Dynamic Systems and Applications 25 (2016), 393-408

A HOMOTOPY APPROACH TO COINCIDENCE THEORY

MOHAMED JLELI, DONAL O'REGAN, AND BESSEM SAMET

Department of Mathematics, College of Science,
King Saud University, P.O. Box 2455, Riyadh, 11451, Saudi Arabia
jleli@ksu.edu.sa; bsamet@ksu.edu.sa
School of Mathematics, Statistics and Applied Mathematics
National University of Ireland, Galway, Ireland
donal.oregan@nuigalway.ie

ABSTRACT. We use homotopy type arguments to establish new coincidence theory for general classes of maps. Our theory is based on the notions of Φ-essential or d-Φ-essential maps.

AMS (MOS) Subject Classification. 47H04

Keywords: essential maps, coincidence points, homotopy.

1. INTRODUCTION

The notion of essential maps was introduced by Granas [3] and extended in the literature by many authors (see [2, 4, 5, 6, 7, 9] and the references therein). In Section 2, using the notions of homotopy and Φ-essential maps we establish a variety of coincidence theorems (in particular we show if G is Φ-essential and if natural conditions are assumed so $F \cong G$ then F is Φ-essential). In Section 3 we discuss d-Φ-essential maps.

2. Φ-ESSENTIAL MAPS

Let E be a completely regular topological space and U an open subset of E.

We consider classes **A** and **B** of maps.

Definition 2.1. We say $F \in A(\overline{U}, E)$ (respectively $F \in B(\overline{U}, E)$) if $F : \overline{U} \to 2^E$ and $F \in \mathbf{A}(\overline{U}, E)$ (respectively $F \in \mathbf{B}(\overline{U}, E)$); here 2^E denotes the family of nonempty subsets of E.

In this section we <u>fix</u> a $\Phi \in B(\overline{U}, E)$.

Definition 2.2. We say $F \in A_{\partial U}(\overline{U}, E)$ if $F \in A(\overline{U}, E)$ with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.

Received July 5, 2016

Definition 2.3. Let $F \in A_{\partial U}(\overline{U}, E)$. We say $F : \overline{U} \to 2^E$ is Φ-essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$.

Theorem 2.4. *Let E be a completely regular (respectively normal) topological space, U an open subset of E, $F \in A_{\partial U}(\overline{U}, E)$ and let $G \in A_{\partial U}(\overline{U}, E)$ be Φ-essential in $A_{\partial U}(\overline{U}, E)$. For any map $R \in A_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = F|_{\partial U}$ assume there exists a map $H^R : \overline{U} \times [0,1] \to 2^E$ with $H^R(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H^R_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$ and $\{x \in \overline{U} : \Phi(x) \cap H^R(x,t) \neq \emptyset$ for some $t \in [0,1]\}$ is compact (respectively closed) and $H^R_0 = G$, $H^R_1 = R$; here $H^R_t(x) = H^R(x,t)$. Then F is Φ-essential in $A_{\partial U}(\overline{U}, E)$.*

Proof. Let $R \in A_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = F|_{\partial U}$. We must show there exists a $x \in U$ with $R(x) \cap \Phi(x) \neq \emptyset$. Choose the map $H^R : \overline{U} \times [0,1] \to 2^E$ as in the statement of Theorem 2.4. Let

$$D = \left\{x \in \overline{U} : \Phi(x) \cap H^R(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}.$$

Note $D \neq \emptyset$ since $H^R_0 (= G)$ is Φ-essential in $A_{\partial U}(\overline{U}, E)$. Also D is compact (respectively closed) if E is a completely regular (respectively normal) topological space. Next note $D \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $J_\mu : \overline{U} \to 2^E$ by $J_\mu(x) = H^R(x, \mu(x)) = H^R_{\mu(x)}(x)$. Note $J_\mu \in A_{\partial U}(\overline{U}, E)$ with $J_\mu|_{\partial U} = H^R_0|_{\partial U}$. Now since H^R_0 is Φ-essential in $A_{\partial U}(\overline{U}, E)$ then there exists a $x \in U$ with $J_\mu(x) \cap \Phi(x) \neq \emptyset$ (i.e. $H^R_{\mu(x)}(x) \cap \Phi(x) \neq \emptyset$), and thus $x \in D$ so $\mu(x) = 1$ and as a result $\emptyset \neq H^R_1(x) \cap \Phi(x) = R(x) \cap \Phi(x)$. □

Remark 2.5. (i). In applications usually one puts conditions on the maps so that D in the proof of Theorem 2.4 is closed and $\overline{D}$ is compact (so as a result D is compact). However in the weak topology case one might need to work a little differently (the case we describe below occurs in applications [8]). Suppose E is a metrizable locally convex linear topological space. Note $E = (E, w)$, the space E endowed with the weak topology, is completely regular. Let $D \subseteq E$ and suppose D is weakly sequentially closed and $\overline{D^w}$ is weakly compact. Then D is weakly compact. To see this let $x \in \overline{D^w}$. Then the Eberlein–Šmulian theorem [1 pg. 549] guarantees that that there is a sequence (x_n) in D with $x_n \rightharpoonup x$ (here $\rightharpoonup$ denotes weak convergence). Now since D is weakly sequentially closed then $x \in D$, so $\overline{D^w} = D$, and D is weakly compact.

(ii). Suppose in the statement of Theorem 2.4 we have that E is a topological vector space and F and G are as in the statement of Theorem 2.4. Assume there exists a map $H : \overline{U} \times [0,1] \to 2^E$ with $H_0 = G$ and $H_1 = F$. Now for any map R with $R|_{\partial U} = F|_{\partial U}$ note

$$H^R(x,t) = \begin{cases} H(x, 2t), & t \in \left[0, \frac{1}{2}\right] \\ 2(1-t)F(x) + 2\left(t - \frac{1}{2}\right)R(x), & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

connects G to R (note as well for $x \in \partial U$ and $t \in \left[\frac{1}{2}, 1\right]$ that $H^R(x,t) = 2(1-t)F(x) + 2\left(t - \frac{1}{2}\right)F(x) = F(x)$).

It is possible to obtain an analogue result if we change Definition 2.3 as follows.

Definition 2.6. Let E be a completely regular (respectively normal) topological space, and U an open subset of E. Let $F, G \in A_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if there exists a map $H : \overline{U} \times [0,1] \to 2^E$ with $H(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $H_t(x) \cap \Phi(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, $H_1 = F$, $H_0 = G$ and $\left\{x \in \overline{U} : \Phi(x) \cap H(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}$ is compact (respectively closed); here $H_t(x) = H(x,t)$.

Definition 2.7. Let $F \in A_{\partial U}(\overline{U}, E)$. We say $F : \overline{U} \to 2^E$ is Φ-essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$.

