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APPROXIMATIONS OF FROBENIUS-PERRON OPERATORS VIA PIECEWISE QUADRATIC FUNCTIONS

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ABSTRACT. We develop a piecewise quadratic projection algorithm for the approximation of an absolutely continuous invariant measure associated with a given chaotic mapping $S : [0,1] \to [0,1]$. The idea is to approximate the corresponding Frobenius-Perron operator $P_S : L^1(0,1) \to L^1(0,1)$ via the projection principle on the subspace of continuous piecewise quadratic functions. The convergence of the method for the Lasota-Yorke class of mappings is proved, and numerical results are also presented.

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1. **Introduction**

The statistical study of chaotic mappings has been shown to be an effective way of investigating complicated deterministic dynamics. Natural measures, in particular absolutely continuous invariant measures, give the quantitative description of the statistical property of the individual orbits of the underlying mappings, but in almost all practical applications of ergodic theory, analytic expressions of such measures are not available or difficult to determine. This makes it a desirable task to develop efficient numerical scheme for the computation of invariant measures [1, 2, 3, 9].

It is Ulam who proposed in 1960 [13] the first numerical method for the computation of the stationary density of the Frobenius-Perron operator associated with the mapping. Stationary densities are density functions which are fixed points of the Frobenius-Perron operator and they are nothing but the density functions of absolutely continuous invariant measures of the underlying mapping. Since the pioneering work [12] of Li on the convergence of Ulam's method for the Lasota-Yorke class of piecewise C^2 and stretching mappings, there have been many works on the numerical

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study of the statistical properties of chaotic systems in mathematical and physical sciences.

Ulam's method belongs to two categories of numerical methods for bounded linear operators defined on an L^1 space. One is the so-called Markov finite approximations, that is, the discrete Frobenius-Perron operator shares the same Markov property of the Frobenius-Perron operator, and the other is the Gelerkin projection principle. In the first kind of approach, both positivity and integral are preserved via the approximation, while the second approach is based on the principle that the residual is "orthogonal" to the finite dimensional subspace from the partition of the domain.

So far, piecewise quadratic functions have been used in both the Markov finite approximations and the projection approximations of Frobenius-Perron operators. However, one can see from the numerical results of [5] that the piecewise quadratic Markov method does not increase the convergence order over the piecewise linear Markov method. The reason is that the piecewise quadratic function is only continuous, but not differentiable on the whole domain, which has the same smoothness feature as the piecewise linear one. Thus the structure-preserving approach to approximating Frobenius-Perron operators via the continuous piecewise quadratic functions may not gain much over that via continuous piecewise linear functions. On the other hand, the piecewise quadratic projection method developed in [6] with numerical results in [4] employed only discontinuous piecewise quadratic functions, so it may not achieve the maximal possible convergence rate for smooth stationary density functions. A Gelerkin projection method using continuous piecewise linear functions has been developed in [8], which gives a better convergence rate than the piecewise linear Markov approximation method in [5] from the numerical comparisons.

In this paper we propose a new piecewise quadratic method for the computation of absolutely continuous invariant measures, using the Gelerkin projection principle applied to subspaces spanned by continuous piecewise quadratic functions, unlike the discontinuous piecewise quadratic polynomials mentioned above. It is expected that because of the increase of continuity of the approximating functions, a better convergence result would be achieved when the stationary density has some regularity.

The paper is organized as follows. In the next section we review basic concepts related to Frobenius-Perron operators. In Section 3 we introduce the continuous piecewise quadratic functions and propose our new algorithm. The convergence theorem will be given in Section 4, following the consistency and stability analysis. Numerical results will be presented in Section 5, and we conclude in Section 6.

2. **Preliminaries**

In order to perform the statistical investigation of chaotic mappings S on the interval $[0, 1]$, one is interested in absolutely continuous invariant probability measures.

The density function of an absolutely continuous invariant measure with respect to the Lebesgue measure m of $[0,1]$, often referred to as a *stationary density*, is a fixed point of the Frobenius-Perron operator $P_S : L^1(0,1) \to L^1(0,1)$ associated with the mapping.

For the rigorous definition of the Frobenius-Perron operator associated with a mapping $S : [0,1] \to [0,1]$, we first assume that S is *nonsingular*, that is, $m(A) = 0$ implies that $m(S^{-1}(A)) = 0$ for any Borel measurable subset A of $[0,1]$. The L^1-norm of a function $f \in L^1(0,1)$ is denoted as $\|f\|_1 = \int_0^1 |f| dm$. A nonnegative function $f \in L^1(0,1)$ with $\|f\|_1 = 1$ is called a *density*. If $g \in L^\infty(0,1)$, then $\|g\|_\infty \equiv \operatorname{ess\,sup}_{x\in[0,1]} |g(x)|$. The set of all L^1-functions of bounded variation is denoted as $BV(0,1)$. This becomes a Banach space under the BV-norm $\|f\|_{BV} \equiv \|f\|_1 + \bigvee_0^1 f$. When $f \in L^1(0,1)$ and $g \in L^\infty(0,1)$ we denote $\langle f, g\rangle = \int_0^1 fg dm$.

The *Frobenius-Perron operator* $P_S : L^1(0,1) \to L^1(0,1)$ associated with S is defined (implicitly) by

$$\int_A P_S f dm = \int_{S^{-1}(A)} f dm \tag{2.1}$$

for all Borel measurable subsets A of $[0,1]$, because of the non-singularity assumption on S and the Radon-Nikodym theorem in measure theory (see, e.g., [9] for more details). Let $A = [0,x]$ and apply the fundamental theorem of calculus to the both sides of (2.1), we have the explicit definition

$$P_S f(x) = \frac{d}{dx} \int_{S^{-1}([0,x])} f(t) dt$$

of P_S. It is well known [9] that an L^1 function f^* is a stationary density of P_S if and only if the absolutely continuous probability measure μ, defined by $\mu(A) = \int_A f^* dm$, is S-invariant, that is, $\mu(S^{-1}(A)) = \mu(A)$ for all measurable sets $A \subset [0,1]$. Invariant measures describe statistical properties of the dynamics of S, and their efficient computation is very important in applications. See [7] for a survey of the state of the arts in this area.

Frobenius-Perron operators constitute a special class of *Markov operators* which are defined as linear operators on L^1 spaces that map densities to densities, thus their norm is 1, which means that they are weak contractions. Various existence results of stationary densities are referred to [9, 10]. Here we are only interested in the computational problem for a stationary density of the Frobenius-Perron operator.

