \cap \Phi(x) \neq \emptyset$), and thus $x \in D$ so $\mu(x) = 1$ and as a result $\emptyset \neq H_1(x) \cap \Phi(x) = R(x) \cap \Phi(x)$.

It remains to show (2.1). Let $Q : \overline{U} \times [0, 1] \to 2^E$ be given by $Q(x, t) = H(x, t\mu(x))$. Note $Q(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$ and

$$\{x \in \overline{U} : \emptyset \neq \Phi(x) \cap Q(x, t) = \Phi(x) \cap H(x, t\mu(x)) \text{ for some } t \in [0, 1]\}$$

is compact (respectively closed). Note $Q_0 = H_0$ and $Q_1 = J_\mu$. Finally if there exists $t \in [0, 1]$ and $x \in \partial U$ with $\Phi(x) \cap Q_t \neq \emptyset$ then $\Phi(x) \cap H_{\mu(x)} \neq \emptyset$ so $x \in D$ and so $\mu(x) = 1$ i.e. $\Phi(x) \cap H_t(x) \neq \emptyset$, a contradiction. Thus (2.1) holds. \hfill \Box

**Remark 2.9.** In Theorem 2.8 note one example of a map $R$ is $F$ itself. It is of interest to note if we consider maps $R$ other than $F$ and if we suppose

$$\cong \text{ is an equivalence relation in } A_{\partial U}(\overline{U}, E),$$

then (see in the statement of Theorem 2.8) if $R \cong F$ in $A_{\partial U}(\overline{U}, E)$ and $R \cong G$ in $A_{\partial U}(\overline{U}, E)$ then $F \cong G$ in $A_{\partial U}(\overline{U}, E)$.

We now show that the ideas in this section can be applied to other natural situations. Let $E$ be a Hausdorff topological vector space (so automatically a completely regular space), $Y$ a topological vector space, and $U$ an open subset of $E$. Also let $L : \text{dom} \ L \subseteq E \to Y$ be a linear single valued map; here $\text{dom} \ L$ is a vector subspace of $E$. Finally $T : E \to Y$ will be a linear single valued map with $L + T : \text{dom} \ L \to Y$ a bijection; for convenience we say $T \in H_L(E, Y)$.

**Definition 2.10.** We say $F \in A(\overline{U}, Y; L, T)$ (respectively $F \in B(\overline{U}, Y; L, T)$) if $F : \overline{U} \to 2^Y$ and $(L+T)^{-1}(F+T) \in A(\overline{U}, E)$ (respectively $(L+T)^{-1}(F+T) \in B(\overline{U}, E)$).

We now fix a $\Phi \in B(\overline{U}, Y; L, T)$.

**Definition 2.11.** We say $F \in A_{\partial U}(\overline{U}, Y; L, T)$ if $F \in A(\overline{U}, Y; L, T)$ with $(L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ for $x \in \partial U$.

**Definition 2.12.** Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$. $F$ is $L$-$\Phi$-essential in $A_{\partial U}(\overline{U}, Y; L, T)$ if for every map $J \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $(L+T)^{-1}(J+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$.

**Theorem 2.13.** Let $E$ be a topological vector space (so automatically completely regular), $Y$ a topological vector space, $U$ an open subset of $E$, $L : \text{dom} \ L \subseteq E \to Y$ a linear single valued map and $T \in H_L(E, Y)$. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$ and let $G \in A_{\partial U}(\overline{U}, Y; L, T)$ be $L$-$\Phi$-essential in $A_{\partial U}(\overline{U}, Y; L, T)$. For any map $R \in A_{\partial U}(\overline{U}, Y; L, T)$ with $R|_{\partial U} = F|_{\partial U}$ assume there exists a map $H : \overline{U} \times [0, 1] \to \overline{U} \times [0, 1]$
$2^Y$ with $(L + T)^{-1}(H(\cdot, \eta(\cdot)) + T(\cdot)) \in A(\overline{U}, E)$ for any continuous function \( \eta : \overline{U} \to [0,1] \) with \( \eta(\partial U) = 0 \), \((L + T)^{-1}(H_t + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) = \emptyset \) for any \( x \in \partial U \) and \( t \in (0,1) \), \( H_0 = G \), \( H_1 = R \) (here \( H_t(x) = H(x,t) \)) and 
\[ \{ x \in \overline{U} : (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_t + T)(x) \neq \emptyset \text{ for some } t \in [0,1] \} \] is compact. Then \( F \) is \( L-\Phi \)-essential in \( A_{\partial U}(\overline{U}, Y; L, T) \).

Proof. Let \( R \in A_{\partial U}(\overline{U}, Y; L, T) \) with \( R|_{\partial U} = F|_{\partial U} \). We must show there exists \( x \in U \) with 
\[ (L + T)^{-1}(R + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) \neq \emptyset \]. Choose the map 
\[ H : \overline{U} \times [0,1] \to 2^Y \] as in the statement of Theorem 2.13. Let 
\[ D = \{ x \in \overline{U} : (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_t + T)(x) \neq \emptyset \text{ for some } t \in [0,1] \} \].

Note \( D \neq \emptyset \), \( D \) is compact, \( D \cap \partial U = \emptyset \) so there exists a continuous map \( \mu : \overline{U} \to [0,1] \) with \( \mu(\partial U) = 0 \) and \( \mu(D) = 1 \). Define \( J_\mu : \overline{U} \to 2^Y \) by \( J_\mu(x) = H(x, \mu(x)) \). Note 
\[ J_\mu \in A_{\partial U}(\overline{U}, Y; L, T) \] and \( J_\mu|_{\partial U} = H_0|_{\partial U} \). Now since \( H_0(= G) \) is \( L-\Phi \)-essential in 
\( A_{\partial U}(\overline{U}, Y; L, T) \) there exists \( x \in U \) with 
\[ (L+T)^{-1}(J_\mu(x)) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset \] (i.e. 
\[ (L+T)^{-1}(H_\mu(x) + T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset \]), and thus \( x \in D \) so \( \mu(x) = 1 \) and we are finished. \( \Box \)

Remark 2.14. If \( E \) is a normal topological vector space then the assumption that 
\( D \) (in the proof of Theorem 2.13) is compact, can be replaced by \( D \) is closed, in the statement (and proof) of Theorem 2.13.