Theorem 2.8. *Let E be a completely regular (respectively normal) topological space, U an open subset of E, $F \in A_{\partial U}(\overline{U}, E)$ and let $G \in A_{\partial U}(\overline{U}, E)$ be Φ-essential in $A_{\partial U}(\overline{U}, E)$. For any map $R \in A_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = F|_{\partial U}$ and $R \cong F$ in $A_{\partial U}(\overline{U}, E)$ assume there exists a map $H : \overline{U} \times [0,1] \to 2^E$ with $H(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$ and*

$$\left\{x \in \overline{U} : \Phi(x) \cap H(x,t) \neq \emptyset \textit{ for some } t \in [0,1]\right\}$$

is compact (respectively closed), $H_0 = G$, $H_1 = R$ (here $H_t(x) = H(x,t)$) and for any continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ assume

$$\left\{x \in \overline{U} : \emptyset \neq \Phi(x) \cap H(x, t\mu(x)) \textit{ for some } t \in [0,1]\right\}$$

is closed. Then F is Φ-essential in $A_{\partial U}(\overline{U}, E)$.

Proof. Let $R \in A_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = F|_{\partial U}$ and $R \cong F$ in $A_{\partial U}(\overline{U}, E)$. We must show there exists a $x \in U$ with $R(x) \cap \Phi(x) \neq \emptyset$. Choose the map $H : \overline{U} \times [0,1] \to 2^E$ as in the statement of Theorem 2.8. Let

$$D = \left\{x \in \overline{U} : \Phi(x) \cap H(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}.$$

Note $D \neq \emptyset$. Also D is compact (respectively closed) if E is a completely regular (respectively normal) topological space. Next note $D \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $J_\mu : \overline{U} \to 2^E$ by $J_\mu(x) = H(x, \mu(x)) = H_{\mu(x)}(x)$. Note $J_\mu \in A_{\partial U}(\overline{U}, E)$ with $J_\mu|_{\partial U} = H_0|_{\partial U}$. We now claim

$$J_\mu \cong H_0 \text{ in } A_{\partial U}(\overline{U}, E). \tag{2.1}$$

If the claim is true then since H_0 is Φ-essential in $A_{\partial U}(\overline{U}, E)$ then there exists a $x \in U$ with $J_\mu(x) \cap \Phi(x) \neq \emptyset$ (i.e. $H_{\mu(x)}(x) \cap \Phi(x) \neq \emptyset$), and thus $x \in D$ so $\mu(x) = 1$ and as a result $\emptyset \neq H_1(x) \cap \Phi(x) = R(x) \cap \Phi(x)$.

It remains to show (2.1). Let $Q : \overline{U} \times [0,1] \to 2^E$ be given by $Q(x,t) = H(x, t\mu(x))$. Note $Q(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$ and

$$\left\{x \in \overline{U} : \emptyset \neq \Phi(x) \cap Q(x,t) = \Phi(x) \cap H(x, t\mu(x)) \text{ for some } t \in [0,1]\right\}$$

is compact (respectively closed). Note $Q_0 = H_0$ and $Q_1 = J_\mu$. Finally if there exists a $t \in [0,1]$ and $x \in \partial U$ with $\Phi(x) \cap Q_t(x) \neq \emptyset$ then $\Phi(x) \cap H_{t\mu(x)}(x) \neq \emptyset$ so $x \in D$ and so $\mu(x) = 1$ i.e. $\Phi(x) \cap H_t(x) \neq \emptyset$, a contradiction. Thus (2.1) holds. □

Remark 2.9. In Theorem 2.8 note one example of a map R is F itself. It is of interest to note if we consider maps R other than F and if we suppose

$$\cong \text{ is an equivalence relation in } A_{\partial U}(\overline{U}, E),$$

then (see in the statement of Theorem 2.8) if $R \cong F$ in $A_{\partial U}(\overline{U}, E)$ and $R \cong G$ in $A_{\partial U}(\overline{U}, E)$ then $F \cong G$ in $A_{\partial U}(\overline{U}, E)$.

We now show that the ideas in this section can be applied to other natural situations. Let E be a Hausdorff topological vector space (so automatically a completely regular space), Y a topological vector space, and U an open subset of E. Also let $L : domL \subseteq E \to Y$ be a linear single valued map; here $domL$ is a vector subspace of E. Finally $T : E \to Y$ will be a linear single valued map with $L + T : domL \to Y$ a bijection; for convenience we say $T \in H_L(E, Y)$.

Definition 2.10. We say $F \in A(\overline{U}, Y; L, T)$ (respectively $F \in B(\overline{U}, Y; L, T)$) if $F : \overline{U} \to 2^Y$ and $(L+T)^{-1}(F+T) \in A(\overline{U}, E)$ (respectively $(L+T)^{-1}(F+T) \in B(\overline{U}, E)$).

We now <u>fix</u> a $\Phi \in B(\overline{U}, Y; L, T)$.

Definition 2.11. We say $F \in A_{\partial U}(\overline{U}, Y; L, T)$ if $F \in A(\overline{U}, Y; L, T)$ with $(L + T)^{-1}(F + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) = \emptyset$ for $x \in \partial U$.

Definition 2.12. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$. F is L-Φ-essential in $A_{\partial U}(\overline{U}, Y; L, T)$ if for every map $J \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $(L + T)^{-1}(J + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) \neq \emptyset$.

Theorem 2.13. *Let E be a topological vector space (so automatically completely regular), Y a topological vector space, U an open subset of E, $L : domL \subseteq E \to Y$ a linear single valued map and $T \in H_L(E, Y)$. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$ and let $G \in A_{\partial U}(\overline{U}, Y; L, T)$ be L-Φ-essential in $A_{\partial U}(\overline{U}, Y; L, T)$. For any map $R \in A_{\partial U}(\overline{U}, Y; L, T)$ with $R|_{\partial U} = F|_{\partial U}$ assume there exists a map $H : \overline{U} \times [0,1] \to$*

2^Y with $(L+T)^{-1}(H(\cdot,\eta(\cdot))+T(\cdot)) \in A(\overline{U},E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U)=0$, $(L+T)^{-1}(H_t+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$, $H_0 = G$, $H_1 = R$ (here $H_t(x) = H(x,t)$) and $\{x \in \overline{U} : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) \neq \emptyset$ for some $t \in [0,1]\}$ is compact. Then F is L-Φ-essential in $A_{\partial U}(\overline{U},Y;L,T)$.

Proof. Let $R \in A_{\partial U}(\overline{U},Y;L,T)$ with $R|_{\partial U} = F|_{\partial U}$. We must show there exists $x \in U$ with $(L+T)^{-1}(R+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$. Choose the map $H : \overline{U} \times [0,1] \to 2^Y$ as in the statement of Theorem 2.13. Let

$$D = \{x \in \overline{U} : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\}.$$

Note $D \neq \emptyset$, D is compact, $D \cap \partial U = \emptyset$ so there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $J_\mu : \overline{U} \to 2^Y$ by $J_\mu(x) = H(x,\mu(x))$. Note $J_\mu \in A_{\partial U}(\overline{U},Y;L,T)$ and $J_\mu|_{\partial U} = H_0|_{\partial U}$. Now since $H_0 (= G)$ is L-Φ-essential in $A_{\partial U}(\overline{U},Y;L,T)$ there exists $x \in U$ with $(L+T)^{-1}(J_\mu+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$ (i.e. $(L+T)^{-1}(H_{\mu(x)}+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$), and thus $x \in D$ so $\mu(x) = 1$ and we are finished. □

Remark 2.14. If E is a normal topological vector space then the assumption that D (in the proof of Theorem 2.13) is compact, can be replaced by D is closed, in the statement (and proof) of Theorem 2.13.