3. **The Piecewise Quadratic Projection Method**

Although the piecewise quadratic Markov approximations method has the advantage of approximating the exact stationary density with density functions, it lacks some good features of the projection method, for example, it does not keep piecewise quadratic functions fixed. In other words, it approximates a piecewise quadratic function with a different piecewise quadratic function. This may cause a relatively large error of the approximate stationary density for a normal partition of the interval.

In the piecewise quadratic Gelerkin projection method, every function is projected onto a finite dimensional subspace of $L^1(0,1)$ consisting of piecewise quadratic functions related to the partition of $[0,1]$, and thus each piecewise quadratic function in the subspace is kept fixed after projection. In the following we first construct the space of continuous piecewise quadratic functions and study their basic properties.

For the sake of simplicity and convenience of computation, we divide the interval $[0,1]$ into n equal subintervals with $x_i = ih$ for $i = 0, 1, \ldots, n$, where $h = 1/n$ is the length of each subinterval. Denote by Δ_n the corresponding space of all continuous piecewise quadratic functions. Then the dimension of Δ_n is $2n+1$, and a canonical basis of Δ_n consists of $e_0, e_1, \ldots, e_{2n}$, where

$$e_{2k} = \tau\left(\frac{x - x_k}{h}\right)$$

for $k = 0, 1, \ldots, n$ and

$$e_{2k-1}(x) = \rho\left(\frac{x - x_{k-1}}{h}\right)$$

for $k = 1, 2, \ldots, n$. Here the two basic functions τ and ρ are

$$\tau(x) = \begin{cases} (x+1)^2, & -1 \le x \le 0, \\ (x-1)^2, & 0 < x \le 1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\rho(x) = \begin{cases} 2x(1-x), & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $\|e_0\|_1 = \|e_{2n}\|_1 = h/3$, $\|e_{2k}\|_1 = 2h/3$ for $k = 1, 2, \ldots, 2(n-1)$, and $\|e_{2k-1}\|_1 = h/3$ for $k = 1, 2, \ldots, n$. By definition, the *support* of a function g, denoted supp$g$, is the closure of the set of all x in the domain of g such that $g(x) \neq 0$. So, supp$e_0 = [0, h]$, supp$e_{2n} = [1-h, 1]$, supp$e_{2k} = [(k-1)h, (k+1)h]$ for $k = 1, 2, \ldots, n-1$, and supp$e_{2k-1} = [(k-1)h, kh]$ for $k = 1, 2, \ldots, n$. In addition, the basis functions satisfy the identity

$$\sum_{i=0}^{2n} e_i(x) \equiv 1, \ \forall\, x \in [0,1], \tag{3.1}$$

which is called the *partition of unity* in the theory of finite elements.

Now we are ready to define a projection operator Q_n from $L^1(0,1)$ onto the subspace Δ_n by requiring that the residual function $f - Q_n f$ is "orthogonal" to Δ_n.

Namely, for a given function $f \in L^1(0,1)$, we let $Q_n f$ be the unique function in Δ_n such that

$$\langle Q_n f, e_i \rangle = \langle f, e_i \rangle, \quad i = 0, 1, \ldots, 2n. \tag{3.2}$$

If we write $Q_n f = \sum_{i=0}^{2n} c_i e_i$, then the coefficients $c_0, c_1, \ldots, c_{2n}$ are the unique solutions of the following linear equations

$$\sum_{j=0}^{2n} c_j \langle e_j, e_i \rangle = \langle f, e_i \rangle, \quad i = 0, 1, \ldots, 2n. \tag{3.3}$$

If we restrict f to be in $L^2(0,1)$, then $Q_n f$ is exactly the orthogonal projection of $L^2(0,1)$ onto $\Delta_n \subset L^2(0,1)$ under the L^2-norm $\|f\|_2 = \sqrt{\int_0^1 |f|^2 dm}$ of $L^2(0,1)$. In this case, the unique solution $Q_n f$ is also the least squares solution to the least squares problem

$$\begin{aligned} \|f - Q_n f\|_2 &= \min\{\|f - g\|_2 : g \in \Delta_n\} \\ &= \min\left\{ \left\| f - \sum_{i=0}^{2n} c_i e_i \right\|_2 : (c_0, c_1, \ldots, c_{2n})^T \in \mathbf{R}^{2n+1} \right\}. \end{aligned}$$

By letting $B = (b_{ij})$, where $b_{ij} = \langle e_j, e_i \rangle$ for $0 \le i, j \le 2n$, we find that

$$B = \frac{h}{30} \begin{bmatrix} 6 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 3 & 4 & 3 & 0 & 0 & & 0 & 0 & \cdots & 0 \\ 1 & 3 & 12 & 3 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 3 & 4 & 3 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 3 & 12 & 3 & 1 & 0 & \cdots & 0 \\ \cdot & & & \cdot & & & \cdot & & & \cdot \\ \cdot & & & \cdot & & & \cdot & & & \cdot \\ 0 & \cdots & & 0 & 0 & 1 & 3 & 12 & 3 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 3 & 4 & 3 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 6 \end{bmatrix}. \tag{3.4}$$

Substituting (3.4) into (3.3), we obtain the system of linear equations

$$\begin{bmatrix} 6 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 3 & 4 & 3 & 0 & 0 & & 0 & 0 & \cdots & 0 \\ 1 & 3 & 12 & 3 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 3 & 4 & 3 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 3 & 12 & 3 & 1 & 0 & \cdots & 0 \\ \cdot & & & \cdot & & & \cdot & & & \cdot \\ \cdot & & & \cdot & & & \cdot & & & \cdot \\ 0 & \cdots & & 0 & 1 & 3 & 0 & 12 & 3 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 3 & 4 & 3 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ c_{2n-2} \\ c_{2n-1} \\ c_{2n} \end{bmatrix}$$

$$= \frac{30}{h} \begin{bmatrix} \int_0^1 f(x)e_0(x)dx \\ \int_0^1 f(x)e_1(x)dx \\ \int_0^1 f(x)e_2(x)dx \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \int_0^1 f(x)e_{2n-2}(x)dx \\ \int_0^1 f(x)e_{2n-1}(x)dx \\ \int_0^1 f(x)e_{2n}(x)dx \end{bmatrix}. \tag{3.5}$$

Our piecewise quadratic projection method for numerically computing a stationary density of the Frobenius-Perron operator $P_S : L^1(0,1) \to L^1(0,1)$ is to solve the finite dimensional fixed point equation

$$P_n f = f, \quad f \in \Delta_n, \tag{3.6}$$

where $P_n \equiv Q_n P_S : \Delta_n \to \Delta_n$ in which Q_n is defined by (3.2).

