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NONLOCAL FRACTIONAL SUM BOUNDARY VALUE PROBLEMS FOR MIXED TYPES OF RIEMANN-LIOUVILLE AND CAPUTO FRACTIONAL DIFFERENCE EQUATIONS

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ABSTRACT. In this article, we study an existence result for a mixed types of Riemann-Liouville and Caputo fractional difference equation with nonlocal three-point fractional sum boundary conditions, by using the Sadovskii's fixed point theorem. Our problem contains a shift-operator of fractional difference operator. Finally, we present an example to shows this result.

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1. INTRODUCTION

Fractional difference equations and discrete fractional calculus have been interested many mathematicians and researchers since they can be used for describing many real-world phenomena problems such as physics, chemistry, mechanics, control systems, flow in porous media, and electrical networks. The good papers dealing with discrete fractional boundary value problems, which has helped to build up some of the basic theory of this area, one may see the textbooks [1] and the papers [2]-[23] and references cited therein. Some real-world phenomena are being studied with the assistance of fractional difference operators, one may see the papers [24]-[25] and the references therein.

Presently, there is a development of boundary value problems for fractional difference equations that show an operation of the investigative function. The study may also have another function that is related to our interested one. These creations are incorporating with nonlocal conditions that are both extensive and more complex. Firstly, we introduce some notations, Δ_C^α is the Caputo fractional difference operator of order α, Δ^β is the Riemann-Liouville fractional difference operator of order β, $\Delta^{-\gamma}$ is the fractional sum of order γ, and the shift-operator $E_{\beta-\alpha}[u(t)] := u(t+\beta-\alpha)$.

In this paper, we consider a mixed types of the Riemann-Liouville and Caputo fractional difference equation with nonlocal three-point fractional sum boundary value

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conditions with a shift of the form

$$
(1.1)\quad \begin{cases} \Delta_C^{\alpha} u(t+\beta-1) + E_{\beta-\alpha}\left[\Delta^{\beta} g(t+\alpha-2)v(t+\alpha-2)\right] \\ \qquad = f(t+\alpha+\beta-2, u(t+\alpha+\beta-2), v(t+\alpha+\beta-2)), \quad t \in \mathbb{N}_{0,T}, \\ u(\eta) = \phi(u,v) + \Delta^{-\alpha}\Delta^{\beta} u(\eta+\beta-2), \\ \Delta^{-\gamma} u(T+\alpha+\beta+\gamma-1) = \varphi(u,v) + \Delta^{-\gamma}\Delta^{-\alpha}\Delta^{\beta} u(T+2\beta+\gamma-1), \end{cases}
$$

where $1 < \alpha \le 2$, $0 < \beta < 1$, $0 < \gamma \le 1$ and $\alpha+\beta-2 \le \eta \le T+\alpha+\beta-2$ are given constants, $f \in C\left(\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\right)$ and $g, v \in C\left(\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1}, \mathbb{R}^+\right)$ are given functions, $\phi, \varphi : C\left(\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1}, \mathbb{R}\right) \times C\left(\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1}, \mathbb{R}\right) \to \mathbb{R}$ are given functionals.

The plan of this paper is as follows. In the next section, we recall some definitions and basic lemmas. In Section 3, using this representation, we prove the existence of solutions of the boundary value problem (1.1) by the help of the Sadovskii's fixed point theorem. An example to illustrate our result is presented in the last section.

2. PRELIMINARIES

In the following, there are notations, definitions, and lemmas which are used in the main results.

Definition 2.1. The generalized falling function is defined by $t^{\underline{\alpha}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$, for any t and α for which the right-hand side is defined. If $t+1-\alpha$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\alpha}} = 0$.

Lemma 2.2 ([16]). *Assume the following factorial functions are well defined. If $t \le r$, then $t^{\underline{\alpha}} \le r^{\underline{\alpha}}$ for any $\alpha > 0$.*

Definition 2.3. For $\alpha > 0$ and f defined on $\mathbb{N}_a := \{a, a+1, \ldots\}$, the α-order fractional sum of f is defined by

$$
\Delta^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}} f(s),
$$

where $t \in \mathbb{N}_{a+\alpha}$ and $\sigma(s) = s+1$.

Definition 2.4. For $\alpha > 0$ and f defined on $\mathbb{N}_a$, the α-order Riemann-Liouville fractional difference of f is defined by

$$
\Delta^{\alpha} f(t) := \Delta^N \Delta^{-(N-\alpha)} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{t+\alpha} (t-\sigma(s))^{\underline{-\alpha-1}} f(s),
$$

where $t \in \mathbb{N}_{a+N-\alpha}$ and $N \in \mathbb{N}$ is chosen so that $0 \le N-1 < \alpha \le N$.

Definition 2.5. For $\alpha > 0$ and f defined on $\mathbb{N}_a$, the α-order Caputo fractional difference of f is defined by

$$\Delta_C^\alpha f(t) := \Delta^{-(N-\alpha)}\Delta^N f(t) = \frac{1}{\Gamma(N-\alpha)}\sum_{s=a}^{t-(N-\alpha)}(t-\sigma(s))^{\underline{N-\alpha-1}}\Delta^N f(s),$$

where $t \in \mathbb{N}_{a+N-\alpha}$ and $N \in \mathbb{N}$ is chosen so that $0 \le N-1 < \alpha < N$. If $\alpha = N$, then $\Delta_C^\alpha f(t) = \Delta^N f(t)$.

Lemma 2.6 ([12]). *Assume that $\alpha > 0$ and $0 \le N-1 < \alpha \le N$. Then*

$$\Delta^{-\alpha}\Delta_C^\alpha y(t) = y(t) + C_0 + C_1 t^{\underline{1}} + C_2 t^{\underline{2}} + \ldots + C_{N-1}t^{\underline{N-1}},$$

for some $C_i \in \mathbb{R}$, $0 \le i \le N-1$.

The following lemma deals with linear variant of the boundary value problem (1.1) and gives a representation of the solution.

Lemma 2.7. *Let $1 < \alpha \le 2$, $0 < \beta < 1$, $0 < \gamma \le 1$ and $\eta \in \mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta-2}$ be given constants, functions $h \in C(\mathbb{N}_{\alpha+\beta-2,T+\alpha+\beta-2}, \mathbb{R})$, $g, v \in C(\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1}, \mathbb{R}^+)$ and functionals $\phi, \varphi : C(\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1}, \mathbb{R}) \times C(\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1}, \mathbb{R}) \to \mathbb{R}$ be given. Then the problem*

$$(2.1)\quad \begin{cases} \Delta_C^\alpha u(t+\beta-1) + E_{\beta-\alpha}\left[\Delta^\beta g(t+\alpha-2)v(t+\alpha-2)\right] = h(t+\alpha+\beta-2), \\ u(\eta) = \phi(u,v) + \Delta^{-\alpha}\Delta^\beta u(\eta+\beta-2), \\ \Delta^{-\gamma}u(T+\alpha+\beta+\gamma-1) = \varphi(u,v) + \Delta^{-\gamma}\Delta^{-\alpha}\Delta^\beta u(T+2\beta+\gamma-1) \end{cases}$$

has the unique solution

$$\begin{aligned} u(t) = {} & \frac{1}{\Lambda\Gamma(\gamma)}\sum_{s=\alpha+\beta-3}^{T+\alpha+\beta-1}(T+\alpha+\beta+\gamma-1-\sigma(s))^{\underline{\gamma-1}}\left(s-t\right) \times \\ & \left[\phi(u,v) + \mathcal{A}(u) + \mathcal{P}(v) - \frac{1}{\Gamma(\alpha)}\sum_{s=\alpha+\beta-2}^{\eta-1}(\eta+\alpha-1-\sigma(s))^{\underline{\alpha-1}}h(s)\right] \\ & + \left(\frac{t-\eta}{\Lambda}\right)\left[\varphi(u,v) + \mathcal{B}(u) + \mathcal{Q}(v) - \frac{1}{\Gamma(\gamma)\Gamma(\alpha)} \times \right. \\ & \left. \sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=\alpha+\beta-2}^{r-1}(T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}}(r+\alpha-1-\sigma(s))^{\underline{\alpha-1}}h(s)\right] \\ & + \frac{1}{\Gamma(\alpha)}\sum_{s=\alpha+\beta-2}^{t-1}(t+\alpha-1-\sigma(s))^{\underline{\alpha-1}}h(s) - \frac{1}{\Gamma(\alpha)\Gamma(-\beta)} \times \\ & \sum_{s=0}^{t-\alpha-\beta+1}(t-\beta+1-\sigma(s))^{\underline{\alpha-1}}E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-1}(s+\alpha-1-\sigma(\xi))^{\underline{-\beta-1}}g(\xi)v(\xi)\right], \end{aligned} \quad (2.2)$$

where

$$\begin{aligned}\Lambda &= \frac{1}{\Gamma(\gamma)}\sum_{s=\alpha+\beta-3}^{T+\alpha+\beta-1}(T+\alpha+\beta+\gamma-1-\sigma(s))^{\underline{\gamma-1}}\big(s-\eta\big)\\ &= \frac{\big[T+2-(\eta-\alpha-\beta+3)(\gamma+1)\big]\Gamma(T+\gamma+3)}{\Gamma(\gamma+2)\Gamma(T+3)}, \end{aligned} \tag{2.3}$$

$$\begin{aligned}\mathcal{P}(v) &= \frac{1}{\Gamma(\alpha)\Gamma(-\beta)}\sum_{s=0}^{\eta-\alpha-\beta+1}(\eta-\beta+1-\sigma(s))^{\underline{\alpha-1}}\times\\ &\quad E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2}(s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}}g(\xi)v(\xi)\right],\end{aligned} \tag{2.4}$$