It is possible to obtain an analogue result if we change Definition 2.12 as follows.

Definition 2.15. Let $F, G \in A_{\partial U}(\overline{U},Y;L,T)$. We say $F \cong G$ in $A_{\partial U}(\overline{U},Y;L,T)$ if there exists a map $H : \overline{U} \times [0,1] \to 2^Y$ with $(L+T)^{-1}(H(\cdot,\eta(\cdot))+T(\cdot)) \in A(\overline{U},E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $(L+T)^{-1}(H_t+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, $H_1 = F$, $H_0 = G$ and

$$\{x \in \overline{U} : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\}$$

is compact; here $H_t(x) = H(x,t)$.

Remark 2.16. If E is a normal topological vector space then the assumption that

$$\{x \in \overline{U} : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\}$$

is compact, can be replaced by

$$\{x \in \overline{U} : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\}$$

is closed, in Definition 2.15.

Definition 2.17. Let $F \in A_{\partial U}(\overline{U},Y;L,T)$. F is L-Φ-essential in $A_{\partial U}(\overline{U},Y;L,T)$ if for every map $J \in A_{\partial U}(\overline{U},Y;L,T)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U},Y;L,T)$ there exists $x \in U$ with $(L+T)^{-1}(J+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$.

Theorem 2.18. *Let E be a topological vector space, Y a topological vector space, U an open subset of E, $L : domL \subseteq E \to Y$ a linear single valued map and $T \in H_L(E,Y)$. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$ and let $G \in A_{\partial U}(\overline{U}, Y; L, T)$ be L-Φ-essential in $A_{\partial U}(\overline{U}, Y; L, T)$. For any map $R \in A_{\partial U}(\overline{U}, Y; L, T)$ with $R|_{\partial U} = F|_{\partial U}$ and $R \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ assume there exists a map $H : \overline{U} \times [0,1] \to 2^Y$ with $(L+T)^{-1}(H(\cdot, \eta(\cdot)) + T(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $(L+T)^{-1}(H_t + T)(x) \cap (L+T)^{-1}(\Phi + T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$, $H_0 = G$, $H_1 = R$ (here $H_t(x) = H(x,t)$) and $\{x \in \overline{U} : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\}$ is compact and for any continuous function $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ assume*

$$\left\{x \in \overline{U} : (L+T)^{-1}(\Phi + T)(x) \cap (L+T)^{-1}(H_{t\mu(x)} + T)(x) \neq \emptyset \text{ for some } t \in [0,1]\right\}$$

is closed. Then F is L-Φ-essential in $A_{\partial U}(\overline{U}, Y; L, T)$.

Proof. Let $R \in A_{\partial U}(\overline{U}, Y; L, T)$ with $R|_{\partial U} = F|_{\partial U}$ and $R \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$. Choose the map $H : \overline{U} \times [0,1] \to 2^Y$ as in the statement of Theorem 2.18. Let

$$D = \{x \in \overline{U} : (L+T)^{-1}(\Phi + T)(x) \cap (L+T)^{-1}(H_t + T)(x) \neq \emptyset \text{ for some } t \in [0,1]\}.$$

Note $D \neq \emptyset$, D is compact, $D \cap \partial U = \emptyset$ so there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $J_\mu : \overline{U} \to 2^Y$ by $J_\mu(x) = H(x, \mu(x))$. Note $J_\mu \in A_{\partial U}(\overline{U}, Y; L, T)$ and $J_\mu|_{\partial U} = H_0|_{\partial U}$. Also note $J_\mu \cong H_0$ in $A_{\partial U}(\overline{U}, Y; L, T)$ (to see this let $Q : \overline{U} \times [0,1] \to 2^Y$ be given by $Q(x,t) = H(x, t\mu(x))$). Now since H_0 is L-Φ-essential in $A_{\partial U}(\overline{U}, Y; L, T)$ there exists $x \in U$ with $(L+T)^{-1}(J_\mu + T)(x) \cap (L+T)^{-1}(\Phi + T)(x) \neq \emptyset$, and thus $x \in D$ so $\mu(x) = 1$ and we are finished. □

Remark 2.19. It is of interest to note if we consider maps R other than F in Theorem 2.18 and if we suppose

$$\cong \text{ is an equivalence relation in } A_{\partial U}(\overline{U}, Y; L, T),$$

then (see in the statement of Theorem 2.18) if $R \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ and $R \cong G$ in $A_{\partial U}(\overline{U}, Y; L, T)$ then $F \cong G$ in $A_{\partial U}(\overline{U}, Y; L, T)$.

Remark 2.20. There is an analogue of Remark 2.14 (for normal topological vector spaces) in the statement of Theorem 2.18.

3. d-Φ-ESSENTIAL MAPS

Let E be a completely regular topological space and U an open subset of E. We will consider the classes **A**, **B**, A and B of maps as in Section 2.

In this section we fix a $\Phi \in B(\overline{U}, E)$.

For any map $F \in A(\overline{U}, E)$ let $F^\star = I \times F : \overline{U} \to 2^{\overline{U} \times E}$, with $I : \overline{U} \to \overline{U}$ given by $I(x) = x$, and let

$$d : \left\{ (F^\star)^{-1} (B) \right\} \cup \{\emptyset\} \to \Omega \tag{3.1}$$

be any map with values in the nonempty set Ω; here $B = \left\{ (x, \Phi(x)) : x \in \overline{U} \right\}$.

Definition 3.1. Let $F \in A_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$. We say $F^\star : \overline{U} \to 2^{\overline{U} \times E}$ is d-Φ-essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J^\star = I \times J$ and $J|_{\partial U} = F|_{\partial U}$ we have that $d\left((F^\star)^{-1} (B) \right) = d\left((J^\star)^{-1} (B) \right) \neq d(\emptyset)$.

Remark 3.2. If $F^\star$ is d-Φ-essential in $A_{\partial U}(\overline{U}, E)$ then

$$\begin{aligned} \emptyset \neq (F^\star)^{-1} (B) &= \{x \in \overline{U} : F^\star(x) \cap B \neq \emptyset\} \\ &= \{x \in \overline{U} : (x, F(x)) \cap (x, \Phi(x)) \neq \emptyset\}, \end{aligned}$$

and this together with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$ implies that there exists $x \in U$ with $(x, \Phi(x)) \cap F^\star(x) \neq \emptyset$ (i.e. $\Phi(x) \cap F(x) \neq \emptyset$).

