BLOW-UP CRITERIA FOR A PARABOLIC PROBLEM DUE TO A CONCENTRATED NONLINEAR SOURCE IN $\mathbb{R}^N$

C. Y. CHAN AND P. TRAGOONSIRISAK MARION

Department of Mathematics, University of Louisiana at Lafayette
Lafayette, Louisiana 70504, USA

Department of Mathematics and Computer Science, Fort Valley State University
Fort Valley, GA 31030, USA

ABSTRACT. Let $x = (x_1, x_2, \ldots, x_N)$ be a point in the $N$-dimensional Euclidean space $\mathbb{R}^N$, $B$ be a $N$-dimensional ball $\{x \in \mathbb{R}^N : |x| < R\}$ centered at the origin with a radius $R$, $\partial B$ be the boundary of $B$, $\nu(x)$ denote the unit inward normal at $x \in \partial B$, and $\chi_B(x)$ be the characteristic function, which is 1 for $x \in B$ and 0 for $x \in \mathbb{R}^N \setminus B$. We study the following multi-dimensional semilinear parabolic problem with a concentrated source on the surface of the ball $\partial B$:

$$
\begin{align*}
   &u_t - \Delta u = \alpha \frac{\partial \chi_B(x)}{\partial \nu} (1 + |x|)^\beta f(u) \quad \text{in } \mathbb{R}^N \times (0, T], \\
   &u(x, 0) = \psi(x) \quad \text{for } x \in \mathbb{R}^N, \quad u(x, t) \to 0 \quad \text{as } |x| \to \infty \quad \text{for } 0 < t \leq T,
\end{align*}
$$

where $\alpha$, $\beta$ and $T$ are real numbers such that $\alpha > 0$ and $T > 0$, $f$ and $\psi$ are given functions. For $N \leq 2$, $u$ always blows up in a finite time. For $N \geq 3$, effects of $\alpha$, $R$, $\beta$, $f(u)$ and $\psi(x)$ for $u$ to blow up are given.

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1. INTRODUCTION

Let $H = \partial / \partial t - \Delta$, $T$ be a positive real number, $x = (x_1, x_2, \ldots, x_N)$ be a point in the $N$-dimensional Euclidean space $\mathbb{R}^N$, $B$ be a $N$-dimensional ball $\{x \in \mathbb{R}^N : |x| < R\}$ centered at the origin with a radius $R$, $\partial B$ be the boundary of $B$, $\nu(x)$ denote the unit inward normal at $x \in \partial B$, and $\chi_B(x)$ be the characteristic function, which is 1 for $x \in B$ and 0 for $x \in \mathbb{R}^N \setminus B$. We would like to study the following multi-dimensional semilinear parabolic problem with a concentrated source on the surface of the ball $\partial B$:

$$
\begin{align*}
   &Hu = \alpha \frac{\partial \chi_B(x)}{\partial \nu} (1 + |x|)^\beta f(u) \quad \text{in } \mathbb{R}^N \times (0, T], \\
   &u(x, 0) = \psi(x) \quad \text{for } x \in \mathbb{R}^N, \quad u(x, t) \to 0 \quad \text{as } |x| \to \infty \quad \text{for } 0 < t \leq T,
\end{align*}
$$

where $\alpha$, $\beta$ and $T$ are real numbers such that $\alpha > 0$ and $T > 0$, $f$ is a given function such that $f(0) \geq 0$, $f(u)$ and $f'(u)$ are positive for $u > 0$, and $f''(u) \geq \ldots$
0 for \( u > 0 \), and \( \psi \) is a given function such that \( \psi \) is nontrivial on \( \partial B \), non-negative, and continuous such that \( \psi \to 0 \) as \( |x| \to \infty \), \( \int_{\mathbb{R}^N} \psi(x) \, dx < \infty \), and \( \Delta \psi + \alpha (\partial_X \psi(x)/\partial \nu) (1 + |x|)^\beta f(\psi(x)) \geq 0 \) in \( \mathbb{R}^N \).

A solution \( u \) is said to blow up at the point \((x,t_b)\) if there exists a sequence \( \{(x_n,t_n)\} \) such that \( u(x_n,t_n) \to \infty \) as \( (x_n,t_n) \to (x,t_b) \).