The finite dimensional operator equation (3.6) can be written as

$$\langle Q_n P_S f, e_i \rangle = \langle f, e_i \rangle, \quad i = 0, 1, \ldots, 2n$$

for $f \in \Delta_n$. Since Q_n is an "orthogonal" projection,

$$\langle Q_n P_S f, e_i \rangle = \langle P_S f, e_i \rangle, \quad i = 0, 1, \ldots, 2n \tag{3.7}$$

for all $f \in L^1(0,1)$, so the above equations are simplified to

$$\langle P_S f, e_i \rangle = \langle f, e_i \rangle, \quad i = 0, 1, \ldots, 2n. \tag{3.8}$$

If we write $f = \sum_{j=0}^{2n} v_j e_j$, then (3.8) can be written as

$$\sum_{j=0}^{2n} v_j \langle P_S e_j, e_i \rangle = \sum_{j=0}^{2n} v_j \langle e_j, e_i \rangle, \quad i = 0, 1, \ldots, 2n. \tag{3.9}$$

Define a $(2n+1) \times (2n+1)$ matrix $A = (a_{ij})$ by $a_{ij} = \langle P_S e_j, e_i \rangle$ for $i, j = 0, 1, \ldots, 2n$. Then the linear algebraic equations (3.9) has the matrix form

$$(A - B)v = 0, \quad v = (v_0, v_1, \ldots, v_{2n})^T \in \mathbf{R}^{2n+1}. \tag{3.10}$$

Because of the partition of unity property (3.1) and the orthogonality (3.7) with $f = e_j$ for all indices i and j, and the fact that $\int_0^1 P_S g dm = \int_0^1 g dm$ for any $g \in L^1(0,1)$,

$$\sum_{i=0}^{2n} a_{ij} = \sum_{i=0}^{2n} \langle Q_n P_S e_j, e_i \rangle = \sum_{i=0}^{2n} \langle P_S e_j, e_i \rangle = \left\langle P_S e_j, \sum_{i=0}^{2n} e_i \right\rangle$$

$$= \langle P_S e_j, 1\rangle = \langle e_j, 1\rangle = \sum_{i=0}^{2n} \langle e_j, e_i\rangle = \sum_{i=0}^{2n} b_{ij}.$$

Therefore, $(1, 1, \ldots, 1)$ is a left eigenvector of $A - B$ associated with eigenvalue 0. So there is a right eigenvector v with the same eigenvalue, that is, (3.10) has a nonzero solution. This proves the following result.

Proposition 3.1. *The discrete fixed point equation* (3.6) *has a nontrivial solution* $f_n \in \Delta_n$ *for any* n.

We can normalize the above solution f_n by selecting a nonzero solution $(v_0^*, v_1^*, \ldots, v_{2n}^*)^T$ of (3.10) so that the resulting function

$$f_n = \sum_{i=0}^{2n} v_i^* e_i$$

satisfies the condition $\|f_n\|_1 = 1$, which gives a continuous piecewise quadratic approximation of a stationary density of the original Frobenius-Perron operator. In the next section we shall investigate the convergence of the numerical sequence f_n to the exact stationary density f^* as $n \to \infty$.

4. **Convergence Analysis**

We analyze the convergence of our new piecewise quadratic algorithm in this section that is divided into three parts. First we study the consistency problem of the projection sequence Q_n in subsection 4.1, and the stability issue of Q_n in terms of the variation sequence $\bigvee_0^1 Q_n f$ will be examined in the second subsection. Then we prove the L^1-norm convergence result for the numerical solution sequence f_n in the last subsection.

4.1. **The Strong Convergence of $\{Q_n\}$.**

We show that the sequence Q_n converges strongly to the identity operator by first showing that the sequence $\|Q_n\|_1$ is bounded.

The uniform boundedness of $\|Q_n\|_1$ is equivalent to that of the 1-norm of the inverse of the matrix in (3.5). Unlike the matrix in the case of projections to the space of continuous piecewise linear functions, which is strictly diagonally dominant so that the 1-norm of its inverse matrix is uniformly bounded by 1 for all partitions of $[0, 1]$, the matrix of (3.5) is not strictly diagonally dominant. To establish the uniform boundedness of the inverse matrix, we rearrange the order of the basis functions of Δ_n so that the corresponding matrix has a better structure to estimate the norm of its inverse.

To do so, we put all $e_1, e_3, \ldots, e_{2n-1}$ before all the remaining basis functions. Thus, we let $\phi_k = e_{2k+1}$ for $k = 0, 1, \ldots, n-1$ and $\phi_{n+k} = e_{2k}$ for $k = 0, \ldots, n$. If we write $Q_n f = \sum_{i=0}^{2n} d_i \phi_i$, then the vector $d = (d_0, d_1, \ldots, d_{2n})^T$ satisfies the following system of linear equations

$$Md = \frac{30}{h}\hat{f}, \tag{4.1}$$

where $\hat{f} = (\langle f, \phi_0\rangle, \langle f, \phi_1\rangle, \ldots, \langle f, \phi_{2n}\rangle)^T$ and M is the 2×2 block matrix

$$M = \begin{bmatrix} 4I & 3J \\ 3J^T & H \end{bmatrix}$$

in which I is the $n \times n$ identity matrix, J is the $n \times (n+1)$ matrix whose main diagonal and sup-diagonal entries are 1 and all other entries are zero, and H is the $(n+1) \times (n+1)$ matrix

$$\begin{bmatrix} 6 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 12 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 12 & 1 & 0 & \cdots & 0 \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ 0 & \cdots & 0 & 0 & 1 & 12 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 1 & 6 \end{bmatrix}.$$