Theorem 3.3. *Let E be a completely regular (respectively normal) topological space, U an open subset of E, $B = \{(x, \Phi(x)) : x \in \overline{U}\}$, d a map defined in (3.1) , $F \in A_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$, $G \in A_{\partial U}(\overline{U}, E)$ and $G^\star$ is d-Φ-essential in $A_{\partial U}(\overline{U}, E)$ (here $G^\star = I \times G$). For any map $R \in A_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = F|_{\partial U}$ assume there exists a map $H : \overline{U} \times [0, 1] \to 2^E$ with $H(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0, 1)$, and $\left\{ x \in \overline{U} : (x, \Phi(x)) \cap H^\star(x, t) \neq \emptyset \text{ for some } t \in [0, 1] \right\}$ is compact (respectively closed), $H_0 = G$, $H_1 = R$ (here $H_t(x) = H(x, t)$ and $H^\star(x, t) = (x, H(x, t))$) and also suppose there exists a map $\Psi : \overline{U} \times [0, 1] \to 2^E$ with $\Psi(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap \Psi_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0, 1)$, $\Psi_0 = H_0 (= G)$, $\Psi_1 = F$ and $\left\{ x \in \overline{U} : (x, \Phi(x)) \cap \Psi^\star(x, t) \neq \emptyset \text{ for some } t \in [0, 1] \right\}$ is compact (respectively closed); here $\Psi_t(x) = \Psi(x, t)$ and $\Psi^\star(x, t) = (x, \Psi(x, t))$. Then $F^\star$ is d-Φ-essential in $A_{\partial U}(\overline{U}, E)$.*

Proof. Let $R \in A_{\partial U}(\overline{U}, E)$ with $R^\star = I \times R$ and $R|_{\partial U} = F|_{\partial U}$. We must show $d\left((F^\star)^{-1} (B) \right) = d\left((R^\star)^{-1} (B) \right) \neq d(\emptyset)$. Choose the maps $H : \overline{U} \times [0, 1] \to 2^E$ and $\Psi : \overline{U} \times [0, 1] \to 2^E$ as in the statement of Theorem 3.3. Let

$$D = \left\{ x \in \overline{U} : (x, \Phi(x)) \cap H^\star(x, t) \neq \emptyset \text{ for some } t \in [0, 1] \right\}$$

where $H^\star(x, t) = (x, H(x, t))$. Notice $D \neq \emptyset$ since $H_0^\star$ is d-Φ-essential. Also D is compact (respectively closed) if E is a completely regular (respectively normal) topological space and $D \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $J_\mu : \overline{U} \to 2^E$ by $J_\mu(x) = H(x, \mu(x))$ and let

$J_\mu^\star = I \times J_\mu$. Note $J_\mu \in A_{\partial U}(\overline{U}, E)$ with $J_\mu|_{\partial U} = H_0|_{\partial U}$. Now since $H_0^\star$ is d-Φ-essential in $A_{\partial U}(\overline{U}, E)$ then

$$d\left(\left(J_\mu^\star\right)^{-1}(B)\right) = d\left(\left(H_0^\star\right)^{-1}(B)\right) \neq d(\emptyset). \tag{3.2}$$

Next we note since $\mu(D) = 1$ that

$$\begin{aligned} \left(J_\mu^\star\right)^{-1}(B) &= \left\{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, \mu(x)) \neq \emptyset\right\} \\ &= \left\{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, 1) \neq \emptyset\right\} = \left(R^\star\right)^{-1}(B), \end{aligned}$$

so from (3.2) we have

$$d\left(\left(R^\star\right)^{-1}(B)\right) = d\left(\left(H_0^\star\right)^{-1}(B)\right) \neq d(\emptyset). \tag{3.3}$$

Let

$$D_1 = \left\{x \in \overline{U} : (x, \Phi(x)) \cap \Psi^\star(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\right\}$$

where $\Psi^\star(x, t) = (x, \Psi(x, t))$. Notice $D_1 \neq \emptyset$ since $\Psi_0 = H_0$ and $H_0^\star$ is d-Φ-essential. Also D_1 is compact (respectively closed) if E is a completely regular (respectively normal) topological space and $D_1 \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(D_1) = 1$. Define $G_\mu : \overline{U} \to 2^E$ by $G_\mu(x) = \Psi(x, \mu(x))$ and let $G_\mu^\star = I \times G_\mu$. Note $G_\mu \in A_{\partial U}(\overline{U}, E)$ with $G_\mu|_{\partial U} = H_0|_{\partial U}$. Now since $H_0^\star$ is d-Φ-essential in $A_{\partial U}(\overline{U}, E)$ then

$$d\left(\left(G_\mu^\star\right)^{-1}(B)\right) = d\left(\left(H_0^\star\right)^{-1}(B)\right) \neq d(\emptyset). \tag{3.4}$$

Next we note since $\mu(D_1) = 1$ that

$$\begin{aligned} \left(G_\mu^\star\right)^{-1}(B) &= \left\{x \in \overline{U} : (x, \Phi(x)) \cap (x, \Psi(x, \mu(x)) \neq \emptyset\right\} \\ &= \left\{x \in \overline{U} : (x, \Phi(x)) \cap (x, \Psi(x, 1) \neq \emptyset\right\} = \left(F^\star\right)^{-1}(B), \end{aligned}$$

so from (3.4) we have

$$d\left(\left(F^\star\right)^{-1}(B)\right) = d\left(\left(H_0^\star\right)^{-1}(B)\right) \neq d(\emptyset). \tag{3.5}$$

Now (3.3) and (3.5) yield $d\left(\left(F^\star\right)^{-1}(B)\right) = d\left(\left(R^\star\right)^{-1}(B)\right) \neq d(\emptyset)$. □

It is possible to obtain an analogue result if we change Definition 3.1 as follows.

Definition 3.4. Let E be a completely regular (respectively normal) topological space, and U an open subset of E. Let $F, G \in A_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if there exists a map $H : \overline{U} \times [0, 1] \to 2^E$ with $H(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$, $H_t(x) \cap \Phi(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0, 1]$, $H_1 = F$, $H_0 = G$ and $\left\{x \in \overline{U} : (x, \Phi(x)) \cap H^\star(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\right\}$ is compact (respectively closed); here $H^\star(x, t) = (x, H(x, t))$ and $H_t(x) = H(x, t)$.

Definition 3.5. Let $F \in A_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$. We say $F^\star : \overline{U} \to 2^{\overline{U} \times E}$ is d-Φ-essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J^\star = I \times J$ and $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ we have that $d\left(\left(F^\star\right)^{-1}(B)\right) = d\left(\left(J^\star\right)^{-1}(B)\right) \neq d(\emptyset)$.

Theorem 3.6. *Let E be a completely regular (respectively normal) topological space, U an open subset of E, $B = \{(x, \Phi(x)) : x \in \overline{U}\}$, d a map defined in (3.1) , $F \in A_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$, $G \in A_{\partial U}(\overline{U}, E)$ and $G^\star$ is d-Φ-essential in $A_{\partial U}(\overline{U}, E)$ (here $G^\star = I \times G$). For any map $R \in A_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = F|_{\partial U}$ and $R \cong F$ in $A_{\partial U}(\overline{U}, E)$ assume there exists a map $H : \overline{U} \times [0, 1] \to 2^E$ with $H(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0, 1)$, and*

$$\left\{x \in \overline{U} : (x, \Phi(x)) \cap H^\star(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\right\}$$

is compact (respectively closed), $H_0 = G$, $H_1 = R$ (here $H_t(x) = H(x, t)$ and $H^\star(x, t) = (x, H(x, t))$) and for any continuous function $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ assume

$$\left\{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, t\mu(x))) \neq \emptyset \text{ for some } t \in [0, 1]\right\}$$

is closed and also suppose there exists a map $\Psi : \overline{U} \times [0, 1] \to 2^E$ with $\Psi(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap \Psi_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0, 1)$, and

$$\left\{x \in \overline{U} : (x, \Phi(x)) \cap \Psi^\star(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\right\}$$

is compact (respectively closed), $\Psi_0 = H_0 (= G)$, $\Psi_1 = F$ (here $\Psi_t(x) = \Psi(x, t)$ and $\Psi^\star(x, t) = (x, \Psi(x, t))$) and for any continuous function $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ assume

$$\left\{x \in \overline{U} : (x, \Phi(x)) \cap (x, \Psi(x, t\mu(x))) \neq \emptyset \text{ for some } t \in [0, 1]\right\}$$

is closed. Then $F^\star$ is d-Φ-essential in $A_{\partial U}(\overline{U}, E)$.