It is possible to obtain an analogue result if we change Definition 2.12 as follows.

Definition 2.15. Let \( F, G \in A_{\partial U}(\overline{U}, Y; L, T) \). We say \( F \cong G \) in \( A_{\partial U}(\overline{U}, Y; L, T) \) if there exists a map \( H : \overline{U} \times [0,1] \to 2^Y \) with 
\[ (L + T)^{-1}(H(\cdot, \eta(\cdot)) + T(\cdot)) \in A(\overline{U}, E) \] for any continuous function \( \eta : \overline{U} \to [0,1] \) with \( \eta(\partial U) = 0 \), 
\[ (L + T)^{-1}(H_t + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) = \emptyset \text{ for any } x \in \partial U \text{ and } t \in [0,1], H_1 = F, H_0 = G \text{ and} \]
\[ \{ x \in \overline{U} : (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_t + T)(x) \neq \emptyset \text{ for some } t \in [0,1] \} \]
is compact; here \( H_t(x) = H(x,t) \).

Remark 2.16. If \( E \) is a normal topological vector space then the assumption that 
\[ \{ x \in \overline{U} : (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_t + T)(x) \neq \emptyset \text{ for some } t \in [0,1] \} \]
is compact, can be replaced by 
\[ \{ x \in \overline{U} : (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_t + T)(x) \neq \emptyset \text{ for some } t \in [0,1] \} \]
is closed, in Definition 2.15.

Definition 2.17. Let \( F \in A_{\partial U}(\overline{U}, Y; L, T) \). \( F \) is \( L-\Phi \)-essential in \( A_{\partial U}(\overline{U}, Y; L, T) \) if for every map \( J \in A_{\partial U}(\overline{U}, Y; L, T) \) with \( J|_{\partial U} = F|_{\partial U} \) and \( J \cong F \) in \( A_{\partial U}(\overline{U}, Y; L, T) \) there exists \( x \in U \) with 
\[ (L+T)^{-1}(J + T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset \].
Theorem 2.18. Let $E$ be a topological vector space, $Y$ a topological vector space, $U$ an open subset of $E$, $L : \text{dom} L \subseteq E \to Y$ a linear single valued map and $T \in H_L(E, Y)$. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$ and let $G \in A_{\partial U}(\overline{U}, Y; L, T)$ be $L$-$\Phi$-essential in $A_{\partial U}(\overline{U}, Y; L, T)$. For any map $R \in A_{\partial U}(\overline{U}, Y; L, T)$ with $R|_{\partial U} = F|_{\partial U}$ and $R \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ assume there exists a map $H : \overline{U} \times [0, 1] \to 2^Y$ with $(L + T)^{-1}(H(\cdot, \eta(\cdot)) + T(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$, $(L + T)^{-1}(H_t + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0, 1)$, $H_0 = G$, $H_1 = R$ (here $H_t(x) = H(x, t)$) and \( \{x \in \overline{U} : (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_t + T)(x) \neq \emptyset \text{ for some } t \in [0, 1]\} \) is compact and for any continuous function $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ assume
\[
\{x \in \overline{U} : (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_{\mu(x)} + T)(x) \neq \emptyset \text{ for some } t \in [0, 1]\}
\]
is closed. Then $F$ is $L$-$\Phi$-essential in $A_{\partial U}(\overline{U}, Y; L, T)$.

Proof. Let $R \in A_{\partial U}(\overline{U}, Y; L, T)$ with $R|_{\partial U} = F|_{\partial U}$ and $R \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$. Choose the map $H : \overline{U} \times [0, 1] \to 2^Y$ as in the statement of Theorem 2.18. Let
\[
D = \{x \in \overline{U} : (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_t + T)(x) \neq \emptyset \text{ for some } t \in [0, 1]\}.
\]
Note $D \neq \emptyset$, $D$ is compact, $D \cap \partial U = \emptyset$ so there exists a continuous map $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $J_\mu : \overline{U} \to 2^Y$ by $J_\mu(x) = H(x, \mu(x))$. Note $J_\mu \in A_{\partial U}(\overline{U}, Y; L, T)$ and $J_\mu|_{\partial U} = H_0|_{\partial U}$. Also note $J_\mu \cong H_0$ in $A_{\partial U}(\overline{U}, Y; L, T)$ (to see this let $Q : \overline{U} \times [0, 1] \to 2^Y$ be given by $Q(x, t) = H(x, t \mu(x))$). Now since $H_0$ is $L$-$\Phi$-essential in $A_{\partial U}(\overline{U}, Y; L, T)$ there exists $x \in U$ with $(L + T)^{-1}(J_\mu + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) \neq \emptyset$, and thus $x \in D$ so $\mu(x) = 1$ and we are finished.

Remark 2.19. It is of interest to note if we consider maps $R$ other than $F$ in Theorem 2.18 and if we suppose
\[
\cong \text{ is an equivalence relation in } A_{\partial U}(\overline{U}, Y; L, T),
\]
then (see in the statement of Theorem 2.18) if $R \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ and $R \cong G$ in $A_{\partial U}(\overline{U}, Y; L, T)$ then $F \equiv G$ in $A_{\partial U}(\overline{U}, Y; L, T)$.

Remark 2.20. There is an analogue of Remark 2.14 (for normal topological vector spaces) in the statement of Theorem 2.18.

3. $d$-$\Phi$-ESSENTIAL MAPS

Let $E$ be a completely regular topological space and $U$ an open subset of $E$. We will consider the classes $A$, $B$, $A$ and $B$ of maps as in Section 2.

In this section we fix a $\Phi \in B(\overline{U}, E)$. 
For any map $F \in A(\overline{U}, E)$ let $F^* = I \times F : \overline{U} \to 2^{\overline{U} \times E}$, with $I : \overline{U} \to \overline{U}$ given by $I(x) = x$, and let

$$d : \{(F^*)^{-1}(B)\} \cup \{\emptyset\} \to \Omega$$

be any map with values in the nonempty set $\Omega$; here $B = \{(x, \Phi(x)) : x \in \overline{U}\}$.

**Definition 3.1.** Let $F \in A_{\partial U}(\overline{U}, E)$ with $F^* = I \times F$. We say $F^* : \overline{U} \to 2^{\overline{U} \times E}$ is $d$-$\Phi$-essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J^* = I \times J$ and $J|_{\partial U} = F|_{\partial U}$ we have that $d ((F^*)^{-1}(B)) = d ((J^*)^{-1}(B)) \neq d(\emptyset)$.