The generalized Gaussian elimination gives the block matrix decomposition

$$M = \begin{bmatrix} 4I & 3J \\ 3J^T & H \end{bmatrix} = \begin{bmatrix} I & 0 \\ \frac{3}{4}J^T & I \end{bmatrix} \begin{bmatrix} 4I & 3J \\ 0 & H - \frac{9}{4}J^T J \end{bmatrix},$$

where I is an identity matrix of size n or $n+1$, depending on its location in the block matrices. Denote $V = H - \frac{9}{4}J^T J$. Then

$$V = \frac{5}{4}\begin{bmatrix} 3 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 6 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 6 & -1 & 0 & \cdots & 0 \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ 0 & \cdots & 0 & 0 & -1 & 6 & -1 \\ 0 & \cdots & 0 & 0 & 0 & -1 & 3 \end{bmatrix},$$

so it is invertible because of its strictly diagonal dominance. It follows that

$$M^{-1} = \begin{bmatrix} \frac{1}{4}I & -\frac{3}{4}JV^{-1} \\ 0 & V^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\frac{3}{4}J^T & I \end{bmatrix}. \tag{4.2}$$

Lemma 1. *Let C be an m-by-m matrix such that*

$$|c_{jj}| > \sum_{i \neq j} |c_{ij}|, \quad \textit{for any } j.$$

Then

$$\left\|C^{-1}\right\|_1 \leq \frac{1}{\min_{1\leq j\leq m}\left(|c_{jj}| - \sum_{i\neq j}|c_{ij}|\right)}.$$

Proof. Since C is columnwise diagonally dominant, it is invertible. Note that there exists $y = (y_1, \ldots, y_n)^T$ with $\|y\|_1 = 1$ such that $\|C^{-1}\|_1 = \|C^{-1}y\|_1$. Let $x = C^{-1}y$. Note that

$$\begin{aligned}
\|Cx\|_1 &= \sum_{i=1}^m \sum_{j=1}^m |c_{ij}x_j| \geq \sum_{i=1}^m \left[|c_{ii}x_i| - \sum_{j\neq i}|c_{ij}x_j|\right] \\
&= \sum_{i=i}^m |c_{ii}|\,|x_i| - \sum_{i=1}^m \sum_{j\neq i}|c_{ij}|\,|x_j| = \sum_{j=1}^m \left[|c_{jj}| - \sum_{i\neq j}|c_{ij}|\right]|x_j| \\
&\geq \min_{1\leq j\leq m}\left(|c_{jj}| - \sum_{i\neq j}|c_{ij}|\right)\sum_{j=1}^m |x_j| = \min_{1\leq j\leq m}\left(|c_{jj}| - \sum_{i\neq j}|c_{ij}|\right)\|x\|_1.
\end{aligned}$$

Since $Cx = y$, we have

$$\|y\|_1 \geq \min_{1\leq j\leq m}\left(|c_{jj}| - \sum_{i\neq j}|c_{ij}|\right)\|x\|_1.$$

Because $\|y\|_1 = 1$ and $\|x\|_1 = \|C^{-1}y\|_1 = \|C^{-1}\|_1$, the result follows. $\square$

Lemma 4.1 implies that $\|V^{-1}\|_1 \leq 2/5$. Partition $d^T = (\bar{d}^T, \tilde{d}^T)$ and $\hat{f}^T = (\bar{f}^T, \tilde{f}^T)$ into two blocks of size n and $n+1$. Then (4.1) and (4.2) give

$$\begin{aligned}
\frac{h}{30}\begin{bmatrix}\bar{d}\\ \tilde{d}\end{bmatrix} &= M^{-1}\begin{bmatrix}\bar{f}\\ \tilde{f}\end{bmatrix} = \begin{bmatrix}\frac{1}{4}I & -\frac{3}{4}JV^{-1}\\ 0 & V^{-1}\end{bmatrix}\begin{bmatrix}I & 0\\ -\frac{3}{4}J^T & I\end{bmatrix}\begin{bmatrix}\bar{f}\\ \tilde{f}\end{bmatrix} \\
&= \begin{bmatrix}\frac{1}{4}I & -\frac{3}{4}JV^{-1}\\ 0 & V^{-1}\end{bmatrix}\begin{bmatrix}\bar{f}\\ \tilde{f} - \frac{3}{4}J^T\bar{f}\end{bmatrix} \\
&= \begin{bmatrix}\frac{1}{4}\bar{f} - \frac{3}{4}JV^{-1}\left(\tilde{f} - \frac{3}{4}J^T\bar{f}\right)\\ V^{-1}\left(\tilde{f} - \frac{3}{4}J^T\bar{f}\right)\end{bmatrix} \\
&= \begin{bmatrix}\left(\frac{1}{4} + \frac{9}{16}JV^{-1}J^T\right)\bar{f} - \frac{3}{4}JV^{-1}\tilde{f}\\ V^{-1}\left(\tilde{f} - \frac{3}{4}J^T\bar{f}\right)\end{bmatrix},
\end{aligned}$$

from which we see that

$$\frac{h}{30}\|\bar{d}\|_1 \leq \left(\frac{1}{4} + \frac{9}{16}\cdot 2\cdot\frac{2}{5}\cdot 2\right)\|\bar{f}\|_1 + \frac{3}{4}\cdot 2\cdot\frac{2}{5}\|\tilde{f}\|_1 = \frac{23}{20}\|\bar{f}\|_1 + \frac{3}{5}\|\tilde{f}\|_1$$

and

$$\frac{h}{30}\|\tilde{d}\|_1 \leq \frac{2}{5}\left(\|\tilde{f}\|_1 + \frac{3}{4}\cdot 2\|\bar{f}\|_1\right) = \frac{3}{5}\|\bar{f}\|_1 + \frac{2}{5}\|\tilde{f}\|_1.$$

Lemma 2. $\left\|\widehat{f}\right\|_1 \leq \|f\|_1$ *for any* $f \in L^1(0,1)$.