Proof. Let $R \in A_{\partial U}(\overline{U}, E)$ with $R^\star = I \times R$ and $R|_{\partial U} = F|_{\partial U}$ and $R \cong F$ in $A_{\partial U}(\overline{U}, E)$. We must show $d\left((F^\star)^{-1}(B)\right) = d\left((R^\star)^{-1}(B)\right) \neq d(\emptyset)$. Choose the maps $H : \overline{U} \times [0, 1] \to 2^E$ and $\Psi : \overline{U} \times [0, 1] \to 2^E$ as in the statement of Theorem 3.6. Let

$$D = \left\{x \in \overline{U} : (x, \Phi(x)) \cap H^\star(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\right\}$$

where $H^\star(x, t) = (x, H(x, t))$. Notice $D \neq \emptyset$. Also D is compact (respectively closed) if E is a completely regular (respectively normal) topological space and $D \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $J_\mu : \overline{U} \to 2^E$ by $J_\mu(x) = H(x, \mu(x))$ and let $J_\mu^\star = I \times J_\mu$. Note $J_\mu \in A_{\partial U}(\overline{U}, E)$ with $J_\mu|_{\partial U} = H_0|_{\partial U}$. Also note $J_\mu \cong H_0$ in $A_{\partial U}(\overline{U}, E)$ (to see this let $Q : \overline{U} \times [0, 1] \to 2^E$ be given by $Q(x, t) = H(x, t\mu(x))$). Now since $H_0^\star$ is d-Φ-essential in $A_{\partial U}(\overline{U}, E)$ then

$$d\left(\left(J_\mu^\star\right)^{-1}(B)\right) = d\left((H_0^\star)^{-1}(B)\right) \neq d(\emptyset)$$

and as in Theorem 3.3 (note $\mu(D)=1$) we have $\left(J_\mu^\star\right)^{-1}(B)=(R^\star)^{-1}(B)$, so

$$d\left((R^\star)^{-1}(B)\right)=d\left((H_0^\star)^{-1}(B)\right)\neq d(\emptyset). \tag{3.6}$$

Let

$$D_1=\left\{x\in\overline{U}:(x,\Phi(x))\cap\Psi^\star(x,t)\neq\emptyset \text{ for some } t\in[0,1]\right\}$$

where $\Psi^\star(x,t)=(x,\Psi(x,t))$. Notice $D_1\neq\emptyset$. Also D_1 is compact (respectively closed) if E is a completely regular (respectively normal) topological space and $D_1\cap\partial U=\emptyset$. Thus there exists a continuous map $\mu:\overline{U}\to[0,1]$ with $\mu(\partial U)=0$ and $\mu(D_1)=1$. Define $G_\mu:\overline{U}\to 2^E$ by $G_\mu(x)=\Psi(x,\mu(x))$ and let $G_\mu^\star=I\times G_\mu$. Note $G_\mu\in A_{\partial U}(\overline{U},E)$ with $G_\mu|_{\partial U}=H_0|_{\partial U}$. Also note $G_\mu\cong H_0(=G)$ in $A_{\partial U}(\overline{U},E)$ (to see this let $Q^1:\overline{U}\times[0,1]\to 2^E$ be given by $Q^1(x,t)=\Psi(x,t\mu(x))$). Now since $H_0^\star$ is d-Φ-essential in $A_{\partial U}(\overline{U},E)$ then

$$d\left(\left(G_\mu^\star\right)^{-1}(B)\right)=d\left((H_0^\star)^{-1}(B)\right)\neq d(\emptyset)$$

and as in Theorem 3.3 (note $\mu(D)=1$) we have $\left(G_\mu^\star\right)^{-1}(B)=(F^\star)^{-1}(B)$, so

$$d\left((F^\star)^{-1}(B)\right)=d\left((H_0^\star)^{-1}(B)\right)\neq d(\emptyset). \tag{3.7}$$

Now (3.6) and (3.7) yield $d\left((F^\star)^{-1}(B)\right)=d\left((R^\star)^{-1}(B)\right)\neq d(\emptyset)$. □

Remark 3.7. It is of interest to note if we consider maps R other than F in Theorem 3.6 and if we suppose

$$\cong \text{ is an equivalence relation in } A_{\partial U}(\overline{U},E),$$

then (see in the statement of Theorem 3.6) if $R\cong F$ in $A_{\partial U}(\overline{U},E)$ and $R\cong G$ in $A_{\partial U}(\overline{U},E)$ then (since $F\cong G$ in $A_{\partial U}(\overline{U},E)$) there exists a map $\Psi:\overline{U}\times[0,1]\to 2^E$ with $\Psi(\cdot,\eta(\cdot))\in A(\overline{U},E)$ for any continuous function $\eta:\overline{U}\to[0,1]$ with $\eta(\partial U)=0$, $\Phi(x)\cap\Psi_t(x)=\emptyset$ for any $x\in\partial U$ and $t\in(0,1)$, and $\left\{x\in\overline{U}:(x,\Phi(x))\cap\Psi^\star(x,t)\neq\emptyset\right.$ for some $\left.t\in[0,1]\right\}$ is compact (respectively closed), $\Psi_0=G$ and $\Psi_1=F$ (here $\Psi_t(x)=\Psi(x,t)$ and $\Psi^\star(x,t)=(x,\Psi(x,t))$)

Remark 3.8. Suppose the following conditions holds (which is common in the literature on topological degree):

$$\begin{cases} \text{if } F,G\in A_{\partial U}(\overline{U},E) \text{ with } F|_{\partial U}=G|_{\partial U} \text{and } F\cong G \\ \text{in } A_{\partial U}(\overline{U},E) \text{ then } d\left((F^\star)^{-1}(B)\right)=d\left((G^\star)^{-1}(B)\right). \end{cases}$$

Then Definition 3.5 reduces to the following. Let $F\in A_{\partial U}(\overline{U},E)$ with $F^\star=I\times F$. We say $F^\star:\overline{U}\to 2^{\overline{U}\times E}$ is d-Φ-essential in $A_{\partial U}(\overline{U},E)$ if $d\left((F^\star)^{-1}(B)\right)\neq d(\emptyset)$.

Let E be a topological vector space, Y a topological vector space, U an open subset of E, $L:domL\subseteq E\to Y$ a linear single valued map and $T\in H_L(E,Y)$.

We now <u>fix</u> a $\Phi\in B(\overline{U},Y;L,T)$.