**Remark 3.2.** If $F^*$ is $d$-$\Phi$-essential in $A_{\partial U}(\overline{U}, E)$ then

$$\emptyset \neq (F^*)^{-1}(B) = \{x \in \overline{U} : F^*(x) \cap B \neq \emptyset\}$$

$$\emptyset \neq \{x \in \overline{U} : (x, F(x)) \cap (x, \Phi(x)) \neq \emptyset\},$$

and this together with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$ implies that there exists $x \in U$ with $(x, \Phi(x)) \cap F^*(x) \neq \emptyset$ (i.e. $\Phi(x) \cap F(x) \neq \emptyset$).

**Theorem 3.3.** Let $E$ be a completely regular (respectively normal) topological space, $U$ an open subset of $E$, $B = \{(x, \Phi(x)) : x \in \overline{U}\}$, $d$ a map defined in (3.1), $F \in A_{\partial U}(\overline{U}, E)$ with $F^* = I \times F$, $G \in A_{\partial U}(\overline{U}, E)$ and $G^*$ is $d$-$\Phi$-essential in $A_{\partial U}(\overline{U}, E)$ (here $G^* = I \times G$). For any map $R \in A_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = F|_{\partial U}$ assume there exists a map $H : \overline{U} \times [0, 1] \to 2^E$ with $H(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0, 1)$, and $x \in \overline{U} : (x, \Phi(x)) \cap H^*(x, t) \neq \emptyset$ for some $t \in [0, 1]$ is compact (respectively closed), $H_0 = G$, $H_1 = R$ (here $H_t(x) = H(x, t)$ and $H^*(x, t) = (x, H(x,t))$) and also suppose there exists a map $\Psi : \overline{U} \times [0, 1] \to 2^E$ with $\Psi(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap \Psi_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0, 1)$, $\Psi_0 = H_0(= G)$, $\Psi_1 = F$ and $x \in \overline{U} : (x, \Phi(x)) \cap \Psi^*(x, t) \neq \emptyset$ for some $t \in [0, 1]$ is compact (respectively closed); here $\Psi_t(x) = \Psi(x, t)$ and $\Psi^*(x, t) = (x, \Psi(x, t))$. Then $F^*$ is $d$-$\Phi$-essential in $A_{\partial U}(\overline{U}, E)$.

**Proof.** Let $R \in A_{\partial U}(\overline{U}, E)$ with $R^* = I \times R$ and $R|_{\partial U} = F|_{\partial U}$. We must show $d ((F^*)^{-1}(B)) = d ((R^*)^{-1}(B)) \neq d(\emptyset)$. Choose the maps $H : \overline{U} \times [0, 1] \to 2^E$ and $\Psi : \overline{U} \times [0, 1] \to 2^E$ as in the statement of Theorem 3.3. Let

$$D = \{x \in \overline{U} : (x, \Phi(x)) \cap H^*(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\}$$

where $H^*(x, t) = (x, H(x, t))$. Notice $D \neq \emptyset$ since $H_0^*$ is $d$-$\Phi$-essential. Also $D$ is compact (respectively closed) if $E$ is a completely regular (respectively normal) topological space and $D \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $J_\mu : \overline{U} \to 2^E$ by $J_\mu(x) = H(x, \mu(x))$ and let
$J_\mu^* = I \times J_\mu$. Note $J_\mu \in A_{\partial U}(\overline{U}, E)$ with $J_\mu|_{\partial U} = H_0|_{\partial U}$. Now since $H_0^*$ is $d$-$\Phi$-essential in $A_{\partial U}(\overline{U}, E)$ then

(3.2) \[ d \left( (J_\mu^*)^{-1} (B) \right) = d \left( (H_0^*)^{-1} (B) \right) \neq d(\emptyset). \]

Next we note since $\mu(D) = 1$ that

$$ (J_\mu^*)^{-1} (B) = \{ x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, \mu(x)) \neq \emptyset \}$$

$$ = \{ x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, 1) \neq \emptyset \} = (R^*)^{-1} (B), $$

so from (3.2) we have

(3.3) \[ d \left( (R^*)^{-1} (B) \right) = d \left( (H_0^*)^{-1} (B) \right) \neq d(\emptyset). \]

Let

$$ D_1 = \{ x \in \overline{U} : (x, \Phi(x)) \cap \Psi^*(x, t) \neq \emptyset \text{ for some } t \in [0, 1] \} $$

where $\Psi^*(x, t) = (x, \Psi(x, t))$. Notice $D_1 \neq \emptyset$ since $\Psi_0 = H_0$ and $H_0^*$ is $d$-$\Phi$-essential. Also $D_1$ is compact (respectively closed) if $E$ is a completely regular (respectively normal) topological space and $D_1 \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(D_1) = 1$. Define $G_\mu : \overline{U} \to 2^E$ by $G_\mu(x) = \Psi(x, \mu(x))$ and let $G_\mu^* = I \times G_\mu$. Note $G_\mu \in A_{\partial U}(\overline{U}, E)$ with $G_\mu|_{\partial U} = H_0|_{\partial U}$. Now since $H_0^*$ is $d$-$\Phi$-essential in $A_{\partial U}(\overline{U}, E)$ then

(3.4) \[ d \left( (G_\mu^*)^{-1} (B) \right) = d \left( (H_0^*)^{-1} (B) \right) \neq d(\emptyset). \]

Next we note since $\mu(D_1) = 1$ that

$$ (G_\mu^*)^{-1} (B) = \{ x \in \overline{U} : (x, \Phi(x)) \cap (x, \Psi(x, \mu(x)) \neq \emptyset \}$$

$$ = \{ x \in \overline{U} : (x, \Phi(x)) \cap (x, \Psi(x, 1) \neq \emptyset \) = (F^*)^{-1} (B), $$

so from (3.4) we have

(3.5) \[ d \left( (F^*)^{-1} (B) \right) = d \left( (H_0^*)^{-1} (B) \right) \neq d(\emptyset). \]

Now (3.3) and (3.5) yield $d \left( (F^*)^{-1} (B) \right) = d \left( (R^*)^{-1} (B) \right) \neq d(\emptyset)$. \hfill \Box

It is possible to obtain an analogue result if we change Definition 3.1 as follows.