$$\begin{aligned}\mathcal{Q}(v) &= \frac{1}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(-\beta)}\sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=0}^{r-\alpha-\beta+1}(T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}}\times\\ &\quad (r-\beta+1-\sigma(s))^{\underline{\alpha-1}}E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2}(s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}}g(\xi)v(\xi)\right],\end{aligned} \tag{2.5}$$

$$\begin{aligned}\mathcal{A}(u) =& \frac{1}{\Theta}\Bigg\{\big[1+\mathcal{P}_B\big]\Bigg[\mathcal{Q}_A\Bigg(\phi(u,v)+\mathcal{P}(v)-\sum_{s=\alpha+\beta-2}^{\eta-1}\frac{(\eta+\alpha-1-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)}h(s)\Bigg)\\ &+\mathcal{P}_A\Bigg(\varphi(u,v)+\mathcal{Q}(v)-\sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=\alpha+\beta-2}^{r-1}\frac{(T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}}}{\Gamma(\gamma)}\times\\ &\frac{(r+\alpha-1-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)}h(s)\Bigg)+\mathcal{R}_A\Bigg]+\mathcal{P}_A\Bigg[\mathcal{Q}_B\Bigg(\phi(u,v)+\mathcal{P}(v)-\frac{1}{\Gamma(\alpha)}\times\\ &\sum_{s=\alpha+\beta-2}^{\eta-1}(\eta+\alpha-1-\sigma(s))^{\underline{\alpha-1}}h(s)\Bigg)+\mathcal{P}_B\Bigg(\varphi(u,v)+\mathcal{Q}(v)-\frac{1}{\Gamma(\gamma)\Gamma(\alpha)}\times\\ &\sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=\alpha+\beta-2}^{r-1}(T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}}\times\\ &(r+\alpha-1-\sigma(s))^{\underline{\alpha-1}}h(s)\Bigg)+\mathcal{R}_B\Bigg]\Bigg\},\end{aligned} \tag{2.6}$$

$$\begin{aligned}\mathcal{B}(u) =& \frac{1}{\Theta}\Bigg\{\mathcal{Q}_B\Bigg[\mathcal{Q}_A\Bigg(\phi(u,v)+\mathcal{P}(v)-\frac{1}{\Gamma(\alpha)}\sum_{s=\alpha+\beta-2}^{\eta-1}(\eta+\alpha-1-\sigma(s))^{\underline{\alpha-1}}h(s)\Bigg)\\ &+\mathcal{P}_A\Bigg(\varphi(u,v)+\mathcal{Q}(v)-\sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=\alpha+\beta-2}^{r-1}\frac{(T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}}}{\Gamma(\gamma)}\times\\ &\frac{(r+\alpha-1-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)}h(s)\Bigg)+\mathcal{R}_A\Bigg]\end{aligned}$$

$$+\left[1+\mathcal{Q}_A\right]\Bigg[\mathcal{Q}_B\Bigg(\phi(u,v)+\mathcal{P}(v)-\frac{1}{\Gamma(\alpha)}\sum_{s=\alpha+\beta-2}^{\eta-1}(\eta+\alpha-1-\sigma(s))^{\underline{\alpha-1}}h(s)\Bigg)$$
$$+\mathcal{P}_B\Bigg(\varphi(u,v)+\mathcal{Q}(v)-\frac{1}{\Gamma(\gamma)\Gamma(\alpha)}\times$$
$$\sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=\alpha+\beta-2}^{r-1}(T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}}\times$$
$$(r+\alpha-1-\sigma(s))^{\underline{\alpha-1}}h(s)\Bigg)+\mathcal{R}_B\Bigg]\Bigg\},$$

(2.7)

and

$$\mathcal{P}_A=\sum_{s=0}^{\eta-\alpha-\beta+1}\frac{(\eta-\beta+1-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)\Gamma(-\beta)}E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2}(s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}}\left(\frac{\xi-\eta}{\Lambda}\right)\right],$$

$$\mathcal{P}_B=\frac{1}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(-\beta)}\sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=0}^{r-\alpha-\beta+1}(T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}}\times$$
$$(r-\beta+1-\sigma(s))^{\underline{\alpha-1}}E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-1}(s+\alpha-1-\sigma(\xi))^{\underline{-\beta-1}}\left(\frac{\xi-\eta}{\Lambda}\right)\right],$$

$$\mathcal{Q}_A=\sum_{s=0}^{\eta-\alpha-\beta+1}\frac{(\eta-\beta+1-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)\Gamma(-\beta)}E_{\beta-\alpha}\left[\sum_{\theta=\alpha+\beta-3}^{s+\alpha+\beta-2}(s+\alpha-2-\sigma(\theta))^{\underline{-\beta-1}}\Phi(\theta-\xi)\right]$$

$$\mathcal{Q}_B=\frac{1}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(-\beta)}\sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=0}^{r-\alpha-\beta+1}(T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}}\times$$
$$(r-\beta+1-\sigma(s))^{\underline{\alpha-1}}E_{\beta-\alpha}\left[\sum_{\theta=\alpha+\beta-3}^{s+\alpha+\beta-2}(s+\alpha-2-\sigma(\theta))^{\underline{-\beta-1}}\Phi(\theta-\xi)\right],$$

$$\mathcal{R}_A=\sum_{s=0}^{\eta-\alpha-\beta+1}\frac{(\eta-\beta+1-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)\Gamma(-\beta)}\times$$
$$E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2}(s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}}\big(\Psi(\xi)-\Upsilon(\xi)\big)\right],$$

$$\mathcal{R}_B=\frac{1}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(-\beta)}\sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=0}^{r-\alpha-\beta+1}(T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}}\times$$

$$(r-\beta+1-\sigma(s))^{\underline{\alpha-1}}E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2}(s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}}\big(\Psi(\xi)-\Upsilon(\xi)\big)\right],$$

$$\Theta=\big[1-\mathcal{Q}_A\big]\big[1-\mathcal{P}_B\big]-\mathcal{P}_A\mathcal{Q}_B,$$

$$\Phi(\theta-\xi)=\frac{1}{\Lambda\Gamma(\gamma)}\sum_{\theta=\alpha+\beta-3}^{T+\alpha+\beta-1}(T+\alpha+\beta+\gamma-1-\sigma(s))^{\underline{\gamma-1}}\big(\theta-\xi\big),$$

$$\mathcal{I}(\eta)=\frac{1}{\Gamma(\alpha)}\sum_{s=\alpha+\beta-2}^{\eta-1}(\eta+\alpha-1-\sigma(s))^{\underline{\alpha-1}}h(s),$$

$$\mathcal{J}(T)=\sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=\alpha+\beta-2}^{r-1}\frac{(T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}}(r+\alpha-1-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\gamma)\Gamma(\alpha)}h(s),$$

$$\Psi(\xi)=\frac{1}{\Gamma(\alpha)}\sum_{s=\alpha+\beta-2}^{\xi-1}(\xi+\alpha-1-\sigma(s))^{\underline{\alpha-1}}h(s),$$

$$\Upsilon(\xi)=\sum_{s=0}^{\xi-\alpha-\beta+1}\frac{(\xi-\beta+1-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)\Gamma(-\beta)}E_{\beta-\alpha}\times \left[\sum_{\tau=\alpha+\beta-3}^{s+\alpha+\beta-2}(s+\alpha-2-\sigma(\tau))^{\underline{-\beta-1}}g(\tau)v(\tau)\right].$$

Proof. Using lemma 2.6 and the fractional sum of order $1<\alpha\le 2$ for (2.1), we obtain

$$u(t+\beta-1)=C_1+C_2t+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}h(s+\alpha+\beta-2)-\frac{1}{\Gamma(\alpha)\Gamma(-\beta)}\times$$

$$\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}{}_sE_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2}(s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}}g(\xi)v(\xi)\right], \tag{2.8}$$

for $t\in\mathbb{N}_{\alpha-2,T+\alpha}$.

Changing the variable from $t+\beta-1$ to t, we have

$$\begin{aligned}u(t)&=C_1+C_2(t-\beta+1)+\frac{1}{\Gamma(\alpha)}\sum_{s=\alpha+\beta-2}^{t-1}(t+\alpha-1-\sigma(s))^{\underline{\alpha-1}}h(s)\\&\quad-\frac{1}{\Gamma(\alpha)\Gamma(-\beta)}\sum_{s=0}^{t-\alpha-\beta+1}(t-\beta+1-\sigma(s))^{\underline{\alpha-1}}\times\\&\quad E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2}(s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}}g(\xi)v(\xi)\right],\ t\in\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1}.\end{aligned} \tag{2.9}$$

Applying the first boundary condition of (2.1) implies

$$C_1+(\eta-\beta+1)C_2=\phi(u,v)-\frac{1}{\Gamma(\alpha)}\sum_{s=\alpha+\beta-2}^{\eta-1}(\eta+\alpha-1-\sigma(s))^{\underline{\alpha-1}}h(s)$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(-\beta)}\sum_{s=0}^{\eta-\alpha-\beta+1}(\eta-\beta+1-\sigma(s))^{\underline{\alpha-1}}\times$$

$$E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2}(s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}}\Big(u(\xi)+g(\xi)v(\xi)\Big)\right]. \tag{2.10}$$