Proof. Since $\hat{f} = (\langle f, \phi_0\rangle, \langle f, \phi_1\rangle, \ldots, \langle f, \phi_{2n}\rangle)^T$,

$$\|\hat{f}\|_1 = \sum_{i=0}^{2n} |\langle f, \phi_i\rangle| \leq \sum_{i=0}^{2n}\langle |f|, \phi_i\rangle = \left\langle |f|, \sum_{i=0}^{2n} \phi_i \right\rangle = \langle |f|, 1\rangle = \|f\|_1.$$

□

Proposition 4.1. *The sequence $Q_n : L^1(0,1) \to L^1(0,1)$ is uniformly bounded by $47/2$ in norm and converges to the identity operator strongly.*

Proof. Let $f \in L^1(0,1)$. Since $\|\phi_i\|_1 = h/3$ for $i = 0, \ldots, n-1$ and $\|\phi_i\|_1 = 2h/3$ for $i = n, \ldots, 2n$, the above lemmas imply that

$$\begin{aligned}
\|Q_n f\|_1 &= \left\| \sum_{i=0}^{2n} d_i\phi_i \right\|_1 \leq \sum_{i=0}^{2n} |d_i|\|\phi_i\|_1 \leq \frac{h}{3}(\|\bar{d}\|_1 + 2\|\tilde{d}\|_1) \\
&\leq \frac{h}{3} \cdot \frac{30}{h}\left(\frac{23}{20}\|\bar{f}\|_1 + \frac{3}{5}\|\tilde{f}\|_1 + \frac{6}{5}\|\bar{f}\|_1 + \frac{4}{5}\|\tilde{f}\|_1\right) \\
&= 10\left(\frac{47}{20}\|\bar{f}\|_1 + \frac{7}{5}\|\tilde{f}\|_1\right) \leq 10 \cdot \frac{47}{20}\|\hat{f}\|_1 \leq \frac{47}{2}\|f\|_1,
\end{aligned}$$

which shows that $\|Q_n\|_1 \leq 47/2$ uniformly.

Now let $f \in C^3[0,1]$. Then by the Cauchy-Schwartz inequality and the least squares property of the Q_n for L^2 functions,

$$\|Q_n f - f\|_1 \leq \|Q_n f - f\|_2 \leq \|L_n f - f\|_2 \leq \|L_n f - f\|_\infty = O(h^3),$$

where $L_n f$ is the piecewise quadratic Lagrange interpolation function for f with nodes x_i's and the midpoints of all the subintervals $[x_{i-1}, x_i]$, and the last equality is from the standard error result for Lagrange interpolations. In fact, one can verify that

$$L_n f = \sum_{i=0}^{n} f(x_i)e_{2i} + \sum_{i=1}^{n}\left[-\frac{1}{2}f(x_{i-1}) + 2f(x_{i-\frac{1}{2}}) - \frac{1}{2}f(x_i)\right] e_{2i-1},$$

where $x_{i-1/2}$ is the midpoint of the subinterval $[x_{i-1}, x_i]$ for $1 \leq i \leq n$.

Finally, since $C^3[0,1]$ is dense in $L^1(0,1)$, that $\lim_{n\to\infty} \|Q_n f - f\|_1 = 0$ for any $f \in L^1(0,1)$ comes from the uniform boundedness of $\|Q_n\|_1$. □

4.2. **A Variation Inequality for $\{Q_n\}$.** Next we establish an upper bound on $\bigvee_0^1 Q_n f$ in terms of $\bigvee_0^1 f$. Since $Q_n f$ is continuous and piecewise quadratic,

$$\bigvee_0^1 Q_n f = \int_0^1 |(Q_n f)'(x)|dx = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} |(Q_n f)'(x)|dx.$$

Since $Q_n f = d_i\phi_i + d_{n+i}\phi_{n+i} + d_{n+i+1}\phi_{n+i+1}$ on $[x_i, x_{i+1}]$, and since $\phi_i(x) = \rho((x - x_i)/h)$ for $i = 0, \ldots, n-1$ and $\phi_{n+i}(x) = \tau((x - x_i)/h)$ for $i = 0, \ldots, n$,

$$\int_{x_i}^{x_{i+1}} |(Q_n f)'(x)|dx = \int_{x_i}^{x_{i+1}} |d_i\phi_i'(x) + d_{n+i}\phi_{n+i}'(x) + d_{n+i+1}\phi_{n+i+1}'(x)|dx$$

$$
\begin{aligned}
&= \frac{1}{h}\int_{x_i}^{x_{i+1}} \left| d_i\rho'\left(\frac{x-x_i}{h}\right) + d_{n+i}\tau'\left(\frac{x-x_i}{h}\right) + d_{n+i+1}\tau'\left(\frac{x-x_{i+1}}{h}\right)\right| dx \\
&= \int_0^1 |d_i\rho'(t) + d_{n+i}\tau'(t) + d_{n+i+1}\tau'(t-1)|dt \\
&= \int_0^1 |d_i(2-4t) + d_{n+i}2(t-1) + d_{n+i+1}2t|dt \\
&= 2\int_0^1 |(d_{n+i} + d_{n+i+1} - 2d_i)(t-1) + d_{n+i+1} - d_i|dt \\
&\le |d_{n+i} + d_{n+i+1} - 2d_i| + 2|d_{n+i+1} - d_i| \le |d_i - d_{n+i}| + 3|d_{n+i+1} - d_i|.
\end{aligned}
$$

Summing up the above for $i = 0, \dots, n-1$, we obtain

$$
\bigvee_0^1 Q_n f \le \sum_{i=0}^{n-1} |d_i - d_{n+i}| + 3\sum_{i=0}^{n-1} |d_{n+i+1} - d_i|. \tag{4.3}
$$

From the system (4.1) we obtain

$$
\begin{aligned}
&\begin{bmatrix}
3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
5 & 10 & 4 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
1 & 4 & 10 & 5 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 5 & 10 & 4 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 4 & 10 & 5 & 0 & 0 & \cdots & 0 \\
\vdots & & & \vdots & & & \vdots & & & \vdots \\
0 & \cdots & 0 & 0 & 0 & 0 & 5 & 10 & 4 & 1 \\
0 & \cdots & 0 & 0 & 0 & 0 & 1 & 4 & 10 & 5 \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3
\end{bmatrix}
\begin{bmatrix}
d_0 - d_n \\ d_{n+1} - d_0 \\ d_1 - d_{n+1} \\ d_{n+2} - d_1 \\ d_2 - d_{n+2} \\ \vdots \\ d_{2n-1} - d_{n-2} \\ d_{n-1} - d_{2n-1} \\ d_{2n} - d_{n-1}
\end{bmatrix} \\
&= \frac{30}{h}
\begin{bmatrix}
\hat f_0 - \hat f_n \\ \hat f_{n+1} - 2\hat f_0 \\ 2\hat f_1 - \hat f_{n+1} \\ \hat f_{n+2} - 2\hat f_1 \\ 2\hat f_2 - \hat f_{n+2} \\ \vdots \\ \hat f_{2n-1} - 2\hat f_{n-2} \\ 2\hat f_{n-1} - \hat f_{2n-1} \\ \hat f_{2n} - \hat f_{n-1}
\end{bmatrix}.
\end{aligned} \tag{4.4}
$$