**Definition 3.4.** Let $E$ be a completely regular (respectively normal) topological space, and $U$ an open subset of $E$. Let $F, G \in A_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if there exists a map $H : \overline{U} \times [0, 1] \to 2^E$ with $H(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$, $H_t(x) \cap \Phi(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0, 1]$, $H_1 = F$, $H_0 = G$ and $\{ x \in \overline{U} : (x, \Phi(x)) \cap H^*(x, t) \neq \emptyset \text{ for some } t \in [0, 1] \}$ is compact (respectively closed); here $H^*(x, t) = (x, H(x, t))$ and $H_t(x) = H(x, t)$.

**Definition 3.5.** Let $F \in A_{\partial U}(\overline{U}, E)$ with $F^* = I \times F$. We say $F^* : \overline{U} \to 2^{U \times E}$ is $d$-$\Phi$-essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J^* = I \times J$ and $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ we have that $d \left( (F^*)^{-1} (B) \right) = d \left( (J^*)^{-1} (B) \right) \neq d(\emptyset)$. 

Theorem 3.6. Let $E$ be a completely regular (respectively normal) topological space, $U$ an open subset of $E$, $B = \{ (x, \Phi(x)) : x \in \overline{U} \}$, $d$ a map defined in (3.1), $F \in A_{\partial U}(\overline{U}, E)$ with $F^* = I \times F$, $G \in A_{\partial U}(\overline{U}, E)$ and $G^*$ is $d$-$\Phi$-essential in $A_{\partial U}(\overline{U}, E)$ (here $G^* = I \times G$). For any map $R \in A_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = F|_{\partial U}$ and $R \cong F$ in $A_{\partial U}(\overline{U}, E)$ assume there exists a map $H : \overline{U} \times [0, 1] \rightarrow 2^E$ with $H(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \rightarrow [0, 1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0, 1)$, and

$$\{ x \in \overline{U} : (x, \Phi(x)) \cap H^*(x, t) \neq \emptyset \text{ for some } t \in [0, 1] \}$$

is compact (respectively closed), $H_0 = G$, $H_1 = R$ (here $H_t(x) = H(x, t)$ and $H^*(x, t) = (x, H(x, t))$) and for any continuous function $\mu : \overline{U} \rightarrow [0, 1]$ with $\mu(\partial U) = 0$ assume

$$\{ x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, t\mu(x))) \neq \emptyset \text{ for some } t \in [0, 1] \}$$

is closed and also suppose there exists a map $\Psi : \overline{U} \times [0, 1] \rightarrow 2^E$ with $\Psi(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \rightarrow [0, 1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap \Psi_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0, 1)$, and

$$\{ x \in \overline{U} : (x, \Phi(x)) \cap \Psi^*(x, t) \neq \emptyset \text{ for some } t \in [0, 1] \}$$

is compact (respectively closed), $\Psi_0 = H_0(= G)$, $\Psi_1 = F$ (here $\Psi_t(x) = \Psi(x, t)$ and $\Psi^*(x, t) = (x, \Psi(x, t))$) and for any continuous function $\mu : \overline{U} \rightarrow [0, 1]$ with $\mu(\partial U) = 0$ assume

$$\{ x \in \overline{U} : (x, \Phi(x)) \cap (x, \Psi(x, t\mu(x))) \neq \emptyset \text{ for some } t \in [0, 1] \}$$

is closed. Then $F^*$ is $d$-$\Phi$-essential in $A_{\partial U}(\overline{U}, E)$.

Proof. Let $R \in A_{\partial U}(\overline{U}, E)$ with $R^* = I \times R$ and $R|_{\partial U} = F|_{\partial U}$ and $R \cong F$ in $A_{\partial U}(\overline{U}, E)$. We must show $d((F^*)^{-1}(B)) = d((R^*)^{-1}(B)) \neq d(\emptyset)$. Choose the maps $H : \overline{U} \times [0, 1] \rightarrow 2^E$ and $\Psi : \overline{U} \times [0, 1] \rightarrow 2^E$ as in the statement of Theorem 3.6. Let

$$D = \{ x \in \overline{U} : (x, \Phi(x)) \cap H^*(x, t) \neq \emptyset \text{ for some } t \in [0, 1] \}$$

where $H^*(x, t) = (x, H(x, t))$. Notice $D \neq \emptyset$. Also $D$ is compact (respectively closed) if $E$ is a completely regular (respectively normal) topological space and $D \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \rightarrow [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $J_\mu : \overline{U} \rightarrow 2^E$ by $J_\mu(x) = H(x, \mu(x))$ and let $J_\mu^* = I \times J_\mu$. Note $J_\mu \in A_{\partial U}(\overline{U}, E)$ with $J_\mu|_{\partial U} = H_0|_{\partial U}$. Also note $J_\mu \cong H_0$ in $A_{\partial U}(\overline{U}, E)$ (to see this let $Q : \overline{U} \times [0, 1] \rightarrow 2^E$ be given by $Q(x, t) = H(x, t\mu(x))$). Now since $H_0^*$ is $d$-$\Phi$-essential in $A_{\partial U}(\overline{U}, E)$ then

$$d\left( (J_\mu^*)^{-1}(B) \right) = d\left( (H_0^*)^{-1}(B) \right) \neq d(\emptyset)$$
and as in Theorem 3.3 (note $\mu(D) = 1$) we have $(J_\mu^*)^{-1}(B) = (R^*)^{-1}(B)$, so
\begin{equation}
(3.6) \quad d((R^*)^{-1}(B)) = d((H_0^*)^{-1}(B)) \neq d(\emptyset).
\end{equation}

Let
\[ D_1 = \{ x \in \overline{U} : (x, \Phi(x)) \cap \Psi^*(x, t) \neq \emptyset \text{ for some } t \in [0, 1] \} \]
where $\Psi^*(x, t) = (x, \Psi(x, t))$. Notice $D_1 \neq \emptyset$. Also $D_1$ is compact (respectively closed) if $E$ is a completely regular (respectively normal) topological space and $D_1 \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(D_1) = 1$.