The integral equation corresponding to the problem (1.1) is given by

\[
u(x,t) = \int_{\mathbb{R}^N} g(x,t;\xi,0) \psi(\xi) \, d\xi + \alpha \int_0^t \int_{\partial B} g(x,t;\xi,\tau) (1 + |\xi|)^\beta f(u(\xi,\tau)) \, dS \, d\tau
\]

(cf. Chan and Tragoonristaks [2]), where

\[
g(x,t;\xi,\tau) = \frac{1}{(4\pi(t-\tau))^{N/2}} \exp\left(-\frac{|x-\xi|^2}{4(t-\tau)}\right).
\]

Let \( M(t) \) denote sup\(_{x \in \mathbb{R}^N} u(x,t) \), and \( t_b \) denote the supremum of all \( t_1 \) such that the integral equation (1.2) has a unique continuous nonnegative solution for \( 0 \leq t \leq t_1 \).

The results given in the next two theorems were proved by Chan and Tragoonristaks [2].

**Theorem 1.1.** There exists some \( t_b \) such that for \( 0 \leq t < t_b \), the integral equation (1.2) has a unique continuous nonnegative solution \( u \). Furthermore, \( u \) is the solution of the problem (1.1), and is a nondecreasing function of \( t \). If

\[
\psi(x) = M(0) > \psi(y) \text{ for } x \in \partial B \text{ and } y \notin \partial B,
\]

then for any \( t > 0 \),

\[
u(x,t) = M(t) \text{ for } x \in \partial B, \ M(t) > u(y,t) \text{ for any } y \notin \partial B.
\]

If \( t_b \) is finite, then at \( t_b \), \( u \) blows up everywhere on \( \partial B \); if in addition, \( \psi \) is radially symmetric about the origin, then \( u \) blows up everywhere on \( \partial B \) only.

We note that a quenching problem in \( \mathbb{R}^N \) with a concentrated nonlinear source \( \alpha (1 + |x|)^\beta f(u) \) was investigated by Chan and Tragoonristaks [1]. Quenching phenomena in \( \mathbb{R}^N \) without a concentrated source was studied by Dai and Zeng [3] for the source \( \alpha (1 + |x|)^\beta / (1 - u) \).

Since for given \( \alpha, R \) and \( \beta \), the term \( \alpha(1+R)^\beta \) is a constant, it follows from Theorem 3.1 of Chan and Tragoonristaks [2] that we have the following result.

**Theorem 1.2.** For \( N \leq 2 \), \( u \) always blows up in a finite time.

Henceforth, we consider \( N \geq 3 \). In Section 2, we give a formula for the critical value \( \alpha^* \) such that \( u \) exists globally for \( \alpha \leq \alpha^* \) and blows up in a finite time for \( \alpha > \alpha^* \). Effects of \( R \) and \( \beta \) on the blow-up problem are investigated. We study the
case $f(u) = u^p$ where $p > 1$ in Section 3. We prove that the solution exists globally when the initial value $M(0)$ from (1.3) is small enough and the solution blows up in a finite time when $M(0)$ is large enough. The effect of the exponent $p$ is also studied.

2. EFFECTS OF $\alpha$, $R$ AND $\beta$

The following result follows from Theorem 4.2 of Chan and Tragoonrisaks [2].

**Theorem 2.1.** If $u(x, t) \leq C$ for some positive constant $C$, then $u(x, t)$ converges from below to a solution $U(x) = \lim_{t \to \infty} u(x, t)$ of the nonlinear integral equation,

$$U(x) = \alpha (1 + R)^{\beta} \int_{\partial B} G(x - \xi) f(U(\xi)) dS_\xi,$$

where

$$G(x) = \frac{\Gamma \left(\frac{N}{2} + 1\right)}{N (N - 2) \pi^{N/2}} \cdot \frac{1}{|x|^{N-2}}.$$

The next result follows from Theorems 4.3 to 4.4 of Chan and Tragoonrisaks [2].