Taking the fractional sum of order $0<\gamma\le 1$ for (2.9), we obtain

$$\begin{aligned}\Delta^{-\gamma}u(t) &= \frac{C_1}{\Gamma(\gamma)}\sum_{s=\alpha+\beta-3}^{t-\gamma}(t-\sigma(s))^{\underline{\gamma-1}}+\frac{C_2}{\Gamma(\gamma)}\sum_{s=\alpha+\beta-3}^{t-\gamma}(t-\sigma(s))^{\underline{\gamma-1}}(s-\beta+1)\\ &+\frac{1}{\Gamma(\gamma)\Gamma(\alpha)}\sum_{s=\alpha+\beta-3}^{t-\gamma}\sum_{\xi=\alpha+\beta-2}^{s-1}(t-\sigma(s))^{\underline{\gamma-1}}(s+\alpha-1-\sigma(\xi))^{\underline{\alpha-1}}h(\xi)\\ &-\frac{1}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(-\beta)}\sum_{r=\alpha+\beta-1}^{t-\gamma}\sum_{s=0}^{r-\alpha-\beta+1}(t-\sigma(s))^{\underline{\gamma-1}}(r-\beta+1-\sigma(s))^{\underline{\alpha-1}}\times\end{aligned}$$

$$E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2}(s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}}g(\xi)v(\xi)\right]. \tag{2.11}$$

The second condition of (2.1) implies

$$\begin{aligned}&\frac{C_1}{\Gamma(\gamma)}\sum_{s=\alpha+\beta-3}^{T+\alpha+\beta-1}(T+\alpha+\beta+\gamma-1-\sigma(s))^{\underline{\gamma-1}}\\ &+\frac{C_2}{\Gamma(\gamma)}\sum_{s=\alpha+\beta-3}^{T+\alpha+\beta-1}(T+\alpha+\beta+\gamma-1-\sigma(s))^{\underline{\gamma-1}}(s-\beta+1)\\ =\ &\varphi(u,v)-\frac{1}{\Gamma(\gamma)\Gamma(\alpha)}\sum_{s=\alpha+\beta-3}^{T+\alpha+\beta-1}\sum_{\xi=\alpha+\beta-2}^{s-1}(T+\alpha+\beta+\gamma-1-\sigma(s))^{\underline{\gamma-1}}\times\\ &(s+\alpha-1-\sigma(\xi))^{\underline{\alpha-1}}h(\xi)+\frac{1}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(-\beta)}\times\\ &\sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=0}^{r-\alpha-\beta+1}(T+\alpha+\beta+\gamma-1-\sigma(s))^{\underline{\gamma-1}}(r-\beta+1-\sigma(s))^{\underline{\alpha-1}}\times\end{aligned}$$

$$E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2}(s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}}\Big(u(\xi)+g(\xi)v(\xi)\Big)\right]. \tag{2.12}$$

The constants C_1, C_2 can be obtained by solving the system of equations (2.10) and (2.12), so

$$\begin{aligned}C_1 &= \frac{1}{\Lambda\Gamma(\gamma)}\sum_{s=\alpha+\beta-3}^{T+\alpha+\beta-1}(T+\alpha+\beta+\gamma-1-\sigma(s))^{\underline{\gamma-1}}\big(s-\beta+1\big)\times\\ &\Bigg\{\phi(u,v)+\frac{1}{\Gamma(\alpha)\Gamma(-\beta)}\sum_{s=0}^{\eta-\alpha-\beta+1}(\eta-\beta+1-\sigma(s))^{\underline{\alpha-1}}\times\end{aligned}$$

$$E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2}(s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}}\Big(u(\xi)+g(\xi)v(\xi)\Big)\right]$$

$$-\frac{1}{\Gamma(\alpha)}\sum_{s=\alpha+\beta-2}^{\eta-1}(\eta+\alpha-1-\sigma(s))^{\underline{\alpha-1}}h(s)\Bigg\}-\left(\frac{\eta-\beta+1}{\Lambda}\right)\times$$

$$\Bigg\{\varphi(u,v)+\frac{1}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(-\beta)}\sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=0}^{r-\alpha-\beta+1}(T+\alpha+\beta+\gamma-1-\sigma(s))^{\underline{\gamma-1}}\times$$

$$(r-\beta+1-\sigma(s))^{\underline{\alpha-1}}\times$$

$$E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2}(s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}}\Big(u(\xi)+g(\xi)v(\xi)\Big)\right]$$

$$-\sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=\alpha+\beta-2}^{r-1}\frac{(T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}}(r+\alpha-1-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\gamma)\Gamma(\alpha)}h(s)\Bigg\},\tag{2.13}$$

and

$$\begin{aligned}C_2 \;=\; & \frac{1}{\Lambda}\Bigg\{\varphi(u,v)+\frac{1}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(-\beta)}\sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=0}^{r-\alpha-\beta+1}(T+\alpha+\beta+\gamma-1-\sigma(s))^{\underline{\gamma-1}}\times\\
& (r-\beta+1-\sigma(s))^{\underline{\alpha-1}}E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2}(s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}}\Big(u(\xi)+g(\xi)v(\xi)\Big)\right]\\
& -\frac{1}{\Gamma(\gamma)\Gamma(\alpha)}\sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=\alpha+\beta-2}^{r-1}(T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}}\times\\
& (r+\alpha-1-\sigma(s))^{\underline{\alpha-1}}h(s)\Bigg\}-\frac{1}{\Lambda\Gamma(\gamma)}\sum_{s=\alpha+\beta-3}^{T+\alpha+\beta-1}(T+\alpha+\beta+\gamma-1-\sigma(s))^{\underline{\gamma-1}}\times\\
& \Bigg\{\phi(u,v)+\frac{1}{\Gamma(\alpha)\Gamma(-\beta)}\sum_{s=0}^{\eta-\alpha-\beta+1}(\eta-\beta+1-\sigma(s))^{\underline{\alpha-1}}\times\\
& E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2}(s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}}\Big(u(\xi)+g(\xi)v(\xi)\Big)\right]\\
& -\frac{1}{\Gamma(\alpha)}\sum_{s=\alpha+\beta-2}^{\eta-1}(\eta+\alpha-1-\sigma(s))^{\underline{\alpha-1}}h(s)\Bigg\},\end{aligned}\tag{2.14}$$

where Λ is defined on (2.3).

Substituting the constants C_1, C_2 into (2.9), and let

$$\mathcal{A}(u) = \sum_{s=0}^{\eta-\alpha-\beta+1} \frac{(\eta-\beta+1-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)\Gamma(-\beta)} E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2} (s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}} u(\xi)\right],$$

$$\mathcal{B}(u) = \frac{1}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(-\beta)} \sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1} \sum_{s=0}^{r-\alpha-\beta+1} (T+\alpha+\beta+\gamma-1-\sigma(s))^{\underline{\gamma-1}} \times (r-\beta+1-\sigma(s))^{\underline{\alpha-1}} E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2} (s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}} u(\xi)\right],$$

then we have

$$\begin{aligned}
&\mathcal{A}(u) \\
&= \sum_{s=0}^{\eta-\alpha-\beta+1} \frac{(\eta-\beta+1-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)\Gamma(-\beta)} E_{\beta-\alpha} \sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2} (s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}} \times \\
&\Bigg\{ \frac{1}{\Lambda\Gamma(\gamma)} \sum_{\theta=\alpha+\beta-3}^{T+\alpha+\beta-1} (T+\alpha+\beta+\gamma-1-\sigma(\theta))^{\underline{\gamma-1}} \times \\
&\big(\theta-\xi\big)\left[\phi(u,v)+\mathcal{A}(u)+\mathcal{P}(v) - \frac{1}{\Gamma(\alpha)} \sum_{\theta=\alpha+\beta-2}^{\eta-1} (\eta+\alpha-1-\sigma(\theta))^{\underline{\alpha-1}} h(\theta)\right] \\
&+ \left(\frac{\xi-\eta}{\Lambda}\right)\Bigg[\varphi(u,v)+\mathcal{B}(u)+\mathcal{Q}(v) - \frac{1}{\Gamma(\gamma)\Gamma(\alpha)} \times \\
&\sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1} \sum_{\theta=\alpha+\beta-2}^{r-1} (T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}} (r+\alpha-1-\sigma(\theta))^{\underline{\alpha-1}} h(\theta)\Bigg] \\
&+ \sum_{\theta=\alpha+\beta-2}^{\xi-1} \frac{(\xi+\alpha-1-\sigma(\theta))^{\underline{\alpha-1}}}{\Gamma(\alpha)} h(\theta) - \sum_{\theta=0}^{\xi-\alpha-\beta+1} \frac{(\xi-\beta+1-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)\Gamma(-\beta)} \times \\
&E_{\beta-\alpha}\left[\sum_{\tau=\alpha+\beta-3}^{\theta+\alpha+\beta-2} (\theta+\alpha-2-\sigma(\tau))^{\underline{-\beta-1}} g(\tau) v(\tau)\right] \Bigg\},
\end{aligned} \tag{2.15}$$

and $\mathcal{B}(u)$

$$\begin{aligned}
&= \frac{1}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(-\beta)} \sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1} \sum_{s=0}^{r-\alpha-\beta+1} (T+\alpha+\beta+\gamma-1-\sigma(s))^{\underline{\gamma-1}} \times \\
&(r-\beta+1-\sigma(s))^{\underline{\alpha-1}} E_{\beta-\alpha} \sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2} (s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}} \Bigg\{ \frac{1}{\Lambda\Gamma(\gamma)} \times
\end{aligned}$$

$$\sum_{\theta=\alpha+\beta-3}^{T+\alpha+\beta-1}(T+\alpha+\beta+\gamma-1-\sigma(\theta))^{\underline{\gamma-1}}\left(\theta-\xi\right)\Bigg[\phi(u,v)+\mathcal{A}(u)+\mathcal{P}(v)$$

$$-\frac{1}{\Gamma(\alpha)}\sum_{\theta=\alpha+\beta-2}^{\eta-1}(\eta+\alpha-1-\sigma(\theta))^{\underline{\alpha-1}}h(\theta)\Bigg]+\left(\frac{\xi-\eta}{\Lambda}\right)\Bigg[\varphi(u,v)+\mathcal{B}(u)+\mathcal{Q}(v)$$

$$-\sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{\theta=\alpha+\beta-2}^{r-1}\frac{(T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}}}{\Gamma(\gamma)\Gamma(\alpha)}(r+\alpha-1-\sigma(\theta))^{\underline{\alpha-1}}h(\theta)\Bigg]$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{\theta=\alpha+\beta-2}^{\xi-1}(\xi+\alpha-1-\sigma(\theta))^{\underline{\alpha-1}}h(\theta)-\sum_{\theta=0}^{\xi-\alpha-\beta+1}\frac{(\xi-\beta+1-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)\Gamma(-\beta)}\times$$

$$E_{\beta-\alpha}\left[\sum_{\tau=\alpha+\beta-3}^{\theta+\alpha+\beta-2}(\theta+\alpha-2-\sigma(\tau))^{\underline{-\beta-1}}g(\tau)v(\tau)\right]\Bigg\}. \tag{2.16}$$