Define $G_\mu : \overline{U} \to 2^E$ by $G_\mu(x) = \Psi(x, \mu(x))$ and let $G^*_\mu = I \times G_\mu$. Note $G_\mu \in A_{\partial U}(\overline{U}, E)$ with $G_{\mu|\partial U} = H_0|\partial U$. Also note $G_\mu \equiv H_0(= G)$ in $A_{\partial U}(\overline{U}, E)$ (to see this let $Q^1 : \overline{U} \times [0, 1] \to 2^E$ be given by $Q^1(x, t) = \Psi(x, t\mu(x))$). Now since $H_0^*$ is $d$-$\Phi$-essential in $A_{\partial U}(\overline{U}, E)$ then
\[ d\left((G_\mu^*)^{-1}(B)\right) = d\left((H_0^*)^{-1}(B)\right) \neq d(\emptyset) \]
and as in Theorem 3.3 (note $\mu(D) = 1$) we have $(G^*_\mu)^{-1}(B) = (F^*)^{-1}(B)$, so
\begin{equation}
(3.7) \quad d\left((F^*)^{-1}(B)\right) = d\left((H_0^*)^{-1}(B)\right) \neq d(\emptyset).
\end{equation}

Now (3.6) and (3.7) yield $d\left((F^*)^{-1}(B)\right) = d\left((R^*)^{-1}(B)\right) \neq d(\emptyset)$.

**Remark 3.7.** It is of interest to note if we consider maps $R$ other than $F$ in Theorem 3.6 and if we suppose
\[ \equiv \] is an equivalence relation in $A_{\partial U}(\overline{U}, E),
\]
then (see in the statement of Theorem 3.6) if $R \equiv F$ in $A_{\partial U}(\overline{U}, E)$ and $R \equiv G$ in $A_{\partial U}(\overline{U}, E)$ then (since $F \equiv G$ in $A_{\partial U}(\overline{U}, E)$) there exists a map $\Psi : \overline{U} \times [0, 1] \to 2^E$ with $\Psi(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap \Psi_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0, 1)$, and $\{ x \in \overline{U} : (x, \Phi(x)) \cap \Psi^*(x, t) \neq \emptyset \text{ for some } t \in [0, 1] \}$ is compact (respectively closed), $\Psi_0 = G$ and $\Psi_1 = F$ (here $\Psi_t(x) = \Psi(x, t)$ and $\Psi^*(x, t) = (x, \Psi(x, t))$).

**Remark 3.8.** Suppose the following conditions holds (which is common in the literature on topological degree):
\[ \begin{cases} 
\text{if } F, G \in A_{\partial U}(\overline{U}, E) \text{ with } F|_{\partial U} = G|_{\partial U} \text{ and } F \equiv G \\
\text{in } A_{\partial U}(\overline{U}, E) \text{ then } d\left((F^*)^{-1}(B)\right) = d\left((G^*)^{-1}(B)\right). 
\end{cases} \]

Then Definition 3.5 reduces to the following. Let $F \in A_{\partial U}(\overline{U}, E)$ with $F^* = I \times F$. We say $F^* : \overline{U} \to 2^{\overline{U} \times E}$ is $d$-$\Phi$-essential in $A_{\partial U}(\overline{U}, E)$ if $d\left((F^*)^{-1}(B)\right) \neq d(\emptyset)$.

Let $E$ be a topological vector space, $Y$ a topological vector space, $U$ an open subset of $E$, $L : \text{dom} L \subseteq E \to Y$ a linear single valued map and $T \in H_L(E, Y)$.

We now fix a $\Phi \in B(\overline{U}, Y; L, T)$. 


For any map $F \in A(\overline{U}, Y; L, T)$ let $F^* = I \times (L + T)^{-1}(F + T) : \overline{U} \to 2^{\overline{U} \times E}$, with $I : \overline{U} \to \overline{U}$ given by $I(x) = x$, and let

$$d : \{(F^*)^{-1}(B)\} \cup \{\emptyset\} \to \Omega$$

be any map with values in the nonempty set $\Omega$; here

$$B = \{(x, (L + T)^{-1}(\Phi + T)(x)) : x \in \overline{U}\}.$$

**Definition 3.9.** Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$ with $F^* = I \times (L + T)^{-1}(F + T)$. We say $F^* : \overline{U} \to 2^{\overline{U} \times E}$ is $d$-$L$-$\Phi$-essential in $A_{\partial U}(\overline{U}, Y; L, T)$ if for every map $J \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J^* = I \times (L + T)^{-1}(J + T)$ and $J|_{\partial U} = F|_{\partial U}$ we have that $d ((F^*)^{-1}(B)) = d ((J^*)^{-1}(B)) \neq d(\emptyset)$.

**Theorem 3.10.** Let $E$ be a Hausdorff topological vector space, $Y$ a topological vector space, $U$ an open subset of $E$, $B = \{(x, (L + T)^{-1}(\Phi + T)(x)) : x \in \overline{U}\}$, $L : \text{dom} L \subseteq E \to Y$ a linear single valued map, $T \in H_L(E, Y)$, $d$ a map defined in (3.8), $F \in A_{\partial U}(\overline{U}, Y; L, T)$ with $F^* = I \times (L + T)^{-1}(F + T)$, $G \in A_{\partial U}(\overline{U}, Y; L, T)$ and $G^*$ is $d$-$L$-$\Phi$-essential in $A_{\partial U}(\overline{U}, Y; L, T)$ (here $G^* = I \times (L + T)^{-1}(G + T)$). For any map $R \in A_{\partial U}(\overline{U}, Y; L, T)$ with $R^* = I \times (L + T)^{-1}(R + T)$ and $R|_{\partial U} = F|_{\partial U}$ assume there exists a map $H : \overline{U} \times [0, 1] \to 2^Y$ with $(L + T)^{-1}(H(\cdot, \eta(\cdot)) + T(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$, $(L + T)^{-1}((H_t + T)(x) \cap (L + T)^{-1}(\Phi + T)(x)) = \emptyset$ for any $x \in \partial U$ and $t \in (0, 1)$, and

$$\{x \in \overline{U} : (x, (L + T)^{-1}(\Phi + T)(x)) \cap H^*(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\}$$

is compact, $H_0 = G$, $H_1 = R$ (here $H_t(x) = H(x, t)$ and $H^*(x, \lambda) = (x, (L + T)^{-1}(H + T)(x, \lambda))$) and also suppose there exists a map $\Psi : \overline{U} \times [0, 1] \to 2^Y$ with $(L + T)^{-1}(\Psi(\cdot, \eta(\cdot)) + T(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$, $(L + T)^{-1}((\Psi_t + T)(x) \cap (L + T)^{-1}(\Phi + T)(x)) = \emptyset$ for any $x \in \partial U$ and $t \in (0, 1)$, and

$$\{x \in \overline{U} : (x, (L + T)^{-1}(\Phi + T)(x)) \cap \Psi^*(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\}$$

is compact, $\Psi_0 = H_0$ and $\Psi_1 = F$ (here $\Psi_t(x) = \Psi(x, t)$ and $\Psi^*(x, \lambda) = (x, (L + T)^{-1}(\Psi + T)(x, \lambda))$). Then $F^*$ is $d$-$L$-$\Phi$-essential in $A_{\partial U}(\overline{U}, Y; L, T)$.