**Theorem 2.2.** There exists a unique

$$\alpha^* = \frac{(N - 2) \pi^{(N-3)/2}}{R (1 + R)^{\beta} \Gamma \left(\frac{N-1}{2}\right) \left(\prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi \, d\varphi\right)} \cdot \sup_{M(0) < s < \infty} \left(\frac{s}{f(s)}\right),$$

where for $N = 3$, $\prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi \, d\varphi = 1$, such that $u$ exists globally for $\alpha \leq \alpha^*$, and $u$ blows up in a finite time for $\alpha > \alpha^*$. If $f(0) = 0$, then

$$\alpha^* = \frac{(N - 2) \pi^{(N-3)/2}}{R (1 + R)^{\beta} \Gamma \left(\frac{N-1}{2}\right) \left(\prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi \, d\varphi\right)} \cdot \left(\frac{M(0)}{f(M(0))}\right).$$

Let $\sup_{0 < s < \infty} (s / f(s))$ occur at $s = \bar{s} \in (0, \infty)$. If $f(0) > 0$, then

$$\alpha^* = \begin{cases} \frac{(N - 2) \pi^{(N-3)/2}}{R (1 + R)^{\beta} \Gamma \left(\frac{N-1}{2}\right) \left(\prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi \, d\varphi\right)} \cdot \left(\frac{\bar{s}}{f(\bar{s})}\right) & \text{if } M(0) < \bar{s}, \\ \frac{(N - 2) \pi^{(N-3)/2}}{R (1 + R)^{\beta} \Gamma \left(\frac{N-1}{2}\right) \left(\prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi \, d\varphi\right)} \cdot \left(\frac{M(0)}{f(M(0))}\right) & \text{if } M(0) \geq \bar{s}. \end{cases}$$

We study the effects of $R$ and $\beta$.

**Lemma 2.3.** (i) If

$$R (1 + R)^{\beta} \leq \frac{(N - 2) \pi^{(N-3)/2}}{\alpha \Gamma \left(\frac{N-1}{2}\right) \left(\prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi \, d\varphi\right)} \cdot \sup_{M(0) < s < \infty} \left(\frac{s}{f(s)}\right),$$

then $u$ exists globally.

(ii) If

$$R (1 + R)^{\beta} > \frac{(N - 2) \pi^{(N-3)/2}}{\alpha \Gamma \left(\frac{N-1}{2}\right) \left(\prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi \, d\varphi\right)} \cdot \sup_{M(0) < s < \infty} \left(\frac{s}{f(s)}\right),$$

then $u$ blows up in a finite time.
Proof. (i) (2.1) is equivalent to \( \alpha \leq \alpha^* \). By Theorem 2.2, \( u \) exists globally.

(ii) Since (2.2) is equivalent to \( \alpha > \alpha^* \), it follows from Theorem 2.2 that \( u \) blows up in a finite time.

Let \( \varphi(R) = R(1+R)^\beta \). We have

\[
\varphi'(R) = (1+R)^{\beta-1} [1 + (\beta+1) R].
\]

**Theorem 2.4.** For a given \( \alpha \), if \( \beta > -1 \), then there exists a unique \( R^* \) such that \( u \) exists globally for \( R \leq R^* \) and blows up in a finite time for \( R > R^* \).

Proof. Using (2.3), we have for \( \beta > -1 \),

\[
\left\{ \begin{array}{l}
\varphi'(R) > 0 \text{ for } R > 0, \\
\varphi(0) = 0, \\
\lim_{R \to \infty} \varphi(R) = \infty.
\end{array} \right.
\]

By solving

\[
R(1+R)^\beta = \frac{(N-2)\pi^{(N-3)/2}}{\alpha \Gamma\left(\frac{N-1}{2}\right) \left( \prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi d\varphi \right)} \cdot \sup_{M(0) < s < \infty} \left( \frac{s}{f(s)} \right),
\]

for \( R \), it follows from (2.4) that there exists exactly one solution, denoted by \( R^* \). The theorem then follows from Lemma 2.3.

**Theorem 2.5.** For \( \beta = -1 \),

(i) if

\[
\alpha \leq \frac{(N-2)\pi^{(N-3)/2}}{\Gamma\left(\frac{N-1}{2}\right) \left( \prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi d\varphi \right)} \cdot \sup_{M(0) < s < \infty} \left( \frac{s}{f(s)} \right),
\]

then \( u \) exists globally for any \( R \).

(ii) if

\[
\alpha > \frac{(N-2)\pi^{(N-3)/2}}{\Gamma\left(\frac{N-1}{2}\right) \left( \prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi d\varphi \right)} \cdot \sup_{M(0) < s < \infty} \left( \frac{s}{f(s)} \right),
\]

then there exists a unique \( R^* \) such that \( u \) exists globally for \( R \leq R^* \) and blows up in a finite time for \( R > R^* \).