Rearranging the above equations according to the permutation $(0, 2, \dots, 2n-2, 1, 3, \dots, 2n-1)$ of $(0, 1, \dots, 2n-1)$ and reordering the columns of the above coefficient matrix with the same permutation, the system (4.4) becomes

$$
Zv \equiv \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \begin{bmatrix} \bar v \\ \tilde v \end{bmatrix} = \frac{30}{h} \begin{bmatrix} \bar w \\ \tilde w \end{bmatrix}, \tag{4.5}
$$

where

$$P = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 10 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 10 & 0 & 0 & \cdots & 0 \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ 0 & \cdots & 0 & 0 & 1 & 10 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & 10 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 4 & 5 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 4 & 5 & 0 & 0 & \cdots & 0 \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ 0 & \cdots & 0 & 0 & 4 & 5 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 4 & 5 \end{bmatrix},$$

$$R = \begin{bmatrix} 5 & 4 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 5 & 4 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 5 & 4 & 0 & \cdots & 0 \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ 0 & \cdots & 0 & 0 & 0 & 5 & 4 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad S = \begin{bmatrix} 10 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 10 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 10 & 1 & 0 & \cdots & 0 \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ 0 & \cdots & 0 & 0 & 0 & 10 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 3 \end{bmatrix},$$

$$\bar{v}^T = (d_0 - d_n, d_1 - d_{n+1}, \ldots, d_{n-1} - d_{2n-1}),$$
$$\tilde{v}^T = (d_{n+1} - d_0, d_{n+2} - d_1, \ldots, d_{2n} - d_{n-1}),$$
$$\bar{w}^T = (\hat{f}_0 - \hat{f}_n, 2\hat{f}_1 - \hat{f}_{n+1}, 2\hat{f}_2 - \hat{f}_{n+2}, \ldots, 2\hat{f}_{n-2} - \hat{f}_{2n-2}, 2\hat{f}_{n-1} - \hat{f}_{2n-1}),$$
$$\tilde{w}^T = (\hat{f}_{n+1} - 2\hat{f}_0, \hat{f}_{n+2} - 2\hat{f}_1, \hat{f}_{n+3} - 2\hat{f}_2, \ldots, \hat{f}_{2n-1} - 2\hat{f}_{n-2}, \hat{f}_{2n} - \hat{f}_{n-1}).$$

Partitioning

$$P = \begin{bmatrix} 3 & 0 \\ e_1 & C \end{bmatrix},$$

where e_1 is the first column of the identity matrix of size $n-1$, and using the generalized Gaussian elimination, we have

$$P^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3}C^{-1}e_1 & C^{-1} \end{bmatrix},$$

where C^{-1} is a lower triangular matrix with its (i,j)-entry $(-1)^{j-i} \times 10^{j-i-1}$ for $1 \le j \le i \le n-1$.

Denote $K = S - RP^{-1}Q$. After some computations we find the following about the entries of K:

- All the entries above the super-diagonal are zero.
- All of the super-diagonal entries are -1.
- All of the diagonal entries are 6.1 except $K_{11} = 16/3$ and $K_{nn} = 2$.
- $K_{i1} = (-1)^{i-1} \times (23/15) \times 10^{-i+2}$ for $2 \le i \le n-1$ and $K_{n1} = (-1)^{n-1} \times (20/3) \times 10^{1-n}$ (the entries in the first column except K_{11}).
- $K_{nj} = (-1)^{n-j} \times 7 \times 10^{j-n}$ for $2 \le j \le n-1$ (the entries in the last row except K_{n1} and K_{nn}).

- $K_{ij} = (-1)^{i-j} \times 1.61 \times 10^{j-i+1}$ for $2 \le j \le i-1 \le n-2$ (the entries in the triangular region bounded by the diagonal, the first column and the last row).

Using Lemma 4.1, we see that

$$\|K^{-1}\|_1 \le 1$$

uniformly. It is easy to see that

$$\|P^{-1}\|_1 \le \frac{10}{27}, \quad \|P^{-1}Q\|_1 \le \frac{28}{27}, \quad \|RP^{-1}\|_1 \le \frac{46}{27}$$

uniformly. Since

$$Z^{-1} = \begin{bmatrix} P & Q \\ 0 & K \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ -RP^{-1} & I \end{bmatrix} = \begin{bmatrix} P^{-1} & -P^{-1}QK^{-1} \\ 0 & K^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -RP^{-1} & I \end{bmatrix},$$

from (4.5) we have

$$\begin{aligned} \frac{h}{30}\begin{bmatrix} \bar{v} \\ \tilde{v} \end{bmatrix} = Z^{-1}\begin{bmatrix} \bar{w} \\ \tilde{w} \end{bmatrix} &= \begin{bmatrix} P^{-1} & -P^{-1}QK^{-1} \\ 0 & K^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -RP^{-1} & I \end{bmatrix} \begin{bmatrix} \bar{w} \\ \tilde{w} \end{bmatrix} \\ &= \begin{bmatrix} P^{-1} & -P^{-1}QK^{-1} \\ 0 & K^{-1} \end{bmatrix} \begin{bmatrix} \bar{w} \\ \tilde{w} - RP^{-1}\bar{w} \end{bmatrix} \\ &= \begin{bmatrix} (P^{-1} + P^{-1}QK^{-1}RP^{-1})\bar{w} - P^{-1}QK^{-1}\tilde{w}] \\ K^{-1}(\tilde{w} - RP^{-1}\bar{w}) \end{bmatrix}, \end{aligned}$$

from which

$$\begin{aligned} \frac{h}{30}\|\bar{v}\|_1 &\le (\|P^{-1}\|_1 + \|P^{-1}Q\|_1\|K^{-1}\|_1\|RP^{-1}\|_1)\|\bar{w}\|_1 + \|P^{-1}Q\|_1\|K^{-1}\|_1\|\tilde{w}\|_1 \\ &\le \left(\frac{10}{27} + \frac{28}{27}\cdot\frac{46}{27}\right)\|\bar{w}\|_1 + \frac{28}{27}\|\tilde{w}\|_1 = \frac{1558}{729}\|\bar{w}\|_1 + \frac{28}{27}\|\tilde{w}\|_1 \end{aligned}$$

and

$$\frac{h}{30}\|\tilde{v}\|_1 \le \frac{46}{27}\|\bar{w}\|_1 + \|\tilde{w}\|_1.$$