We simplify (2.15)–(2.16) into (2.6)–(2.7), respectively. Finally, substituting $\mathcal{A}(u)$ and $\mathcal{B}(u)$ into (2.2). This complete the proof. □

In the following, we give the existence result for problem (1.1), by the help of the Sadovskii's fixed point theorem.

Definition 2.8. Let M be a bounded set in metric space $(X;d)$; then Kuratowskii measure of noncompactness, $\alpha(M)$ is defined as

$$\inf\{\epsilon : \text{M covered by a finitely many sets such that the diameter of each set} \leq \epsilon\}.$$

Definition 2.9. Let $\Phi : D(\Phi) \subseteq X \to X$ be a bounded and continuous operator on Banach space X. Then Φ is called a condensing map if $\alpha(\Phi(B)) < \alpha(B)$ for all bounded sets $B \subset D(\Phi)$, where α denotes the Kuratowski measure of noncompactness.

Lemma 2.10 ([27])**.** *The map $K + C$ is a k-set contraction with $0 \leq k < 1$, and thus also condensing, if*

(i) *$K, C : D \subseteq X \to X$ are operators on the Banach space X,*
(ii) *K is k-contractive, i.e., $\|Kx - Ky\| \leq k\|x - y\|$ for all $x, y \in D$,*
(iii) *C is compact.*

Theorem 2.11 (the Sadovskii's fixed point theorem)**.** [26] *Let B be a convex, bounded and closed subset of a Banach space X and $\Phi : B \to B$ be a condensing map. Then Φ has a fixed point.*

3. MAIN RESULT

Now, we wish to establish the existence result for problem (1.1). To accomplish this, we denote $\mathcal{C} = C(\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1}, \mathbb{R})$, the Banach space of all function u with the norm defined by

$$\|u\|_{\mathcal{C}} = \|u\| + \|v\|,$$

where $\|u\| = \max_{u\in\mathcal{C}} |u(t)|$ and $\|v\| = \max_{v\in\mathcal{C}} |v(t)|$ of any given function v. Also define an operator $\mathcal{F}:\mathcal{C}\to\mathcal{C}$ by

$$(\mathcal{F}u)(t) = (\mathcal{F}_1u)(t) + (\mathcal{F}_2u)(t), \tag{3.1}$$

$$\begin{aligned}(\mathcal{F}_1u)(t) = {} & \frac{1}{\Lambda\Gamma(\gamma)}\sum_{s=\alpha+\beta-3}^{T+\alpha+\beta-1}(T+\alpha+\beta+\gamma-1-\sigma(s))^{\underline{\gamma-1}}\big(s-t\big)\Bigg[\phi(u,v)+\widehat{\mathcal{A}}(u)\\ & +\mathcal{P}(v)-\frac{1}{\Gamma(\alpha)}\sum_{s=\alpha+\beta-2}^{\eta-1}(\eta+\alpha-1-\sigma(s))^{\underline{\alpha-1}}f(s,u(s),v(s))\Bigg]\\ & +\left(\frac{t-\eta}{\Lambda}\right)\Bigg[\varphi(u,v)+\widehat{\mathcal{B}}(u)+\mathcal{Q}(v)-\frac{1}{\Gamma(\gamma)\Gamma(\alpha)}\times\\ & \sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=\alpha+\beta-2}^{r-1}(T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}}\times\\ & (r+\alpha-1-\sigma(s))^{\underline{\alpha-1}}f(s,u(s),v(s))\Bigg],\end{aligned} \tag{3.2}$$

$$\begin{aligned}(\mathcal{F}_2u)(t) = {} & \frac{1}{\Gamma(\alpha)}\sum_{s=\alpha+\beta-2}^{t-1}(t+\alpha-1-\sigma(s))^{\underline{\alpha-1}}f(s,u(s),v(s))\\ & -\sum_{s=0}^{t-\alpha-\beta+1}\frac{(t-\beta+1-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)\Gamma(-\beta)}\times\\ & E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2}(s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}}g(\xi)v(\xi)\right],\end{aligned} \tag{3.3}$$

where $\Lambda, \mathcal{P}(v), \mathcal{Q}(v)$ are defined on (2.3) – (2.5), respectively,

$$\begin{aligned}\widehat{\mathcal{A}}(u) = {} & \frac{1}{\Theta}\Bigg\{\big(1-\mathcal{P}_B\big)\Bigg[\mathcal{Q}_A\Bigg(\phi(u,v)+\mathcal{P}(v)-\frac{1}{\Gamma(\alpha)}\sum_{s=\alpha+\beta-2}^{\eta-1}(\eta+\alpha-1-\sigma(s))^{\underline{\alpha-1}}\times\\ & f(s,u(s),v(s))\Bigg)+\mathcal{P}_A\Bigg(\varphi(u,v)+\mathcal{Q}(v)-\frac{1}{\Gamma(\gamma)\Gamma(\alpha)}\times\\ & \sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=\alpha+\beta-2}^{r-1}(T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}}(r+\alpha-1-\sigma(s))^{\underline{\alpha-1}}\times\\ & f(s,u(s),v(s))\Bigg)+\mathcal{R}_A\Bigg]+\mathcal{P}_A\Bigg[\mathcal{Q}_B\Bigg(\phi(u,v)+\mathcal{P}(v)-\frac{1}{\Gamma(\alpha)}\times\\ & \sum_{s=\alpha+\beta-2}^{\eta-1}(\eta+\alpha-1-\sigma(s))^{\underline{\alpha-1}}f(s,u(s),v(s))\Bigg)+\mathcal{P}_B\Bigg(\varphi(u,v)+\mathcal{Q}(v)\end{aligned}$$

$$-\frac{1}{\Gamma(\gamma)\Gamma(\alpha)}\sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=\alpha+\beta-2}^{r-1}(T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}}\times$$

$$(r+\alpha-1-\sigma(s))^{\underline{\alpha-1}}f(s,u(s),v(s))\Bigg)+\mathcal{R}_B\Bigg]\Bigg\}, \tag{3.4}$$

$$\begin{aligned}\widehat{\mathcal{B}}(u) = {} & \frac{1}{\Theta}\Bigg\{\mathcal{Q}_B\Bigg[\mathcal{Q}_A\Bigg(\phi(u,v)+\mathcal{P}(v)-\frac{1}{\Gamma(\alpha)}\sum_{s=\alpha+\beta-2}^{\eta-1}(\eta+\alpha-1-\sigma(s))^{\underline{\alpha-1}}\times \\ & f(s,u(s),v(s))\Bigg)+\mathcal{P}_A\Bigg(\varphi(u,v)+\mathcal{Q}(v)-\frac{1}{\Gamma(\gamma)\Gamma(\alpha)}\times \\ & \sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=\alpha+\beta-2}^{r-1}(T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}}(r+\alpha-1-\sigma(s))^{\underline{\alpha-1}}\times \\ & f(s,u(s),v(s))\Bigg)+\mathcal{R}_A\Bigg]+\big(1-\mathcal{Q}_A\big)\Bigg[\mathcal{Q}_B\Bigg(\phi(u,v)+\mathcal{P}(v)-\frac{1}{\Gamma(\alpha)}\times \\ & \sum_{s=\alpha+\beta-2}^{\eta-1}(\eta+\alpha-1-\sigma(s))^{\underline{\alpha-1}}f(s,u(s),v(s))\Bigg)+\mathcal{P}_B\Bigg(\varphi(u,v)+\mathcal{Q}(v) \\ & -\frac{1}{\Gamma(\gamma)\Gamma(\alpha)}\sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=\alpha+\beta-2}^{r-1}(T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}}\times \\ & (r+\alpha-1-\sigma(s))^{\underline{\alpha-1}}f(s,u(s),v(s))\Bigg)+\mathcal{R}_B\Bigg]\Bigg\},\end{aligned} \tag{3.5}$$

and

$$\widehat{\mathcal{I}}(\eta) = \frac{1}{\Gamma(\alpha)}\sum_{s=\alpha+\beta-2}^{\eta-1}(\eta+\alpha-1-\sigma(s))^{\underline{\alpha-1}}f(s,u(s),v(s)), \tag{3.6}$$

$$\begin{aligned}\widehat{\mathcal{J}}(T) = {} & \frac{1}{\Gamma(\gamma)\Gamma(\alpha)}\sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=\alpha+\beta-2}^{r-1}(T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}}\times \\ & (r+\alpha-1-\sigma(s))^{\underline{\alpha-1}}f(s,u(s),v(s)),\end{aligned} \tag{3.7}$$