**Proof.** Let $R \in A_{\partial U}(\overline{U}, Y; L, T)$ with $R^* = I \times (L + T)^{-1}(R + T)$ and $R|_{\partial U} = F|_{\partial U}$. Choose the maps $H : \overline{U} \times [0, 1] \to 2^Y$ and $\Psi : \overline{U} \times [0, 1] \to 2^Y$ as in the statement of Theorem 3.10. Let

$$D = \{x \in \overline{U} : (x, (L + T)^{-1}(\Phi + T)(x)) \cap H^*(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\}$$

where $H^*(x, \lambda) = (x, (L + T)^{-1}(H + T)(x, \lambda))$. Notice $D \neq \emptyset$, $D$ is compact and $D \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $J_\mu : \overline{U} \to 2^Y$ by $J_\mu(x) = H(x, \mu(x)) = H_\mu(x)(x)$ and let
$J^*_\mu = I \times (L + T)^{-1}(J + T)$. Note $J_\mu \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J_\mu|_{\partial U} = H_0|_{\partial U}$. Also since $H_0^*$ is $d$-$L$-$\Phi$-essential in $A_{\partial U}(\overline{U}, Y; L, T)$ we have

\[(3.9) \quad d \left( (J^*_\mu)^{-1}(B) \right) = d \left( (H_0^*)^{-1}(B) \right) \neq d(\emptyset).\]

Next we note since $\mu(D) = 1$ that

\[(J^*_\mu)^{-1}(B) = \{ x \in \overline{U} : (x, (L + T)^{-1}(\Phi + T)(x)) \cap (L + T)^{-1}(H_\mu(x) + T)(x) \neq \emptyset \} = \{ x \in \overline{U} : (x, (\Phi + T)(x)) \cap (L + T)^{-1}(H_\mu(x) + T)(x) \neq \emptyset \} = (R^*)^{-1}(B),\]

so from (3.9) we have

\[(3.10) \quad d \left( (R^*)^{-1}(B) \right) = d \left( (H_0^*)^{-1}(B) \right) \neq \emptyset.\]

Let

\[D_1 = \{ x \in \overline{U} : (x, (L + T)^{-1}(\Phi + T)(x)) \cap \Psi^*(x, t) \neq \emptyset \text{ for some } t \in [0, 1] \}\]

where $\Psi^*(x, \lambda) = (x, (L + T)^{-1}(\Phi + T)(x, \lambda))$. Notice $D_1 \neq \emptyset$, $D_1$ is compact and $D_1 \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \rightarrow [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(D_1) = 1$. Define $G_\mu : \overline{U} \rightarrow 2^Y$ by $G_\mu(x) = \Psi(x, \mu(x))$ and let $G^*_\mu = I \times (L + T)^{-1}(G_\mu + T)$. Note $G_\mu \in A_{\partial U}(\overline{U}, Y; L, T)$ with $G_\mu|_{\partial U} = H_0|_{\partial U}$. Also since $H_0^*$ is $d$-$L$-$\Phi$-essential in $A_{\partial U}(\overline{U}, Y; L, T)$ we have

\[(3.11) \quad d \left( (G^*_\mu)^{-1}(B) \right) = d \left( (H_0^*)^{-1}(B) \right) \neq d(\emptyset).\]

Next we note since $\mu(D_1) = 1$ that

\[(G^*_\mu)^{-1}(B) = \{ x \in \overline{U} : (x, (L + T)^{-1}(\Phi + T)(x)) \cap (L + T)^{-1}(\Psi_\mu(x) + T)(x) \neq \emptyset \} = \{ x \in \overline{U} : (x, (\Phi + T)(x)) \cap (L + T)^{-1}(\Psi_1 + T)(x) \neq \emptyset \} = (F^*)^{-1}(B),\]

so from (3.11) we have

\[(3.12) \quad d \left( (F^*)^{-1}(B) \right) = d \left( (H_0^*)^{-1}(B) \right) \neq d(\emptyset).\]

Now (3.10) and (3.12) yield $d \left( (F^*)^{-1}(B) \right) = d \left( (R^*)^{-1}(B) \right) \neq \emptyset$. \hfill \Box

Remark 3.11. If $E$ is a normal topological vector space then the assumption that $D$ (in the proof of Theorem 3.10) is compact, can be replaced by $D$ is closed, in the statement (and proof) of Theorem 3.10.
It is possible to obtain an analogue result if we change Definition 3.9 as follows.

**Definition 3.12.** Let \( F, G \in A_{\partial U}(\overline{U}, Y; L, T) \). We say \( F \cong G \) in \( A_{\partial U}(\overline{U}, Y; L, T) \) if there exists a map \( H : \overline{U} \times [0, 1] \rightarrow 2^Y \) with \( (L + T)^{-1}(\Phi + T)(x) \cap H^*(x, t) = \emptyset \) for any \( x \in \partial U \) and \( t \in [0, 1] \). Let \( H_1 = F, H_0 = G \) and
\[
\{ x \in \overline{U} : \Phi + T(x) \cap H^*(x, t) = \emptyset \text{ for some } t \in [0, 1] \}
\]
is compact; here \( H_t(x) = H(x, t) \) and \( H^*(x, \lambda) = (x, (L + T)^{-1}(H + T)(x, \lambda)) \).