For any map $F \in A(\overline{U}, Y; L, T)$ let $F^\star = I \times (L+T)^{-1}(F+T) : \overline{U} \to 2^{\overline{U}\times E}$, with $I : \overline{U} \to \overline{U}$ given by $I(x) = x$, and let

$$d : \left\{ (F^\star)^{-1}(B) \right\} \cup \{\emptyset\} \to \Omega \tag{3.8}$$

be any map with values in the nonempty set Ω; here

$$B = \left\{ (x, (L+T)^{-1}(\Phi + T)(x)) : x \in \overline{U} \right\}.$$

Definition 3.9. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$ with $F^\star = I \times (L+T)^{-1}(F+T)$. We say $F^\star : \overline{U} \to 2^{\overline{U}\times E}$ is d-L-Φ-essential in $A_{\partial U}(\overline{U}, Y; L, T)$ if for every map $J \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J^\star = I \times (L+T)^{-1}(J+T)$ and $J|_{\partial U} = F|_{\partial U}$ we have that $d\left((F^\star)^{-1}(B) \right) = d\left((J^\star)^{-1}(B) \right) \neq d(\emptyset)$.

Theorem 3.10. *Let E be a Hausdorff topological vector space, Y a topological vector space, U an open subset of E, $B = \left\{ (x, (L+T)^{-1}(\Phi+T)(x)) : x \in \overline{U} \right\}$, $L : domL \subseteq E \to Y$ a linear single valued map, $T \in H_L(E, Y)$, d a map defined in* (3.8)*, $F \in A_{\partial U}(\overline{U}, Y; L, T)$ with $F^\star = I \times (L+T)^{-1}(F+T)$, $G \in A_{\partial U}(\overline{U}, Y; L, T)$ and $G^\star$ is d-L-Φ-essential in $A_{\partial U}(\overline{U}, Y; L, T)$ (here $G^\star = I \times (L+T)^{-1}(G+T)$). For any map $R \in A_{\partial U}(\overline{U}, Y; L, T)$ with $R^\star = I \times (L+T)^{-1}(R+T)$ and $R|_{\partial U} = F|_{\partial U}$ assume there exists a map $H : \overline{U} \times [0,1] \to 2^Y$ with $(L+T)^{-1}(H(\cdot, \eta(\cdot)) + T(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $(L+T)^{-1}(H_t + T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$, and*

$$\left\{ x \in \overline{U} : (x, (L+T)^{-1}(\Phi+T)(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1] \right\}$$

is compact, $H_0 = G$, $H_1 = R$ (here $H_t(x) = H(x,t)$ and $H^\star(x,\lambda) = (x, (L+T)^{-1}(H+T)(x,\lambda))$) and also suppose there exists a map $\Psi : \overline{U} \times [0,1] \to 2^Y$ with $(L+T)^{-1}(\Psi(\cdot, \eta(\cdot)) + T(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $(L+T)^{-1}(\Psi_t + T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$, and

$$\left\{ x \in \overline{U} : (x, (L+T)^{-1}(\Phi+T)(x)) \cap \Psi^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1] \right\}$$

is compact, $\Psi_0 = H_0$ and $\Psi_1 = F$ (here $\Psi_t(x) = \Psi(x,t)$ and $\Psi^\star(x,\lambda) = (x, (L+T)^{-1}(\Psi+T)(x,\lambda))$). Then $F^\star$ is d-L-Φ-essential in $A_{\partial U}(\overline{U}, Y; L, T)$.

Proof. Let $R \in A_{\partial U}(\overline{U}, Y; L, T)$ with $R^\star = I \times (L+T)^{-1}(R+T)$ and $R|_{\partial U} = F|_{\partial U}$. Choose the maps $H : \overline{U} \times [0,1] \to 2^Y$ and $\Psi : \overline{U} \times [0,1] \to 2^Y$ as in the statement of Theorem 3.10. Let

$$D = \left\{ x \in \overline{U} : (x, (L+T)^{-1}(\Phi+T)(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1] \right\}$$

where $H^\star(x,\lambda) = (x, (L+T)^{-1}(H+T)(x,\lambda))$. Notice $D \neq \emptyset$, D is compact and $D \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $J_\mu : \overline{U} \to 2^Y$ by $J_\mu(x) = H(x, \mu(x)) = H_{\mu(x)}(x)$ and let

$J_\mu^\star = I \times (L+T)^{-1}(J_\mu + T)$. Note $J_\mu \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J_\mu|_{\partial U} = H_0|_{\partial U}$. Also since $H_0^\star$ is d-L-Φ-essential in $A_{\partial U}(\overline{U}, Y; L, T)$ we have

$$d\left(\left(J_\mu^\star\right)^{-1}(B)\right) = d\left(\left(H_0^\star\right)^{-1}(B)\right) \neq d(\emptyset). \tag{3.9}$$

Next we note since $\mu(D) = 1$ that

$$\begin{aligned}
\left(J_\mu^\star\right)^{-1}(B) &= \{x \in \overline{U} : (x, (L+T)^{-1}(\Phi + T)(x)) \\
&\quad \cap (x, (L+T)^{-1}(H_{\mu(x)} + T)(x)) \neq \emptyset\} \\
&= \{x \in \overline{U} : (x, (L+T)^{-1}(\Phi + T)(x)) \\
&\quad \cap (x, (L+T)^{-1}(H_1 + T)(x)) \neq \emptyset\} \\
&= (R^\star)^{-1}(B),
\end{aligned}$$

so from (3.9) we have

$$d\left((R^\star)^{-1}(B)\right) = d\left((H_1^\star)^{-1}(B)\right) \neq \emptyset. \tag{3.10}$$

Let

$$D_1 = \left\{x \in \overline{U} : (x, (L+T)^{-1}(\Phi + T)(x)) \cap \Psi^\star(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\right\}$$

where $\Psi^\star(x, \lambda) = (x, (L+T)^{-1}(\Psi + T)(x, \lambda))$. Notice $D_1 \neq \emptyset$, D_1 is compact and $D_1 \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(D_1) = 1$. Define $G_\mu : \overline{U} \to 2^Y$ by $G_\mu(x) = \Psi(x, \mu(x))$ and let $G_\mu^\star = I \times (L + T)^{-1}(G_\mu + T)$. Note $G_\mu \in A_{\partial U}(\overline{U}, Y; L, T)$ with $G_\mu|_{\partial U} = H_0|_{\partial U}$. Also since $H_0^\star$ is d-L-Φ-essential in $A_{\partial U}(\overline{U}, Y; L, T)$ we have

$$d\left(\left(G_\mu^\star\right)^{-1}(B)\right) = d\left(\left(H_0^\star\right)^{-1}(B)\right) \neq d(\emptyset). \tag{3.11}$$

Next we note since $\mu(D_1) = 1$ that

$$\begin{aligned}
\left(G_\mu^\star\right)^{-1}(B) &= \{x \in \overline{U} : (x, (L+T)^{-1}(\Phi + T)(x)) \\
&\quad \cap (x, (L+T)^{-1}(\Psi_{\mu(x)} + T)(x)) \neq \emptyset\} \\
&= \{x \in \overline{U} : (x, (L+T)^{-1}(\Phi + T)(x)) \\
&\quad \cap (x, (L+T)^{-1}(\Psi_1 + T)(x)) \neq \emptyset\} \\
&= (F^\star)^{-1}(B),
\end{aligned}$$

so from (3.11) we have

$$d\left((F^\star)^{-1}(B)\right) = d\left((H_0^\star)^{-1}(B)\right) \neq d(\emptyset). \tag{3.12}$$

Now (3.10) and (3.12) yield $d\left((F^\star)^{-1}(B)\right) = d\left((R^\star)^{-1}(B)\right) \neq \emptyset$. □

Remark 3.11. If E is a normal topological vector space then the assumption that D (in the proof of Theorem 3.10) is compact, can be replaced by D is closed, in the statement (and proof) of Theorem 3.10.