$$\widehat{\Psi}(\xi) = \frac{1}{\Gamma(\alpha)}\sum_{s=\alpha+\beta-2}^{\xi-1}(\xi+\alpha-1-\sigma(s))^{\underline{\alpha-1}}f(s,u(s),v(s)), \tag{3.8}$$

$$\begin{aligned}\widehat{\mathcal{R}}_A = {} & \frac{1}{\Gamma(\alpha)\Gamma(-\beta)}\sum_{s=0}^{\eta-\alpha-\beta+1}(\eta-\beta+1-\sigma(s))^{\underline{\alpha-1}}\times \\ & E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2}(s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}}\big(\widehat{\Psi}(\xi)-\Upsilon(\xi)\big)\right]\end{aligned} \tag{3.9}$$

$$\widehat{\mathcal{R}}_B = \frac{1}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(-\beta)}\sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=0}^{r-\alpha-\beta+1}(T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}}\times$$

$$(3.10)\ (r-\beta+1-\sigma(s))^{\underline{\alpha-1}}\,E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-1}(s+\alpha-1-\sigma(\xi))^{\underline{-\beta-1}}\left(\widehat{\Psi}(\xi)-\Upsilon(\xi)\right)\right],$$

with $\mathcal{P}_A$, $\mathcal{P}_B$, $\mathcal{Q}_A$, $\mathcal{Q}_B$, Θ, $\Phi(\theta-\xi)$ and $\Upsilon(\xi)$ are defined as Lemma 2.7. The problem (1.1) has solutions if and only if the operator F has fixed points.

Theorem 3.1. *Assume that* $f\in C\big(\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1}\times\mathbb{R},\mathbb{R}\big)$, $g,v\in C\big(\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1},\mathbb{R}^+\big)$ *and* $\phi,\varphi: C\left(\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1},\mathbb{R}\right)\times C\left(\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1},\mathbb{R}\right)\to\mathbb{R}$ *are given. In addition, suppose the following:*

(H_1) *There exist constants* $L_1, L_2>0$ *such that*

$$|f(t,u_1(t),v(t)-f(t,u_2(t),v(t)|\le L_1\,|u_1-u_2|+L_2\,|v|,$$

for each $t\in\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1}$, $u_1,u_2\in\mathcal{C}$.

(H_2) *There exist constant* $\mu_1,\mu_2>0$ *such that*

$$|\phi(u_1,v)-\phi(u_2,v)|\le\mu_1\,|u_1-u_2|+\mu_2\,|v|,$$

for each $u_1,u_2\in\mathcal{C}$.

(H_3) *There exist constant* $\lambda_1,\lambda_2>0$ *such that*

$$|\varphi(u_1,v_1)-\varphi(u_2,v_2)|\le\lambda_1\,|u_1-u_2|+\lambda_2\,|v|,$$

for each $u_1,u_2\in\mathcal{C}$.

(H_4) $0<g(t)<G$ *for each* $t\in\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1}$.

Then problem (1.1) *has at least one solution on* $\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1}$ *provided that*

$$\chi:=L\Omega_1+G\Omega_2+\mu\Omega_3+\lambda\Omega_4\;<\;1, \tag{3.11}$$

where

$$L\;:=\;\max\big\{L_1,L_2\big\}, \tag{3.12}$$

$$\mu\;:=\;\max\big\{\mu_1,\mu_2\big\}, \tag{3.13}$$

$$\lambda\;:=\;\max\big\{\lambda_1,\lambda_2\big\}, \tag{3.14}$$

$$\begin{aligned}\Omega_1\;:=\;&(T+2)|\Phi|\Bigg[|\mathcal{I}|+\frac{\big||\mathcal{P}_B|-1\big|}{|\Theta|}\Big(|\mathcal{Q}_A|\,|\mathcal{I}|+|\mathcal{P}_A|\,|\mathcal{J}|+|\Psi_A|\,|\mathcal{A}|\Big)\\&+\frac{|\mathcal{P}_A|}{|\Theta|}\Big(|\mathcal{Q}_B|\,|\mathcal{I}|+|\mathcal{P}_B|\,|\mathcal{J}|+|\Psi_B|\,|\mathcal{B}|\Big)\Bigg]\\&+\left(\frac{T+\alpha+\beta-1-\eta}{|\Lambda|}\right)\Bigg[|\mathcal{I}|+\frac{\mathcal{Q}_B}{|\Theta|}\Big(|\mathcal{Q}_A|\,|\mathcal{I}|+|\mathcal{P}_A|\,|\mathcal{J}|+|\Psi_A|\,|\mathcal{A}|\Big)\\&+\frac{\big||\mathcal{Q}_A|-1\big|}{|\Theta|}\Big(|\mathcal{Q}_B|\,|\mathcal{I}|+|\mathcal{P}_B|\,|\mathcal{J}|+|\Psi_B|\,|\mathcal{B}|\Big)\Bigg],\end{aligned} \tag{3.15}$$

$$\Omega_2 := (T+2)|\Phi|\left[\frac{\big||\mathcal{P}_B|-1\big|}{|\Theta|}|\Upsilon_A||\mathcal{A}|+\frac{|\mathcal{P}_A|}{|\Theta|}|\Upsilon_B||\mathcal{B}|+|\mathcal{A}|\right]$$

$$+\left(\frac{T+\alpha+\beta-1-\eta}{|\Lambda|}\right)\left[\frac{|\mathcal{Q}_B|}{|\Theta|}|\Upsilon_A||\mathcal{A}|+\frac{\big||\mathcal{Q}_A|-1\big|}{|\Theta|}|\Upsilon_B||\mathcal{B}|+|\mathcal{B}|\right], \tag{3.16}$$

$$\Omega_3 := (T+2)|\Phi|\left[1+\frac{\big||\mathcal{P}_B|-1\big|}{|\Theta|}|\mathcal{Q}_A|+\frac{|\mathcal{P}_A|}{|\Theta|}|\mathcal{Q}_B|\right]$$

$$+\left(\frac{T+\alpha+\beta-1-\eta}{|\Lambda|}\right)\left[\frac{\big||\mathcal{Q}_A|-1\big|}{|\Theta|}|\mathcal{Q}_B|+\frac{|\mathcal{Q}_B|}{|\Theta|}|\mathcal{Q}_A|\right], \tag{3.17}$$

$$\Omega_4 := (T+2)|\Phi|\left[\frac{\big||\mathcal{P}_B|-1\big|}{|\Theta|}|\mathcal{P}_A|+\frac{|\mathcal{P}_A|}{|\Theta|}|\mathcal{P}_B|\right]$$

$$+\left(\frac{T+\alpha+\beta-1-\eta}{|\Lambda|}\right)\left[1+\frac{\big||\mathcal{Q}_A|-1\big|}{|\Theta|}|\mathcal{P}_B|+\frac{|\mathcal{Q}_B|}{|\Theta|}|\mathcal{P}_A|\right]. \tag{3.18}$$

Proof. Let $B_R=\{u\in\mathcal{C}:\|u\|_{\mathcal{C}}\le R\}$, where R will be fixed later. We define a map $\mathcal{F}:B_R\to\mathcal{C}$ as

$$(\mathcal{F}u)(t)=(\mathcal{F}_1u)(t)+(\mathcal{F}_2u)(t),$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are defined by (3.2) and (3.3),respectively. Notice that the problem (1.1) is equivalent to a fixed point problem $\mathcal{F}(u)=u$.

Step I. $(\mathcal{F}u)(B_R)\subset B_R$.

Set $\max_{t\in\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1}}|f(t,0,0)|=K$, $\sup_{u,v\in\mathcal{C}}|\phi(u,v)|=M$, $\sup_{u,v\in\mathcal{C}}|\varphi(u,v)|=N$ and choose a constant R satisfied

$$R\ge\frac{K(\Omega_1+\|\Psi\|)+G(\Omega_2+\|\Upsilon\|)+\mu\Omega_3+\lambda\Omega_4}{1-\big[K(\Omega_1+\|\Psi\|)+M\Omega_3+N\Omega_4\big]}. \tag{3.19}$$

Firstly, we let $\mathcal{A}$, $\mathcal{B}$, Φ, $\mathcal{I}$, $\mathcal{J}$, Ψ and Υ, as follows

$$\mathcal{A}:=\sum_{s=0}^{\eta-\alpha-\beta+1}\frac{(\eta-\beta+1-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)\Gamma(-\beta)}E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2}(s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}}\right]$$

$$\le\frac{\Gamma(\eta-2\alpha-\beta+3)\Gamma(\eta-\beta+2)}{\Gamma(1-\beta)\Gamma(\eta-2\alpha+3)\Gamma(\eta-\alpha-\beta+2)}, \tag{3.20}$$

$$\mathcal{B}:=\sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1}\sum_{s=0}^{r-\alpha-\beta+1}\frac{(T+\alpha+\beta+\gamma-1-\sigma(s))^{\underline{\gamma-1}}}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(-\beta)}(r-\beta+1-\sigma(s))^{\underline{\alpha-1}}\times$$

$$E_{\beta-\alpha}\left[\sum_{\xi=\alpha+\beta-3}^{s+\alpha+\beta-2}(s+\alpha-2-\sigma(\xi))^{\underline{-\beta-1}}\right]$$

$$\le\frac{\Gamma(T-\alpha+2)\Gamma(T+\alpha+\gamma+1)}{\Gamma(1-\beta)\Gamma(T+\beta-\alpha+2)\Gamma(\alpha+\gamma+1)\Gamma(T+1)}, \tag{3.21}$$