**Remark 3.13.** If \( E \) is a normal topological vector space then the assumption that
\[
\{ x \in \overline{U} : \Phi + T(x) \cap H^*(x, t) = \emptyset \text{ for some } t \in [0, 1] \}
\]
is compact, can be replaced by
\[
\{ x \in \overline{U} : \Phi + T(x) \cap H^*(x, t) = \emptyset \text{ for some } t \in [0, 1] \}
\]
is closed, in Definition 3.12; here \( H^*(x, \lambda) = (x, (L + T)^{-1}(H + T)(x, \lambda)) \).

**Definition 3.14.** Let \( F \in A_{\partial U}(\overline{U}, Y; L, T) \) with \( F^* = I \times (L + T)^{-1}(F + T) \). We say \( F^* : \overline{U} \rightarrow 2^{U \times E} \) is \( d - L \)-essential in \( A_{\partial U}(\overline{U}, Y; L, T) \) if for every map \( J \in A_{\partial U}(\overline{U}, Y; L, T) \) with \( J^* = I \times (L + T)^{-1}(J + T) \), and \( J_{|\partial U} = F_{|\partial U} \) and \( J \cong F \) in \( A_{\partial U}(\overline{U}, Y; L, T) \) we have that \( d ((F^*)^{-1}(B)) = d ((J^*)^{-1}(B)) \neq d(\emptyset) \).

**Theorem 3.15.** Let \( E \) be a Hausdorff topological vector space, \( Y \) a topological vector space, \( U \) an open subset of \( E \), \( B = \{(x, (L + T)^{-1}(\Phi + T)(x)) : x \in \overline{U}\} \), \( L : domL \subseteq E \rightarrow Y \) a linear single valued map, \( T \in H_L(E,Y) \), \( d \) a map defined in (3.8), \( F \in A_{\partial U}(\overline{U}, Y; L, T) \) with \( F^* = I \times (L + T)^{-1}(F + T) \), \( G \in A_{\partial U}(\overline{U}, Y; L, T) \) and \( G^* \) is \( d - L \)-essential in \( A_{\partial U}(\overline{U}, Y; L, T) \) (here \( G^* = I \times (L + T)^{-1}(G + T) \)). For any map \( R \in A_{\partial U}(\overline{U}, Y; L, T) \) with \( R^* = I \times (L + T)^{-1}(R + T) \) and \( R_{|\partial U} = F_{|\partial U} \) and \( R \cong F \) in \( A_{\partial U}(\overline{U}, Y; L, T) \) assume there exists a map \( H : \overline{U} \times [0, 1] \rightarrow 2^Y \) with \( (L + T)^{-1}(H(\cdot, \eta(\cdot)) + T(\cdot)) \in A(\overline{U}, E) \) for any continuous function \( \eta : \overline{U} \rightarrow [0, 1] \) with \( \eta(\partial U) = 0 \) and \( H_0 = G \), \( H_1 = R \) (here \( H_t(x) = H(x, t) \) and \( H^*(x, \lambda) = (x, (L + T)^{-1}(H + T)(x, \lambda)) \)) and for any continuous function \( \mu : \overline{U} \rightarrow [0, 1] \) with \( \mu(\partial U) = 0 \) assume
\[
\{ x \in \overline{U} : (x, (L + T)^{-1}(\Phi + T)(x)) \cap H^*(x, t) = \emptyset \text{ for some } t \in [0, 1] \}
\]
is compact, \( H_0 = G \), \( H_1 = R \) (here \( H_t(x) = H(x, t) \) and \( H^*(x, \lambda) = (x, (L + T)^{-1}(H + T)(x, \lambda)) \)) and for any continuous function \( \mu : \overline{U} \rightarrow [0, 1] \) with \( \mu(\partial U) = 0 \) assume
\[
\{ x \in \overline{U} : (x, (L + T)^{-1}(\Phi + T)(x)) \cap (H_{|\partial U})^*(x, t) = \emptyset \text{ for some } t \in [0, 1] \}
\]
is closed and also suppose there exists a map \( \Psi : \overline{U} \times [0, 1] \rightarrow 2^Y \) with \( (L + T)^{-1} \cdot (\Psi(\cdot, \eta(\cdot)) + T(\cdot)) \in A(\overline{U}, E) \) for any continuous function \( \eta : \overline{U} \rightarrow [0, 1] \) with \( \eta(\partial U) = 0 \).
0, \((L + T)^{-1}(\Psi_t + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) = \emptyset\) for any \(x \in \partial U\) and \(t \in (0, 1)\), and
\[
\{ x \in \overline{U} : (x, (L + T)^{-1}(\Phi + T)(x)) \cap \Psi^*(x, t) \neq \emptyset \text{ for some } t \in [0, 1] \}
\]
is compact, \(\Psi_0 = H_0\) and \(\Psi_1 = F\) (here \(\Psi_t(x) = \Psi(x, t)\) and \(\Psi^*(x, \lambda) = (x, (L + T)^{-1}(\Phi + T)(x, \lambda))\)) and for any continuous function \(\mu : \overline{U} \to [0, 1]\) with \(\mu(\partial U) = 0\) assume
\[
\{ x \in \overline{U} : (x, (L + T)^{-1}(\Phi + T)(x)) \cap (L + T)^{-1}(\Psi_{t\mu(x)} + T)(x)) \neq \emptyset \text{ for some } t \in [0, 1] \}
\]
is closed. Then \(F^*\) is d-L-\(\Phi\)-essential in \(A_{\partial U}(\overline{U}, Y; L, T)\).