Applying the above inequalities to (4.3) gives

$$\begin{aligned} \bigvee_0^1 Q_n f &\le \|\bar{v}\|_1 + 3\|\tilde{v}\|_1 \\ &\le \frac{30}{h}\left(\frac{1558}{729}\|\bar{w}\|_1 + \frac{28}{27}\|\tilde{w}\|_1 + \frac{46}{9}\|\bar{w}\|_1 + 3\|\tilde{w}\|_1\right) \\ &= \frac{30}{h}\left(\frac{5284}{729}\|\bar{w}\|_1 + \frac{2943}{729}\|\tilde{w}\|_1\right) \\ &= \frac{52840}{243\,h}\|\bar{w}\|_1 + \frac{29430}{243h}\|\tilde{w}\|_1 \qquad (4.6) \end{aligned}$$

We are ready to prove our stability result in terms of the variation of the sequence $Q_n f$.

Proposition 4.2. *Let $f \in BV(0,1)$ be absolutely continuous. Then*

$$\bigvee_0^1 Q_n f \le 327 \bigvee_0^1 f, \text{ for any } n.$$

Proof. We need to estimate the 1-norm of $w^T = (\bar{w}^T, \tilde{w}^T)$ in terms of the variation of f. First of all, using integration by parts,

$$\begin{aligned}
|\bar{w}_0| = |\hat{f}_0 - \hat{f}_n| &= \left| \int_0^h f(x)(\phi_0(x) - \phi_n(x))dx \right| \\
&= \left| f(x)\Phi(x)|_0^h - \int_0^h f'(x)\Phi(x)dx \right| \\
&= h \left| \int_0^h f'(x) \left[-\frac{x}{h} + 2\left(\frac{x}{h}\right)^2 - \left(\frac{x}{h}\right)^3 \right] dx \right| \\
&\le h \int_0^h |f'(x)| \left| \frac{x}{h} - 2\left(\frac{x}{h}\right)^2 + \left(\frac{x}{h}\right)^3 \right| dx \\
&= h^2 \int_0^1 |f'^2 + y^3| dy \\
&\le \frac{4h^2}{27} \int_0^1 |f'(hy)|dy = \frac{4h}{27} \int_0^h |f'(z)|dz = \frac{4h}{27} \bigvee_{x_0}^{x_1} f.
\end{aligned}$$

Here we used the facts that

- $\Phi(x) = h\left[-x/h + 2(x/h)^2 - (x/h)^3\right]$ is an antiderivative of $\phi_0(x) - \phi_n(x)$ on $[0,h]$.
- $\Phi(0) = \Phi(h) = 0$.
- $y = x/h$.
- $|y - 2y^2 + y^3| \le 4/27$ on $[0,1]$.
- $z = hy$.

For $i = 1, \ldots, n-1$,

$$\begin{aligned}
\bar{w}_i =& 2\hat{f}_i - \hat{f}_{n+i} = 2\int_{x_i}^{x_{i+1}} f(x)\phi_i(x)dx - \int_{x_{i-1}}^{x_{i+1}} f(x)\phi_{n+i}(x)dx \\
&= \int_{x_i}^{x_{i+1}} f(x)\phi_i(x)dx - \int_{x_{i-1}}^{x_i} f(x)\phi_{n+i}(x)dx \\
&\quad + \int_{x_i}^{x_{i+1}} f(x)\phi_i(x)dx - \int_{x_i}^{x_{i+1}} f(x)\phi_{n+i}(x)dx \\
&= \int_{x_{i-1}}^{x_i} \left[f(x+h)\phi_i(x+h) - f(x)\phi_{n+i}(x)\right] dx \\
&\quad + \int_{x_i}^{x_{i+1}} f(x)[\phi_i(x) - \phi_{n+i}(x)]dx \\
&= \int_{x_{i-1}}^{x_i} [f(x+h) - f(x)]\phi_i(x+h)dx + \int_{x_{i-1}}^{x_i} f(x)[\phi_i(x+h) - \phi_{n+i}(x)]dx
\end{aligned}$$

$$+ \int_{x_i}^{x_{i+1}} f(x)[\phi_i(x) - \phi_{n+i}(x)]dx.$$

Note that

$$\left| \int_{x_{i-1}}^{x_i} [f(x+h) - f(x)]\phi_i(x+h)dx \right| \leq \int_{x_{i-1}}^{x_i} |f(x+h) - f(x)|\phi_i(x+h)dx \leq \frac{h}{3} \bigvee_{x_{i-1}}^{x_{i+1}} f.$$

One can verify that

$$\left| \int_{x_{i-1}}^{x_i} f(x)[\phi_i(x+h) - \phi_{n+i}(x)]dx \right| \leq \frac{4h}{27} \bigvee_{x_{i-1}}^{x_i} f$$

and

$$\left| \int_{x_i}^{x_{i+1}} f(x)[\phi_i(x) - \phi_{n+i}(x)]dx \right| \leq \frac{4h}{27} \bigvee_{x_i}^{x_{i+1}} f.$$

It follows that for $i = 1, \ldots, n-1$,

$$|\bar{w}_i| \leq \frac{h}{3} \bigvee_{x_{i-1}}^{x_{i+1}} f + \frac{4h}{27} \bigvee_{x_{i-1}}^{x_i} f + \frac{4h}{27} \bigvee_{x_i}^{x_{i+1}} f = \frac{13h}{27} \bigvee_{x_{i-1}}^{x_{i+1}} f.$$

So we have

$$||\bar{w}||_1 \leq \frac{4h}{27} \bigvee_{x_0}^{x_1} + \sum_{i=1}^{n-1} \frac{13h}{27} \bigvee_{x_{i-1}}^{x_{i+1}} f \leq \frac{26h}{27} \bigvee_0^1 f.$$