$$\Phi := \frac{1}{\Lambda\Gamma(\gamma)} \sum_{s=\alpha+\beta-3}^{T+\alpha+\beta-1} (T+\alpha+\beta+\gamma-1-\sigma(s))^{\underline{\gamma-1}} = \frac{\Gamma(T+\gamma+3)}{\Lambda\Gamma(\gamma+1)\Gamma(T+3)}, \tag{3.22}$$

$$\mathcal{I} := \frac{1}{\Gamma(\alpha)} \sum_{s=\alpha+\beta-2}^{\eta-1} (\eta+\alpha-1-\sigma(s))^{\underline{\alpha-1}} = \frac{\Gamma(\eta-\beta+2)}{\Gamma(\alpha+1)\Gamma(\eta-\alpha-\beta+2)}, \tag{3.23}$$

$$\mathcal{J} := \sum_{r=\alpha+\beta-1}^{T+\alpha+\beta-1} \sum_{s=\alpha+\beta-2}^{r-1} \frac{(T+\alpha+\beta+\gamma-1-\sigma(r))^{\underline{\gamma-1}}(r+\alpha-1-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\gamma)\Gamma(\alpha)}$$

$$= \frac{\Gamma(T+\alpha+\gamma+1)}{\Gamma(\alpha+\gamma+1)\Gamma(T+1)}, \tag{3.24}$$

$$\Psi(\xi) := \frac{1}{\Gamma(\alpha)} \sum_{s=\alpha+\beta-2}^{\xi-1} (\xi+\alpha-1-\sigma(s))^{\underline{\alpha-1}} = \frac{\Gamma(\xi-\beta+2)}{\Gamma(\alpha+1)\Gamma(\xi-\alpha-\beta+2)}, \tag{3.25}$$

$$\Upsilon(\xi) := \sum_{s=0}^{\xi-\alpha-\beta+1} \frac{(\xi-\beta+1-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)\Gamma(-\beta)} E_{\beta-\alpha}\left[\sum_{\tau=\alpha+\beta-3}^{s+\alpha+\beta-2} (s+\alpha-2-\sigma(\tau))^{\underline{-\beta-1}}\right]$$

$$= \frac{\Gamma(\xi-2\alpha-\beta+3)\Gamma(\xi-\beta+2)}{\Gamma(1-\beta)\Gamma(\xi-2\alpha+3)\Gamma(\xi-\alpha-\beta+2)}. \tag{3.26}$$

Substituting (3.20)–(3.26) into $\mathcal{P}_A, \mathcal{P}_B, \mathcal{Q}_A, \mathcal{Q}_B, \Theta, \widehat{\mathcal{I}}(\eta), \widehat{\mathcal{J}}(T), \widehat{\Psi}(\xi), \Upsilon(\xi), \widehat{\mathcal{R}}_A$ and $\widehat{\mathcal{R}}_B$, we obtain

$$|\mathcal{P}_A| \leq |\mathcal{A}| \max\left\{\left|\frac{\eta-\xi}{\Lambda}\right|\right\} = |\mathcal{A}| \cdot \frac{|\eta-\alpha-\beta+3|}{\Lambda}, \tag{3.27}$$

$$|\mathcal{P}_B| \leq |\mathcal{B}| \max\left\{\left|\frac{\eta-\xi}{\Lambda}\right|\right\} = |\mathcal{B}| \cdot \frac{|\eta-\alpha-\beta+3|}{\Lambda}, \tag{3.28}$$

$$|\mathcal{Q}_A| \leq |\mathcal{A}||\Phi| \max\{T+\alpha+\beta-1-\xi\} = |\mathcal{A}| \cdot \frac{(T+2)\Gamma(T+\gamma+3)}{\Lambda\Gamma(\gamma+1)\Gamma(T+3)}, \tag{3.29}$$

$$|\mathcal{Q}_B| \leq |\mathcal{B}||\Phi| \max\{T+\alpha+\beta-1-\xi\} = |\mathcal{B}| \cdot \frac{(T+2)\Gamma(T+\gamma+3)}{\Lambda\Gamma(\gamma+1)\Gamma(T+3)}, \tag{3.30}$$

$$|\Theta| = \Big|[1-|\mathcal{Q}_A|][1-|\mathcal{P}_B|] - |\mathcal{P}_A||\mathcal{Q}_B|\Big|, \tag{3.31}$$

$$\left|\widehat{\mathcal{I}}(\eta)\right| \leq |\mathcal{I}| \cdot \Big(|f(s,u(s),v(s)) - f(s,0,0)| + |f(s,0,0)|\Big)$$

$$\leq |\mathcal{I}| \cdot \Big(L_1\|u\| + L_2\|v\| + K\Big) \leq |\mathcal{I}| \cdot \Big(L\|u\|_{\mathcal{C}} + K\Big), \tag{3.32}$$

$$\left|\widehat{\mathcal{J}}(T)\right| \leq |\mathcal{J}| \cdot \Big(|f(s,u(s),v(s)) - f(s,0,0)| + |f(s,0,0)|\Big)$$

$$\leq |\mathcal{J}| \cdot \Big(L_1\|u\| + L_2\|v\| + K\Big) \leq |\mathcal{J}| \cdot \Big(L\|u\|_{\mathcal{C}} + K\Big), \tag{3.33}$$

$$\left|\widehat{\Psi}(\xi)\right| \leq |\Psi(\xi)| \cdot \Big(|f(s,u(s),v(s)) - f(s,0,0)| + |f(s,0,0)|\Big)$$

$$\leq |\Psi(\xi)| \cdot \Big(L_1\|u\| + L_2\|v\| + K\Big) \leq |\Psi(\xi)| \cdot \Big(L\|u\|_{\mathcal{C}} + K\Big), \tag{3.34}$$

$$|\Upsilon(\xi)| \leq |\Upsilon(\xi)|\cdot G\|v\| \leq |\Upsilon(\xi)|\, G\|u\|_{\mathcal{C}}, \tag{3.35}$$

$$\begin{aligned}\big|\widehat{\mathcal{R}}_A\big| &\leq \big|\mathcal{A}\big| \max\Big\{\big(L\|u\|_{\mathcal{C}}+K\big)\big|\Psi_A\big| + G\|u\|_{\mathcal{C}}\big|\Upsilon_A\big|\Big\}\\ &:= \big|\mathcal{A}\big|\Bigg[\Big(L\|u\|_{\mathcal{C}}+K\Big)\cdot\frac{\Gamma(\eta-\alpha+1)}{\Gamma(\alpha+1)\Gamma(\eta-2\alpha+1)}\\ &\qquad +G\|u\|_{\mathcal{C}}\cdot\frac{\Gamma(\eta-3\alpha+2)\Gamma(\eta-\alpha+1)}{\Gamma(1-\beta)\Gamma(\eta-3\alpha+\beta+2)\Gamma(\eta-2\alpha+1)}\Bigg],\end{aligned} \tag{3.36}$$

$$\begin{aligned}\big|\widehat{\mathcal{R}}_B\big| &\leq \big|\mathcal{B}\big| \max\Big\{\big(L\|u\|_{\mathcal{C}}+K\big)\big|\Psi_B\big| + G\|u\|_{\mathcal{C}}\big|\Upsilon_B\big|\Big\}\\ &:= \big|\mathcal{B}\big|\Bigg[\Big(L\|u\|_{\mathcal{C}}+K\Big)\cdot\frac{\Gamma(T+\alpha)}{\Gamma(\alpha+1)\Gamma(T-\alpha+\beta)}\\ &\qquad +G\|u\|_{\mathcal{C}}\cdot\frac{\Gamma(T-2\alpha+\beta+1)\Gamma(T+\beta)}{\Gamma(1-\beta)\Gamma(T-2\alpha+2\beta+1)\Gamma(T+\beta-\alpha)}\Bigg].\end{aligned} \tag{3.37}$$

Next, we consider

$$\begin{aligned}|\phi(u,v)| &\leq \big|\phi(u,v)-\phi(0,0)\big| + \big|\phi(0,0)\big| \leq \mu_1\|u\| + \mu_2\|v\| + M\\ &\leq \mu\|u\|_{\mathcal{C}} + M,\end{aligned} \tag{3.38}$$

$$\begin{aligned}|\varphi(u,v)| &\leq \big|\varphi(u,v)-\varphi(0,0)\big| + \big|\varphi(0,0)\big| \leq \lambda_1\|u\| + \lambda_2\|v\| + N\\ &\leq \lambda\|u\|_{\mathcal{C}} + N.\end{aligned} \tag{3.39}$$