**Proof.** Let \(R \in A_{\partial U}(\overline{U}, Y; L, T)\) with \(R^* = I \times (L + T)^{-1}(R + T)\) and \(R|_{\partial U} = F|_{\partial U}\) and \(R \cong F\) in \(A_{\partial U}(\overline{U}, Y; L, T)\). Choose the maps \(H : \overline{U} \times [0, 1] \to 2^Y\) and \(\Psi : \overline{U} \times [0, 1] \to 2^Y\) as in the statement of Theorem 3.15. Let
\[
D = \{ x \in \overline{U} : (x, (L + T)^{-1}(\Phi + T)(x)) \cap H^*(x, t) \neq \emptyset \text{ for some } t \in [0, 1] \}
\]
where \(H^*(x, \lambda) = (x, (L + T)^{-1}(H + T)(x, \lambda))\). Notice \(D \neq \emptyset\), \(D\) is compact and \(D \cap \partial U = \emptyset\). Thus there exists a continuous map \(\mu : \overline{U} \to [0, 1]\) with \(\mu(\partial U) = 0\) and \(\mu(D) = 1\). Define \(J_\mu : \overline{U} \to 2^Y\) by \(J_\mu(x) = H(x, \mu(x)) = H_{t\mu(x)}(x)\) and let \(J_\mu^* = I \times (L + T)^{-1}(J_\mu + T)\). Note \(J_\mu \in A_{U}(\overline{U}, Y; L, T)\) with \(J_\mu|_{\partial U} = H_0|_{\partial U}\). Also note \(J_\mu \cong H_0\) in \(A_{\partial U}(\overline{U}, Y; L, T)\) (to see this let \(Q : \overline{U} \times [0, 1] \to 2^Y\) be given by \(Q(x, t) = H(x, t\mu(x))\)). Also since \(H_0^*\) is d-L-\(\Phi\)-essential in \(A_{\partial U}(\overline{U}, Y; L, T)\) we have
\[
d \left( (J_\mu^*)^{-1} (B) \right) = d \left( (H_0^*)^{-1} (B) \right) \neq d(\emptyset),
\]
and as in Theorem 3.10 (note \(\mu(D) = 1\)) we have \((J_\mu^*)^{-1} (B) = (R^*)^{-1} (B)\), so
\[
d \left( (R^*)^{-1} (B) \right) = d \left( (H_0^*)^{-1} (B) \right) \neq \emptyset.
\]
Let
\[
D_1 = \{ x \in \overline{U} : (x, (L + T)^{-1}(\Phi + T)(x)) \cap \Psi^*(x, t) \neq \emptyset \text{ for some } t \in [0, 1] \}
\]
where \(\Psi^*(x, \lambda) = (x, (L + T)^{-1}(\Psi + T)(x, \lambda))\). Notice \(D_1 \neq \emptyset\), \(D_1\) is compact and \(D_1 \cap \partial U = \emptyset\). Thus there exists a continuous map \(\mu : \overline{U} \to [0, 1]\) with \(\mu(\partial U) = 0\) and \(\mu(D_1) = 1\). Define \(G_\mu : \overline{U} \to 2^Y\) by \(G_\mu(x) = \Psi(x, \mu(x))\) and let \(G_\mu^* = I \times (L + T)^{-1}(G_\mu + T)\). Note \(G_\mu \in A_{U}(\overline{U}, Y; L, T)\) with \(G_\mu|_{\partial U} = H_0|_{\partial U}\) and \(G_\mu \cong H_0\) in \(A_{\partial U}(\overline{U}, Y; L, T)\) (to see this let \(Q^1 : \overline{U} \times [0, 1] \to 2^Y\) be given by \(Q^1(x, t) = \Psi(x, t\mu(x))\)). Also since \(H_0^*\) is d-L-\(\Phi\)-essential in \(A_{\partial U}(\overline{U}, Y; L, T)\) we have
\[
d \left( (G_\mu^*)^{-1} (B) \right) = d \left( (H_0^*)^{-1} (B) \right) \neq d(\emptyset),
\]
and as in Theorem 3.10 (note \(\mu(D) = 1\)) we have \((G_\mu^*)^{-1} (B) = (F^*)^{-1} (B)\), so
\[
d \left( (F^*)^{-1} (B) \right) = d \left( (H_0^*)^{-1} (B) \right) \neq \emptyset.
Now (3.13) and (3.14) yield \(d\left((F^*)^{-1}(B)\right) = d\left((R^*)^{-1}(B)\right) \neq \emptyset.\)

**Remark 3.16.** It is of interest to note if we consider maps \(R\) other than \(F\) in Theorem 3.15 and if we suppose

\[\simeq \text{ is an equivalence relation in } A_{\partial U}(\overline{U}, Y; L, T),\]

then (see in the statement of Theorem 3.15) if \(R \simeq F\) in \(A_{\partial U}(\overline{U}, Y; L, T)\) and \(R \simeq G\) in \(A_{\partial U}(\overline{U}, Y; L, T)\) then (since \(F \simeq G\) in \(A_{\partial U}(\overline{U}, Y; L, T)\)) there exists a map \(\Psi : \overline{U} \times [0, 1] \to 2^Y\) with \((L + T)^{-1}(\Psi(x, t)) + T(x)\) \(\in A(\overline{U}, E)\) for any continuous function \(\eta : [0, 1] \to [0, 1]\) with \(\eta(\partial U) = 0, (L + T)^{-1}(\Psi + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) = \emptyset\) for any \(x \in \partial U\) and \(t \in (0, 1),\) and

\[\{x \in \overline{U} : (x, (L + T)^{-1}(\Phi + T)(x)) \cap \Psi(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\}\]

is compact, \(\Psi_0 = G\) and \(\Psi_1 = F\) (here \(\Psi_t(x) = \Psi(x, t)\) and \(\Psi^*(x, \lambda) = (x, (L + T)^{-1}(\Phi + T)(x, \lambda))\)).

**Remark 3.17.** Suppose the following condition holds:

\[
\begin{cases}
\text{if } F, G \in A_{\partial U}(\overline{U}, Y; L, T) \text{ with } F|_{\partial U} = G|_{\partial U} \text{ and } F \simeq G \\
\text{in } A_{\partial U}(\overline{U}, Y; L, T) \text{ then } d\left((F^*)^{-1}(B)\right) = d\left((G^*)^{-1}(B)\right).
\end{cases}
\]

Then Definition 3.14 reduces to the following. Let \(F \in A_{\partial U}(\overline{U}, Y; L, T)\) with \(F^* = I \times (L + T)^{-1}(F + T)\). We say \(F^* : \overline{U} \to 2^{\overline{U} \times E}\) is \(d\)-\(L\)-\(\Phi\)-essential in \(A_{\partial U}(\overline{U}, Y; L, T)\) if \(d\left((F^*)^{-1}(B)\right) \neq d(\emptyset)\).

**Remark 3.18.** There is an analogue of Remark 3.11 (for normal topological vector spaces) in the statement of Theorem 3.15.

**ACKNOWLEDGEMENT**

The authors extend their appreciation to the International Scientific Partnership Program ISPP at King Saud University for funding this research work through ISPP No. 0027.

**REFERENCES**