It is possible to obtain an analogue result if we change Definition 3.9 as follows.

Definition 3.12. Let $F, G \in A_{\partial U}(\overline{U}, Y; L, T)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, Y; L, T)$ if there exists a map $H : \overline{U} \times [0,1] \to 2^Y$ with $(L+T)^{-1}(H(\cdot, \eta(\cdot)) + T(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $(L+T)^{-1}(H_t + T)(x) \cap (L+T)^{-1}(\Phi + T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, $H_1 = F$, $H_0 = G$ and

$$\left\{x \in \overline{U} : (x, (L+T)^{-1}(\Phi + T)(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}$$

is compact; here $H_t(x) = H(x,t)$ and $H^\star(x,\lambda) = (x, (L+T)^{-1}(H+T)(x,\lambda))$.

Remark 3.13. If E is a normal topological vector space then the assumption that

$$\left\{x \in \overline{U} : (x, (L+T)^{-1}(\Phi + T)(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}$$

is compact, can be replaced by

$$\left\{x \in \overline{U} : (x, (L+T)^{-1}(\Phi + T)(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}$$

is closed, in Definition 3.12; here $H^\star(x,\lambda) = (x, (L+T)^{-1}(H+T)(x,\lambda))$.

Definition 3.14. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$ with $F^\star = I \times (L+T)^{-1}(F+T)$. We say $F^\star : \overline{U} \to 2^{\overline{U} \times E}$ is d-L-Φ-essential in $A_{\partial U}(\overline{U}, Y; L, T)$ if for every map $J \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J^\star = I \times (L+T)^{-1}(J+T)$ and $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ we have that $d\left((F^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$.

Theorem 3.15. *Let E be a Hausdorff topological vector space, Y a topological vector space, U an open subset of E, $B = \left\{(x, (L+T)^{-1}(\Phi+T)(x)) : x \in \overline{U}\right\}$, $L : domL \subseteq E \to Y$ a linear single valued map, $T \in H_L(E, Y)$, d a map defined in* (3.8)*, $F \in A_{\partial U}(\overline{U}, Y; L, T)$ with $F^\star = I \times (L+T)^{-1}(F+T)$, $G \in A_{\partial U}(\overline{U}, Y; L, T)$ and $G^\star$ is d-L-Φ-essential in $A_{\partial U}(\overline{U}, Y; L, T)$ (here $G^\star = I \times (L+T)^{-1}(G+T)$). For any map $R \in A_{\partial U}(\overline{U}, Y; L, T)$ with $R^\star = I \times (L+T)^{-1}(R+T)$ and $R|_{\partial U} = F|_{\partial U}$ and $R \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ assume there exists a map $H : \overline{U} \times [0,1] \to 2^Y$ with $(L+T)^{-1}(H(\cdot, \eta(\cdot)) + T(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $(L+T)^{-1}(H_t + T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$, and*

$$\left\{x \in \overline{U} : (x, (L+T)^{-1}(\Phi + T)(x)) \cap H^\star(x,t) \neq \emptyset \textit{ for some } t \in [0,1]\right\}$$

is compact, $H_0 = G$, $H_1 = R$ (here $H_t(x) = H(x,t)$ and $H^\star(x,\lambda) = (x, (L+T)^{-1}(H + T)(x,\lambda))$) and for any continuous function $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ assume

$$\begin{aligned}\{x \in \overline{U} : &(x, (L+T)^{-1}(\Phi + T)(x)) \\ &\cap (x, (L+T)^{-1}(H_{t\mu(x)} + T)(x)) \neq \emptyset \textit{ for some } t \in [0,1]\}\end{aligned}$$

is closed and also suppose there exists a map $\Psi : \overline{U} \times [0,1] \to 2^Y$ with $(L+T)^{-1} \cdot (\Psi(\cdot, \eta(\cdot)) + T(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) =$

0, $(L+T)^{-1}(\Psi_t+T)(x)\cap(L+T)^{-1}(\Phi+T)(x)=\emptyset$ for any $x\in\partial U$ and $t\in(0,1)$, and

$$\left\{x\in\overline{U}:(x,(L+T)^{-1}(\Phi+T)(x))\cap\Psi^\star(x,t)\neq\emptyset \text{ for some } t\in[0,1]\right\}$$

is compact, $\Psi_0=H_0$ and $\Psi_1=F$ (here $\Psi_t(x)=\Psi(x,t)$ and $\Psi^\star(x,\lambda)=(x,(L+T)^{-1}(\Psi+T)(x,\lambda))$) and for any continuous function $\mu:\overline{U}\to[0,1]$ with $\mu(\partial U)=0$ assume

$$\begin{aligned}\{x\in\overline{U}:&(x,(L+T)^{-1}(\Phi+T)(x))\\ &\cap(x,(L+T)^{-1}(\Psi_{t\mu(x)}+T)(x))\neq\emptyset \text{ for some } t\in[0,1]\}\end{aligned}$$

is closed. Then $F^\star$ is d-L-Φ-essential in $A_{\partial U}(\overline{U},Y;L,T)$.

Proof. Let $R\in A_{\partial U}(\overline{U},Y;L,T)$ with $R^\star=I\times(L+T)^{-1}(R+T)$ and $R|_{\partial U}=F|_{\partial U}$ and $R\cong F$ in $A_{\partial U}(\overline{U},Y;L,T)$. Choose the maps $H:\overline{U}\times[0,1]\to 2^Y$ and $\Psi:\overline{U}\times[0,1]\to 2^Y$ as in the statement of Theorem 3.15. Let

$$D=\left\{x\in\overline{U}:(x,(L+T)^{-1}(\Phi+T)(x))\cap H^\star(x,t)\neq\emptyset \text{ for some } t\in[0,1]\right\}$$

where $H^\star(x,\lambda)=(x,(L+T)^{-1}(H+T)(x,\lambda))$. Notice $D\neq\emptyset$, D is compact and $D\cap\partial U=\emptyset$. Thus there exists a continuous map $\mu:\overline{U}\to[0,1]$ with $\mu(\partial U)=0$ and $\mu(D)=1$. Define $J_\mu:\overline{U}\to 2^Y$ by $J_\mu(x)=H(x,\mu(x))=H_{\mu(x)}(x)$ and let $J_\mu^\star=I\times(L+T)^{-1}(J_\mu+T)$. Note $J_\mu\in A_{\partial U}(\overline{U},Y;L,T)$ with $J_\mu|_{\partial U}=H_0|_{\partial U}$. Also note $J_\mu\cong H_0$ in $A_{\partial U}(\overline{U},Y;L,T)$ (to see this let $Q:\overline{U}\times[0,1]\to 2^Y$ be given by $Q(x,t)=H(x,t\mu(x))$). Also since $H_0^\star$ is d-L-Φ-essential in $A_{\partial U}(\overline{U},Y;L,T)$ we have

$$d\left(\left(J_\mu^\star\right)^{-1}(B)\right)=d\left(\left(H_0^\star\right)^{-1}(B)\right)\neq d(\emptyset),$$

and as in Theorem 3.10 (note $\mu(D)=1$) we have $\left(J_\mu^\star\right)^{-1}(B)=\left(R^\star\right)^{-1}(B)$, so

$$d\left(\left(R^\star\right)^{-1}(B)\right)=d\left(\left(H_1^\star\right)^{-1}(B)\right)\neq\emptyset. \tag{3.13}$$