Substituting (3.27)–(3.39) into (3.4)–(3.5), we obtain $\big|\widehat{\mathcal{A}}(u)\big|$ and $|\widehat{\mathcal{B}}(u)|$, as follow

$$\begin{aligned}\big|\widehat{\mathcal{A}}(u)\big| \leq \frac{1}{|\Theta|}\Bigg\{ &\Big|\big|\mathcal{P}_B\big|-1\Big|\Bigg[\big|\mathcal{Q}_A\big|\Bigg(\Big(\mu\|u\|_{\mathcal{C}}+M\Big) + G\,|\mathcal{A}|\,\|u\|_{\mathcal{C}} + \Big(L\|u\|_{\mathcal{C}}+K\Big)\times\\ &\frac{\Gamma(\eta-\beta+2)}{\Gamma(\alpha+1)\Gamma(\eta-\alpha-\beta+2)}\Bigg) + \big|\mathcal{P}_A\big|\Bigg(\Big(\lambda\|u\|_{\mathcal{C}}+M\Big) + G\,|\mathcal{B}|\,\|u\|_{\mathcal{C}}\\ &+\Big(L\|u\|_{\mathcal{C}}+K\Big)\frac{\Gamma(T+\alpha+\gamma+1)}{\Gamma(\alpha+\gamma+1)\Gamma(T+1)}\Bigg) + \big|\widehat{\mathcal{R}}_A\big|\Bigg]\\ &+\big|\mathcal{P}_A\big|\Bigg[\big|\mathcal{Q}_B\big|\Bigg(\Big(\mu\|u\|_{\mathcal{C}}+M\Big) + G\,|\mathcal{A}|\,\|u\|_{\mathcal{C}} + \Big(L\|u\|_{\mathcal{C}}+K\Big)\times\\ &\frac{\Gamma(\eta-\beta+2)}{\Gamma(\alpha+1)\Gamma(\eta-\alpha-\beta+2)}\Bigg) + \big|\mathcal{P}_B\big|\Bigg(\Big(\lambda\|u\|_{\mathcal{C}}+M\Big) + G\,|\mathcal{B}|\,\|u\|_{\mathcal{C}}\\ &+\Big(L\|u\|_{\mathcal{C}}+K\Big)\frac{\Gamma(T+\alpha+\gamma+1)}{\Gamma(\alpha+\gamma+1)\Gamma(T+1)}\Bigg) + \big|\widehat{\mathcal{R}}_B\big|\Bigg]\Bigg\},\end{aligned} \tag{3.40}$$

$$\begin{aligned}|\widehat{\mathcal{B}}(u)| \leq \frac{1}{|\Theta|}\Bigg\{ &\big|\mathcal{Q}_B\big|\Bigg[\big|\mathcal{Q}_A\big|\Bigg(\Big(\mu\|u\|_{\mathcal{C}}+M\Big) + G\,|\mathcal{A}|\,\|u\|_{\mathcal{C}} + \Big(L\|u\|_{\mathcal{C}}+K\Big)\times\\ &\frac{\Gamma(\eta-\beta+2)}{\Gamma(\alpha+1)\Gamma(\eta-\alpha-\beta+2)}\Bigg) + \big|\mathcal{P}_A\big|\Bigg(\Big(\lambda\|u\|_{\mathcal{C}}+M\Big) + G\,|\mathcal{B}|\,\|u\|_{\mathcal{C}}\end{aligned}$$

$$+\big(L\|u\|_{\mathcal{C}}+K\big)\frac{\Gamma(T+\alpha+\gamma+1)}{\Gamma(\alpha+\gamma+1)\Gamma(T+1)}\Big)+\big|\widehat{\mathcal{R}}_A\big|\Big]$$

$$+\Big|\big|\mathcal{Q}_A\big|-1\Big|\Big[\big|\mathcal{Q}_B\big|\Big(\big(\mu\|u\|_{\mathcal{C}}+M\big)+G\,|\mathcal{A}|\,\|u\|_{\mathcal{C}}+\big(L\|u\|_{\mathcal{C}}+K\big)\times$$

$$\frac{\Gamma(\eta-\beta+2)}{\Gamma(\alpha+1)\Gamma(\eta-\alpha-\beta+2)}\Big)+\big|\mathcal{P}_B\big|\Big(\big(\lambda\|u\|_{\mathcal{C}}+M\big)+G\,|\mathcal{B}|\,\|u\|_{\mathcal{C}}$$

$$+\big(L\|u\|_{\mathcal{C}}+K\big)\frac{\Gamma(T+\alpha+\gamma+1)}{\Gamma(\alpha+\gamma+1)\Gamma(T+1)}\Big)+\big|\widehat{\mathcal{R}}_B\big|\Big]\Bigg\}. \tag{3.41}$$

Now, we show that $(\mathcal{F}u)(B_R)\subset B_R$, for each $t\in\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1}$, we have

$$\begin{aligned}
&|(\mathcal{F}_1u)(t)|\\
&\le (T+2)|\Phi|\Big[(\mu\|u\|_{\mathcal{C}}+M)+(L\|u\|_{\mathcal{C}}+K)\,|\mathcal{I}|+\big|\widehat{\mathcal{A}}(u)\big|+G\,|\mathcal{A}|\,\|u\|_{\mathcal{C}}\Big]\\
&+\frac{(T+\alpha+\beta-1-\eta)}{|\Lambda|}\Big[(\lambda\|u\|_{\mathcal{C}}+N)+(L\|u\|_{\mathcal{C}}+K)\,|\mathcal{I}|+\big|\widehat{\mathcal{B}}(u)\big|+G\,|\mathcal{B}|\,\|u\|_{\mathcal{C}}\Big]\\
&=(L\|u\|_{\mathcal{C}}+K)\Bigg\{(T+2)|\Phi|\Bigg[|\mathcal{I}|+\frac{\big||\mathcal{P}_B|-1\big|}{|\Theta|}\Big(\,|\mathcal{Q}_A|\,|\mathcal{I}|+|\mathcal{P}_A|\,|\mathcal{J}|+|\Psi_A|\,|\mathcal{A}|\Big)\\
&+\frac{|\mathcal{P}_A|}{|\Theta|}\Big(\,|\mathcal{Q}_B|\,|\mathcal{I}|+|\mathcal{P}_B|\,|\mathcal{J}|+|\Psi_B|\,|\mathcal{B}|\Big)\Bigg]+\Big(\frac{T+\alpha+\beta-1-\eta}{|\Lambda|}\Big)\Bigg[|\mathcal{I}|+\frac{\mathcal{Q}_B}{|\Theta|}\times\\
&\Big(\,|\mathcal{Q}_A|\,|\mathcal{I}|+|\mathcal{P}_A|\,|\mathcal{J}|+|\Psi_A|\,|\mathcal{A}|\Big)+\frac{\big||\mathcal{Q}_A|-1\big|}{|\Theta|}\Big(\,|\mathcal{Q}_B|\,|\mathcal{I}|+|\mathcal{P}_B|\,|\mathcal{J}|+|\Psi_B|\,|\mathcal{B}|\Big)\Bigg]\Bigg\}\\
&+\|u\|_{\mathcal{C}}G\Bigg\{(T+2)|\Phi|\Bigg[\frac{\big||\mathcal{P}_B|-1\big|}{|\Theta|}\,|\Upsilon_A|\,|\mathcal{A}|+\frac{|\mathcal{P}_A|}{|\Theta|}\,|\Upsilon_B|\,|\mathcal{B}|+|\mathcal{A}|\Bigg]\\
&+\Big(\frac{T+\alpha+\beta-1-\eta}{|\Lambda|}\Big)\Bigg[\frac{|\mathcal{Q}_B|}{|\Theta|}\,|\Upsilon_A|\,|\mathcal{A}|+\frac{\big||\mathcal{Q}_A|-1\big|}{|\Theta|}\,|\Upsilon_B|\,|\mathcal{B}|+|\mathcal{B}|\Bigg]\Bigg\}\\
&+(\mu\|u\|_{\mathcal{C}}+M)\Bigg\{(T+2)|\Phi|\Bigg[1+\frac{\big||\mathcal{P}_B|-1\big|}{|\Theta|}\,|\mathcal{Q}_A|+\frac{|\mathcal{P}_A|}{|\Theta|}\,|\mathcal{Q}_B|\Bigg]\\
&+\Big(\frac{T+\alpha+\beta-1-\eta}{|\Lambda|}\Big)\Bigg[\frac{\big||\mathcal{Q}_A|-1\big|}{|\Theta|}\,|\mathcal{Q}_B|+\frac{|\mathcal{Q}_B|}{|\Theta|}\,|\mathcal{Q}_A|\Bigg]\Bigg\}\\
&+(\lambda\|u\|_{\mathcal{C}}+N)\Bigg\{(T+2)|\Phi|\Bigg[\frac{\big||\mathcal{P}_B|-1\big|}{|\Theta|}\,|\mathcal{P}_A|+\frac{|\mathcal{P}_A|}{|\Theta|}\,|\mathcal{P}_B|\Bigg]
\end{aligned}$$

$$+\left(\frac{T+\alpha+\beta-1-\eta}{|\Lambda|}\right)\left[1+\frac{\left||\mathcal{Q}_A|-1\right|}{|\Theta|}|\mathcal{P}_B|+\frac{|\mathcal{Q}_B|}{|\Theta|}|\mathcal{P}_A|\right]\Bigg\}$$
$$=(L\|u\|_{\mathcal{C}}+K)\,\Omega_1+G\|u\|_{\mathcal{C}}\Omega_2+(\mu\|u\|_{\mathcal{C}}+M)\,\Omega_3+(\lambda\|u\|_{\mathcal{C}}+N)\,\Omega_4,$$

and

$$|(\mathcal{F}_2u)(t)| \;<\; (L\|u\|_{\mathcal{C}}+K)\left|\Psi\right|+G\|u\|_{\mathcal{C}}\left|\Upsilon\right|.$$

Consequently,

$$\begin{aligned}|(\mathcal{F}u)(t)| &= |(\mathcal{F}_1u)(t)|+|(\mathcal{F}_2u)(t)|\\ &= (L\|u\|_{\mathcal{C}}+K)\left(\Omega_1+\left|\Psi\right|\right)+G\|u\|_{\mathcal{C}}\left(\Omega_2+\left|\Upsilon\right|\right)+(\mu\|u\|_{\mathcal{C}}+M)\,\Omega_3\\ &\quad+(\lambda\|u\|_{\mathcal{C}}+N)\,\Omega_4\le R.\end{aligned}$$

Therefore, $(\mathcal{F}u)(B_R)\subset B_R$.

Step II. $\mathcal{F}_1$ is continuous and χ-contractive.