Proof. (i) It follows from (2.3) that for \( \beta = -1 \),

\[
\left\{ \begin{array}{l}
\varphi'(R) > 0 \text{ for } R \geq 0, \\
\varphi(0) = 0, \\
\lim_{R \to -\infty} \varphi(R) = 1.
\end{array} \right.
\]

Then, (2.5) is equivalent to \( \alpha < \alpha^* \). By Theorem 2.2, \( u \) exists globally for any \( R \).
(ii) From (2.6),
\[
\frac{(N - 2)\pi^{(N-3)/2}}{\alpha \Gamma \left(\frac{N-1}{2}\right) \left(\prod_{i=1}^{N-3} \int_0^{\pi} \sin^i \varphi d\varphi\right)} \cdot \sup_{M(0) < s < \infty} \left(\frac{s}{f(s)}\right) < 1.
\]

By solving
\[
\frac{R}{1 + R} = \frac{(N - 2)\pi^{(N-3)/2}}{\alpha \Gamma \left(\frac{N-1}{2}\right) \left(\prod_{i=1}^{N-3} \int_0^{\pi} \sin^i \varphi d\varphi\right)} \cdot \sup_{M(0) < s < \infty} \left(\frac{s}{f(s)}\right)
\]
for $R$, it follows from (2.7) that there exists only one solution denoted by $R^*$. The result then follows from Lemma 2.3.

**Theorem 2.6.** For $\beta < -1$,

(i) if

\[
\alpha \leq \frac{(N - 2)\pi^{(N-3)/2} \left(-\beta - 1\right)}{\Gamma \left(\frac{N-1}{2}\right) \left(\prod_{i=1}^{N-3} \int_0^{\pi} \sin^i \varphi d\varphi\right) \left(\frac{\beta}{\beta + 1}\right)^\beta} \cdot \sup_{M(0) < s < \infty} \left(\frac{s}{f(s)}\right),
\]

then $u$ exists globally for any $R$.

(ii) if

\[
\alpha > \frac{(N - 2)\pi^{(N-3)/2} \left(-\beta - 1\right)}{\Gamma \left(\frac{N-1}{2}\right) \left(\prod_{i=1}^{N-3} \int_0^{\pi} \sin^i \varphi d\varphi\right) \left(\frac{\beta}{\beta + 1}\right)^\beta} \cdot \sup_{M(0) < s < \infty} \left(\frac{s}{f(s)}\right),
\]

then there exist $R^{**}$ and $R^{***}$ such that $u$ exists globally for $R \leq R^{**}$ or $R \geq R^{***}$, and blows up in a finite time for $R^{**} < R < R^{***}$.

**Proof.** (i) From (2.3), we have for $\beta < -1$,

\[
\begin{align*}
\varphi(R) & \text{ attains its maximum at } R = -\frac{1}{\beta + 1}, \\
\varphi'(R) & > 0 \text{ for } R < -\frac{1}{\beta + 1}, \\
\varphi'(R) & < 0 \text{ for } R > -\frac{1}{\beta + 1}.
\end{align*}
\]

(2.8) is equivalent to $\alpha \leq \alpha^*$. By Theorem 2.2, $u$ exists globally.

(ii) From (2.9),

\[
\frac{(N - 2)\pi^{(N-3)/2}}{\alpha \Gamma \left(\frac{N-1}{2}\right) \left(\prod_{i=1}^{N-3} \int_0^{\pi} \sin^i \varphi d\varphi\right)} \cdot \sup_{M(0) < s < \infty} \left(\frac{s}{f(s)}\right) < \varphi \left(-\frac{1}{\beta + 1}\right).
\]

By solving
\[
(1 + R)^\beta R = \frac{(N - 2)\pi^{(N-3)/2}}{\alpha \Gamma \left(\frac{N-1}{2}\right) \left(\prod_{i=1}^{N-3} \int_0^{\pi} \sin^i \varphi d\varphi\right)} \cdot \sup_{M(0) < s < \infty} \left(\frac{s}{f(s)}\right)
\]
for $R$, it follows from (2.10) that there exist two solutions. Let us denote the solution less than $(-\beta - 1)^{-1}$ by $R^{**}$, and the one larger than $(-\beta - 1)^{-1}$ by $R^{***}$. The theorem then follows from Lemma 2.3.
3. EFFECTS OF THE INITIAL CONDITION $M(0)$ AND $f(u)$

From now on, let $f(u) = u^p$ where $p > 1$. We study the effect of the initial value $M(0)$ (in (1.3)) on the boundary of the ball. For convenience, let $M = M(0)$.