Similarly, one can show that

$$||\tilde{w}||_1 \leq \frac{26h}{27} \bigvee_0^1 f.$$

Using these two inequalities together with (4.6) we obtain the result. □

4.3. **The Norm Convergence of the Method.** It is time to prove the convergence of our projection method for the Lasota-Yorke class of piecewise C^2 and stretching interval mappings $S : [0, 1] \to [0, 1]$ with a sufficiently large stretching factor. We also assume that the corresponding Frobenius-Perron operator P_S has a unique stationary density f^*. Then from [9, 10, 11] there are two constants $\alpha = 2/\inf|S'|$ and $\beta > 0$ such that for all functions f of bounded variation

$$\bigvee_0^1 P_S f \leq \alpha \bigvee_0^1 f + \beta \|f\|_1. \tag{4.7}$$

Theorem 4.1. *If the constant α in (4.7) satisfies $\alpha < 1/327$, then the sequence $f_n \in \Delta_n$ from the piecewise quadratic projection method converges to f^* under the L^1-norm as n approaches infinity.*

Proof. Since $f_n = P_n f_n$ and $\|f_n\|_1 = 1$, by Proposition 4.2 and (4.7),

$$\bigvee_0^1 f_n = \bigvee_0^1 P_n f_n = \bigvee_0^1 Q_n P_S f_n \leq 327 \bigvee_0^1 P_S f_n$$

$$\leq 327\alpha \bigvee_0^1 f_n + 327\beta \|f_n\|_1 = 327\alpha \bigvee_0^1 f_n + 327\beta.$$

Since $327\alpha < 1$,

$$\bigvee_0^1 f_n \leq \frac{327\beta}{1 - 327\alpha}, \quad \forall\, n.$$

So Helly's lemma implies that there is a subsequence of f_n that converges to a function of norm 1 under the L^1-norm. Since f^* is the unique stationary density of P_S, a standard argument (see, e.g., [9]) ensures that the sequence f_n itself converges to f^* in norm. $\square$

Remark When the condition $\alpha < 1/327$ is not satisfied for a piecewise C^2 and stretching mapping S, we can consider some iterate S^k of S so that the condition is satisfied by S^k, so the convergence can still be proved, a strategy as used in the classic paper [12].

5. **Numerical Results**

In this section we apply our continuous piecewise quadratic projection method (PQ-PM2) to two test problems for the comparison of the performance against the previous piecewise quadratic methods, including the continuous piecewise quadratic Markov method (PQ-MM) [5] and the discontinuous piecewise quadratic projection method (PQ-PM1) [4]. We also compare our new method with the continuous piecewise linear projection method (PL-PM) [8]. We used such methods for the computation of stationary densities of two chaotic mappings of the unit interval.

The first mapping S_1 is defined by

$$S_1(x) = \begin{cases} \frac{2x}{1-x^2}, & 0 \leq x \leq \sqrt{2} - 1 \\ \frac{1-x^2}{2x}, & \sqrt{2} - 1 \leq x \leq 1 \end{cases},$$

and the stationary density of the corresponding Frobenius-Perron operator P_{S_1} is given by

$$f_1^*(x) = \frac{4}{\pi(1 + x^2)}.$$

The second mapping S_2 is defined by

$$S_2(x) = \begin{cases} \frac{2x}{1-x}, & 0 \leq x \leq \frac{1}{3} \\ \frac{1-x}{2x}, & \frac{1}{3} \leq x \leq 1 \end{cases}$$

with the corresponding stationary density

$$f_2^*(x) = \frac{2}{(1 + x)^2}.$$

In Tables 1 and 2 we present the L^1-norm errors $\|f_n - f^*\|_1$ of the computed densities f_n to the exact density f_1^* and f_2^* for S_1 and S_2, respectively.

TABLE 1. L^1-norm errors comparison for S_1.

n	PL-PM	PQ-MM	PQ-PM1	PQ-PM2
4	2.7×10^{-3}	1.2×10^{-2}	1.0×10^{-2}	4.4×10^{-4}
8	6.5×10^{-4}	5.0×10^{-3}	2.5×10^{-3}	4.6×10^{-5}
16	1.7×10^{-4}	1.9×10^{-3}	6.2×10^{-4}	6.1×10^{-6}
32	4.3×10^{-5}	6.6×10^{-4}	1.5×10^{-4}	8.2×10^{-7}
64	1.1×10^{-5}	2.1×10^{-4}	3.8×10^{-5}	3.4×10^{-7}
128	2.7×10^{-6}	6.4×10^{-5}	*	2.3×10^{-7}
256	6.4×10^{-7}	1.9×10^{-5}	*	1.8×10^{-7}

TABLE 2. L^1-norm errors comparison for S_2.

n	PL-PM	PQ-MM	PQ-PM1	PQ-PM2
4	7.7×10^{-3}	6.9×10^{-2}	3.7×10^{-2}	8.0×10^{-4}
8	1.9×10^{-3}	2.7×10^{-2}	1.0×10^{-2}	1.3×10^{-4}
16	5.4×10^{-4}	9.7×10^{-3}	2.7×10^{-3}	1.9×10^{-5}
32	1.4×10^{-4}	3.3×10^{-3}	7.8×10^{-4}	2.3×10^{-6}
64	3.6×10^{-5}	1.0×10^{-3}	1.8×10^{-4}	5.1×10^{-7}
128	8.5×10^{-6}	3.2×10^{-4}	*	2.5×10^{-7}
256	2.2×10^{-6}	9.5×10^{-5}	*	2.2×10^{-7}

With $n = 4, 8, 16$, and 32 we see that our piecewise quadratic projection method greatly outperformed the piecewise quadratic Markov method. For $n \geq 64$ numerical instability of our algorithm prevented from reaching its optimal accuracy. Also the symbol * in the tables indicates that the results were not reported in [4].

6. **Conclusions**

We have developed a projection method based on the continuous piecewise quadratic functions, and its convergence was established for the Lasota-Yorke class of interval mappings. The numerical results show a faster convergence than the current numerical Markov methods and projection methods.

More precisely speaking, our new method performs much better than the continuous piecewise linear Markov method and projection method under the same partition of the interval so that the dimension of the numerical problem is about the same. On the other hand, even though the dimension of the numerical problem in the discontinuous piecewise quadratic projection method doubles that of the continuous piecewise quadratic projection one, the latter is still considerably faster with less computational cost. This is compatable with our general belief that least squares idea usually leads to a higher order approximation accuracy than more restrictive structure-preserving approaches and higher smooth basis functions normally give better approximations when the exact solution is smooth enough.

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