Let $\epsilon>0$ be given, since f,v,ϕ and φ are continuous, so f,v,ϕ and φ are uniformly continuous on B_R. Therefore, there exists $\delta=\min\left\{\delta_1,\delta_2,\delta_3,\delta_4\right\}>0$ such that

$$\begin{aligned}&\left|f(t,u_1,v)-f(t,u_2,v)\right|<\frac{\epsilon}{4L\Omega_1}\text{ whenever }\max\left\{|u_1-u_2|+|v|\right\}<\delta_1,\\ &\left\|u_1-u_2\right\|_{\mathcal{C}}<\frac{\epsilon}{4G\Omega_2}\text{ whenever }\max\left\{|u_1-u_2|+|v|\right\}<\delta_2,\\ &\left|\phi(u_1,v)-\phi(u_2,v)\right|<\frac{\epsilon}{4\mu\Omega_3}\text{ whenever }\max\left\{|u_1-u_2|+|v|\right\}<\delta_3,\\ &\left|\varphi(u_1,v)-\varphi(u_2,v)\right|<\frac{\epsilon}{4\lambda\Omega_4}\text{ whenever }\max\left\{|u_1-u_2|+|v|\right\}<\delta_4.\end{aligned}$$

Thus for each $t\in\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1}$ and for all $u,v\in B_R$, we obtain

$$\begin{aligned}\|(\mathcal{F}_1u)(t)-(\mathcal{F}_1v)(t)\|_{\mathcal{C}} &< L\Omega_1\left\|f(t,u_1,v)-f(t,u_2,v)\right\|_{\mathcal{C}}+G\Omega_2\left\|u_1-u_2\right\|_{\mathcal{C}}\\ &\quad+\mu\Omega_3\left\|\phi(u_1,v)-\phi(u_2,v)\right\|_{\mathcal{C}}+\lambda\Omega_4\left\|\varphi(u_1,v)-\varphi(u_2,v)\right\|_{\mathcal{C}}\\ &= \frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4}=\epsilon.\end{aligned}$$

This means that $\mathcal{F}_1$ is continuous on B_R.

Next, we show that $\mathcal{F}_1$ is χ-contractive. For any $u_1,u_2\in B_R$ and for each $t\in\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1}$, we have

$$\begin{aligned}&\|(\mathcal{F}_1u_1)(t)-(\mathcal{F}_1u_2)(t)\|_{\mathcal{C}}\\ <\;& L\Omega_1\left\|u_1-u_2\right\|_{\mathcal{C}}+G\Omega_2\left\|u_1-u_2\right\|_{\mathcal{C}}+\mu\Omega_3\left\|u_1-u_2\right\|_{\mathcal{C}}+\lambda\Omega_4\left\|u_1-u_2\right\|_{\mathcal{C}}\\ =\;& \chi\left\|u_1-u_2\right\|_{\mathcal{C}}.\end{aligned}$$

By the given assumption: $\chi<1$, it follows that $\mathcal{F}_1$ is χ-contractive.

Step III. $\mathcal{F}_2$ is compact.

In Step I, it has been shown that $\mathcal{F}_2$ is uniformly bounded. Now we show that $\mathcal{F}_2$ maps bounded sets into equicontinuous sets of $\mathcal{C}$.

Set

$$\max_{(t,u)\in\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1}\times B_R}|f(t,u(t),v(t))| = f_{\max} \text{ and } \max_{t\in\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1}}|v(t)| = v_{\max}.$$

For any $\epsilon > 0$, there exists $\widetilde{\delta} = \min\left\{\widetilde{\delta}_1, \widetilde{\delta}_2\right\} > 0$ such that

$$\begin{aligned}
|\Psi(t_2) - \Psi(t_1)| &\leq \frac{\epsilon}{2f_{max}} \text{ whenever } |t_2 - t_1| < \widetilde{\delta}_1,\\
|\Upsilon(t_2) - \Upsilon(t_1)| &\leq \frac{\epsilon}{2Gv_{max}} \text{ whenever } |t_2 - t_1| < \widetilde{\delta}_2.
\end{aligned}$$

Hence, for any $t_1, t_2 \in \mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1}$ and any $u, v \in B_R$, we have

$$\begin{aligned}
\|\mathcal{F}_2u(t_2) - \mathcal{F}_2u(t_1)\| &\leq f_{max}\,|\Psi(t_2) - \Psi(t_1)| + Gv_{max}\,|\Upsilon(t_2) - \Upsilon(t_1)|\\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus $(\mathcal{F}_2)(B_R)$ is an equicontinuous set. Therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{F}_2$ is completely continuous. Thus $\mathcal{F}_2$ is compact on B_R.

Step IV. $\mathcal{F}$ is condensing.

Since $\mathcal{F}_1$ is continuous, χ-contraction and $\mathcal{F}_2$ is compact, therefore, by Lemma 2.10, $\mathcal{F} : B_R \to B_R$ with $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ is a condensing map on B_R.

Consequently, by Theorem 3.1, the map $\mathcal{F}$ has a fixed point which, implies that the problem (1.1) has a solution. □

4. AN EXAMPLE

In this section, in order to illustrate our result, we consider an example.

Example 4.1. Consider the following fractional sum boundary value problem

$$\begin{aligned}
&\Delta_C^{\frac{3}{2}}u\left(t-\frac{2}{3}\right) + E_{-\frac{7}{6}}\left[\Delta^{\frac{1}{3}}\frac{e^{\cos\left(t-\frac{1}{2}\right)\pi}}{100e + 20\sin^2\left(t-\frac{1}{2}\right)\pi}\right]\\
&\qquad\qquad = \frac{e^{\frac{1}{6}-t}}{\left(t-\frac{599}{6}\right)^2}\cdot\frac{|u| + \left(t+\frac{11}{6}\right)e^{\cos\left(t-\frac{1}{6}\right)\pi}}{|u| + \sin^2\left(t-\frac{1}{6}\right)\pi},\\
&u\left(\frac{17}{6}\right) = \Delta^{-\frac{3}{2}}\Delta^{\frac{1}{3}}u\left(\frac{5}{3}\right) + \sum_{i=0}^{8}C_i\left(\frac{e^{\cos(t_i\pi)+1}}{\|u\|+1} + u(t_i)\right),\\
&\Delta^{-\frac{3}{4}}u\left(\frac{91}{12}\right) = \Delta^{-\frac{3}{4}}\Delta^{-\frac{3}{2}}\Delta^{\frac{1}{3}}u\left(\frac{5}{3}\right) + \sum_{i=0}^{8}D_i\left(\frac{e^{\sin(t_i\pi)-1}}{\|u\|+1} + u(t_i)\right),\ t_i = i - \frac{7}{6},
\end{aligned}$$

where $t \in \mathbb{N}_{0,6}$, C_i and D_i are given positive constants with $\sum_{i=0}^{8}C_i < \frac{1}{200}$ and $\sum_{i=0}^{8}D_i < \frac{1}{40}$.

Here $\alpha = \frac{3}{2}$, $\beta = \frac{1}{3}$, $\gamma = \frac{3}{4}$, $T = 6$, $\eta = \frac{17}{6}$, $v(t) = e^{\cos\pi t}$, $g(t) = \frac{1}{100e+20\sin^2\pi t}$, $\phi(u,v) = \sum_{i=0}^{8}C_i\left[\frac{ev(t)}{\|u\|+1} + u(t_i)\right]$, $\varphi(u,v) = \sum_{i=0}^{8}D_i\left[\frac{v(t)}{e(\|u\|+1)} + u(t_i)\right]$, $t_i = i - \frac{7}{6}$ and $f(t,u(t),v(t)) = \frac{e^{-t}}{(t+100)^2}\cdot\frac{|u|+(t+2)v(t)}{|u|+sin^2\pi t}$.

Let $t \in \mathbb{N}_{-\frac{7}{6},\frac{41}{6}}$ then $|f(t,u_1(t),v(t))-f(t,u_2(t),v(t))| \leq \frac{36e^{7/6}}{351649}|u_1-u_2|+\frac{30e^{7/6}}{351649}|v|$. So, (H_1) holds with $L=\max\{L_1,L_2\}=\max\left\{\frac{36e^{7/6}}{351649},\frac{30e^{7/6}}{351649}\right\}\approx 0.00032$.

Also, we get

$$|\phi(u_1,v)-\phi(u_2,v)| \leq \sum_{i=0}^{8} C_i\ [ev(t)+|u_1(t_i)-u_2(t_i)|] \leq \tfrac{1}{200}|u_1(t_i)-u_2(t_i)|+\tfrac{e}{200}|v|,$$
$$|\varphi(u_1,v)-\varphi(u_2,v) \leq \sum_{i=0}^{8} D_i\left[\tfrac{v(t)}{e}+|u_1(t_i)-u_2(t_i)|\right] \leq \tfrac{1}{40}|u_1(t_i)-u_2(t_i)|+\tfrac{1}{40e}|v|,$$

and $\frac{1}{100e+20}<g(t)<\frac{1}{100e}$. So, (H_2)–(H_4) hold with

$$\mu=\max\left\{\tfrac{1}{200},\tfrac{e}{200}\right\}\approx 0.0136, \lambda=\max\left\{\tfrac{1}{40},\tfrac{1}{40e}\right\}\approx 0.025, G=\tfrac{1}{100e}\approx 0.00368.$$

Finally, we can show that

$$\Omega_1=207.4843,\ \Omega_2=49.3788,\ \Omega_3=31.8052 \text{ and } \Omega_4=2.8419.$$

Therefore, we have

$$\chi=L\Omega_1+G\Omega_2+\mu\Omega_3+\lambda\Omega_4\approx 0.7517<1.$$

Hence, by Theorem 3.1, this problem has at least one solution on $\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta-1}$. □

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