Lemma 3.1. (i) If

\[
M^{p-1} \leq \frac{(N - 2) \pi^{(N-3)/2}}{\alpha R (1 + R)^\beta \Gamma \left(\frac{N-1}{2}\right) \left(\prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi d\varphi\right)},
\]

then $u$ exists globally.

(ii) If

\[
M^{p-1} > \frac{(N - 2) \pi^{(N-3)/2}}{\alpha R (1 + R)^\beta \Gamma \left(\frac{N-1}{2}\right) \left(\prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi d\varphi\right)},
\]

then $u$ blows up in a finite time.

Proof. Since (3.1) is equivalent to $\alpha \leq \alpha^*$, and (3.2) is equivalent to $\alpha > \alpha^*$, the lemma follows from Theorem 2.2.

Let $\mu(M) = M^{p-1}$. We have

\[
\mu'(M) = (p - 1) M^{p-2}.
\]

Theorem 3.2. There exists a unique $M^*$ such that $u$ exists globally for $M \leq M^*$ and blows up in a finite time for $M > M^*$.

Proof. From (3.3), $\mu'(M)$ is positive for $p > 1$ and $M > 0$. By solving

\[
M^{p-1} = \frac{(N - 2) \pi^{(N-3)/2}}{\alpha R (1 + R)^\beta \Gamma \left(\frac{N-1}{2}\right) \left(\prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi d\varphi\right)}
\]

for $M$, there exists exactly one solution, denoted by $M^*$:

\[
M^* = \left[\frac{(N - 2) \pi^{(N-3)/2}}{\alpha R (1 + R)^\beta \Gamma \left(\frac{N-1}{2}\right) \left(\prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi d\varphi\right)}\right]^{\frac{1}{p-1}}.
\]

Let $\omega(p) = M^{p-1}$. We have

\[
\omega'(p) = M^{p-1} \ln M.
\]

Theorem 3.3. If $M > 1$, then there exists a unique $p^*$ such that $u$ exists globally for $p \leq p^*$ and blows up in a finite time for $p > p^*$.
Proof. From (3.5), we have $\omega'(p) > 0$ for $M > 1$. It follows from Lemma 3.1 that there exists exactly one solution, denoted by $p^*$, such that $u$ exists globally for $p \leq p^*$ and blows up in a finite time for $p > p^*$. By solving (3.4) for $p$, we have

\begin{equation}
\tag{3.6}
p^* = 1 + \frac{\ln \left[ (N-2)\pi^{(N-3)/2} \right] - \ln \left[ \alpha R (1 + R)^{\beta} \Gamma \left( \frac{N-1}{2} \right) \left( \prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi \, d\varphi \right) \right]}{\ln M}.
\end{equation}

**Theorem 3.4.** If $M < 1$, then there exists a unique $p^*$ such that $u$ exists globally for $p \geq p^*$ and blows up in a finite time for $p < p^*$.

Proof. From (3.5), we have $\omega'(p) < 0$ for $M < 1$. It follows from Lemma 3.1 that there exists exactly one solution, denoted by $p^*$, such that $u$ exists globally for $p \geq p^*$ and blows up in a finite time for $p < p^*$. By solving (3.4) for $p$, we have (3.6).

For the case $M = 1$, the results do not depend on the exponent $p$. From Lemma 3.1, we have the following results.

**Corollary 3.5.** For $M = 1$,

(i) if

\[ \alpha \leq \frac{(N-2)\pi^{(N-3)/2}}{R (1 + R)^{\beta} \Gamma \left( \frac{N-1}{2} \right) \left( \prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi \, d\varphi \right)} \]

then $u$ exists globally for any $p$.

(ii) if

\[ \alpha > \frac{(N-2)\pi^{(N-3)/2}}{R (1 + R)^{\beta} \Gamma \left( \frac{N-1}{2} \right) \left( \prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi \, d\varphi \right)} \]

then $u$ blows up in a finite time for any $p$.

**REFERENCES**