Let

$$D_1=\left\{x\in\overline{U}:(x,(L+T)^{-1}(\Phi+T)(x))\cap\Psi^\star(x,t)\neq\emptyset \text{ for some } t\in[0,1]\right\}$$

where $\Psi^\star(x,\lambda)=(x,(L+T)^{-1}(\Psi+T)(x,\lambda))$. Notice $D_1\neq\emptyset$, D_1 is compact and $D_1\cap\partial U=\emptyset$. Thus there exists a continuous map $\mu:\overline{U}\to[0,1]$ with $\mu(\partial U)=0$ and $\mu(D_1)=1$. Define $G_\mu:\overline{U}\to 2^Y$ by $G_\mu(x)=\Psi(x,\mu(x))$ and let $G_\mu^\star=I\times(L+T)^{-1}(G_\mu+T)$. Note $G_\mu\in A_{\partial U}(\overline{U},Y;L,T)$ with $G_\mu|_{\partial U}=H_0|_{\partial U}$ and $G_\mu\cong H_0$ in $A_{\partial U}(\overline{U},Y;L,T)$ (to see this let $Q^1:\overline{U}\times[0,1]\to 2^Y$ be given by $Q(x,t)=\Psi(x,t\mu(x))$). Also since $H_0^\star$ is d-L-Φ-essential in $A_{\partial U}(\overline{U},Y;L,T)$ we have

$$d\left(\left(G_\mu^\star\right)^{-1}(B)\right)=d\left(\left(H_0^\star\right)^{-1}(B)\right)\neq d(\emptyset),$$

and as in Theorem 3.10 (note $\mu(D)=1$) we have $\left(G_\mu^\star\right)^{-1}(B)=\left(F^\star\right)^{-1}(B)$, so

$$d\left(\left(F^\star\right)^{-1}(B)\right)=d\left(\left(H_1^\star\right)^{-1}(B)\right)\neq\emptyset. \tag{3.14}$$

Now (3.13) and (3.14) yield $d\left(\left(F^{\star}\right)^{-1}(B)\right)=d\left(\left(R^{\star}\right)^{-1}(B)\right)\neq\emptyset$. □

Remark 3.16. It is of interest to note if we consider maps R other than F in Theorem 3.15 and if we suppose

$$\cong \text{ is an equivalence relation in } A_{\partial U}(\overline{U},Y;L,T),$$

then (see in the statement of Theorem 3.15) if $R\cong F$ in $A_{\partial U}(\overline{U},Y;L,T)$ and $R\cong G$ in $A_{\partial U}(\overline{U},Y;L,T)$ then (since $F\cong G$ in $A_{\partial U}(\overline{U},Y;L,T)$) there exists a map $\Psi:\overline{U}\times[0,1]\to 2^{Y}$ with $(L+T)^{-1}(\Psi(\cdot,\eta(\cdot))+T(\cdot))\in A(\overline{U},E)$ for any continuous function $\eta:\overline{U}\to[0,1]$ with $\eta(\partial U)=0$, $(L+T)^{-1}(\Psi_t+T)(x)\cap(L+T)^{-1}(\Phi+T)(x)=\emptyset$ for any $x\in\partial U$ and $t\in(0,1)$, and

$$\left\{x\in\overline{U}:(x,(L+T)^{-1}(\Phi+T)(x))\cap\Psi^{\star}(x,t)\neq\emptyset \text{ for some } t\in[0,1]\right\}$$

is compact, $\Psi_0=G$ and $\Psi_1=F$ (here $\Psi_t(x)=\Psi(x,t)$ and $\Psi^{\star}(x,\lambda)=(x,(L+T)^{-1}(\Psi+T)(x,\lambda)))$.

Remark 3.17. Suppose the following condition holds:

$$\begin{cases} \text{if } F,G\in A_{\partial U}(\overline{U},Y;L,T) \text{ with } F|_{\partial U}=G|_{\partial U} \text{and } F\cong G \\ \text{in } A_{\partial U}(\overline{U},Y;L,T) \text{ then } d\left(\left(F^{\star}\right)^{-1}(B)\right)=d\left(\left(G^{\star}\right)^{-1}(B)\right). \end{cases}$$

Then Definition 3.14 reduces to the following. Let $F\in A_{\partial U}(\overline{U},Y;L,T)$ with $F^{\star}=I\times(L+T)^{-1}(F+T)$. We say $F^{\star}:\overline{U}\to 2^{\overline{U}\times E}$ is d-L-Φ-essential in $A_{\partial U}(\overline{U},Y;L,T)$ if $d\left(\left(F^{\star}\right)^{-1}(B)\right)\neq d(\emptyset)$.

Remark 3.18. There is an analogue of Remark 3.11 (for normal topological vector spaces) in the statement of Theorem 3.15.

ACKNOWLEDGEMENT

The authors extend their appreciation to the International Scientific Partnership Program ISPP at King Saud University for funding this research work through ISPP No. 0027.

REFERENCES

[1] R. E. Edwards, Functional Analysis, Theory and Applications, Holt, Rinehart and Winston, 1965.

[2] L. Gorniewicz, Topological fixed point theory of multivalued mappings, *Kluwer Acad. Publishers*, Dordrecht, 1999.

[3] A. Granas and J. Dugundji, Fixed Point Theory, *Springer-Verlag*, New York, 2003.

[4] D. O'Regan, A unified theory for homotopy principles for multimaps, *Applicable Analysis* **92** (2013), 1944–1958.

[5] D. O'Regan, Generalized Leray-Schauder principles for general classes of maps in completely regular topological spaces *Applicable Analysis* **93** (2014), 1674–1690.

[6] D. O'Regan, Continuation principles based on essential maps and topological degree, *Applicable Analysis* **94** (2015), 1032–1041.

[7] D. O'Regan, Coincidence points for multivalued maps based on Φ-epi and Φ-essential maps, *Dynamic Systems and Applications* **24** (2015), 143–154.

[8] D. O'Regan, Fixed point results for maps with weakly sequentially closed graphs, *Communications in Applied Analysis* **19** (2015), 235–244.

[9] D. O'Regan and R. Precup, Theorems of Leray–Schauder Type and Applications, *Taylor and Francis Publishers*, London, 2002.