Dynamic Systems and Applications 25 (2016) 575-582

BLOW-UP CRITERIA FOR A PARABOLIC PROBLEM DUE TO A CONCENTRATED NONLINEAR SOURCE IN $\mathbb{R}^N$

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ABSTRACT. Let $x = (x_1, x_2, \ldots, x_N)$ be a point in the N-dimensional Euclidean space $\mathbb{R}^N$, B be a N-dimensional ball $\{x \in \mathbb{R}^N : |x| < R\}$ centered at the origin with a radius R, ∂B be the boundary of B, $\nu(x)$ denote the unit inward normal at $x \in \partial B$, and $\chi_B(x)$ be the characteristic function, which is 1 for $x \in B$ and 0 for $x \in \mathbb{R}^N \setminus B$. We study the following multi-dimensional semilinear parabolic problem with a concentrated source on the surface of the ball ∂B:

$$u_t - \triangle u = \alpha \frac{\partial \chi_B(x)}{\partial \nu} (1 + |x|)^\beta f(u) \text{ in } \mathbb{R}^N \times (0, T],$$
$$u(x, 0) = \psi(x) \text{ for } x \in \mathbb{R}^N,\ u(x, t) \to 0 \text{ as } |x| \to \infty \text{ for } 0 < t \le T,$$

where α, β and T are real numbers such that $\alpha > 0$ and $T > 0$, and f and ψ are given functions. For $N \le 2$, u always blows up in a finite time. For $N \ge 3$, effects of $\alpha,\ R,\ \beta,\ f(u)$ and $\psi(x)$ for u to blow up are given.

AMS (MOS) Subject Classification. 35B35, 35B44, 35K57.

1. INTRODUCTION

Let $H = \partial/\partial t - \triangle$, T be a positive real number, $x = (x_1, x_2, \ldots, x_N)$ be a point in the N-dimensional Euclidean space $\mathbb{R}^N$, B be a N-dimensional ball $\{x \in \mathbb{R}^N : |x| < R\}$ centered at the origin with a radius R, ∂B be the boundary of B, $\nu(x)$ denote the unit inward normal at $x \in \partial B$, and $\chi_B(x)$ be the characteristic function, which is 1 for $x \in B$ and 0 for $x \in \mathbb{R}^N \setminus B$. We would like to study the following multi-dimensional semilinear parabolic problem with a concentrated source on the surface of the ball ∂B:

$$\text{(1.1)} \qquad \begin{cases} Hu = \alpha \dfrac{\partial \chi_B(x)}{\partial \nu} (1 + |x|)^\beta f(u) \text{ in } \mathbb{R}^N \times (0, T], \\ u(x, 0) = \psi(x) \text{ for } x \in \mathbb{R}^N,\ u(x, t) \to 0 \text{ as } |x| \to \infty \text{ for } 0 < t \le T, \end{cases}$$

where α, β and T are real numbers such that $\alpha > 0$ and $T > 0$, f is a given function such that $f(0) \ge 0$, $f(u)$ and $f'(u)$ are positive for $u > 0$, and $f''(u) \ge$

Received June 30, 2016

0 for $u > 0$, and ψ is a given function such that ψ is nontrivial on ∂B, nonnegative, and continuous such that $\psi \to 0$ as $|x| \to \infty$, $\int_{\mathbb{R}^N} \psi(x)\,dx < \infty$, and $\triangle\psi + \alpha\left(\partial\chi_B(x)/\partial\nu\right)(1+|x|)^\beta f(\psi(x)) \geq 0$ in $\mathbb{R}^N$.

A solution u is said to blow up at the point (x, t_b) if there exists a sequence $\{(x_n, t_n)\}$ such that $u(x_n, t_n) \to \infty$ as $(x_n, t_n) \to (x, t_b)$.

The integral equation corresponding to the problem (1.1) is given by

$$\begin{aligned} &u(x,t) \\ &= \int_{\mathbb{R}^N} g(x,t;\xi,0)\,\psi(\xi)\,d\xi + \alpha \int_0^t \int_{\partial B} g(x,t;\xi,\tau)\,(1+|\xi|)^\beta f(u(\xi,\tau))dS_\xi d\tau \end{aligned} \tag{1.2}$$

(cf. Chan and Tragoonsirisak [2]), where

$$g(x,t;\xi,\tau) = \frac{1}{[4\pi(t-\tau)]^{N/2}} \exp\left(-\frac{|x-\xi|^2}{4(t-\tau)}\right).$$

Let $M(t)$ denote $\sup_{x\in\mathbb{R}^N} u(x,t)$, and t_b denote the supremum of all t_1 such that the integral equation (1.2) has a unique continuous nonnegative solution for $0 \leq t \leq t_1$.

The results given in the next two theorems were proved by Chan and Tragoonsirisak [2].

Theorem 1.1. *There exists some t_b such that for $0 \leq t < t_b$, the integral equation* (1.2) *has a unique continuous nonnegative solution u. Furthermore, u is the solution of the problem* (1.1), *and is a nondecreasing function of t. If*

$$\psi(x) = M(0) > \psi(y) \text{ for } x \in \partial B \text{ and } y \notin \partial B, \tag{1.3}$$

then for any $t > 0$,

$$u(x,t) = M(t) \text{ for } x \in \partial B,\ M(t) > u(y,t) \text{ for any } y \notin \partial B.$$

If t_b is finite, then at t_b, u blows up everywhere on ∂B; if in addition, ψ is radially symmetric about the origin, then u blows up everywhere on ∂B only.

We note that a quenching problem in $\mathbb{R}^N$ with a concentrated nonlinear source $\alpha(1+|x|)^\beta f(u)$ was investigated by Chan and Tragoonsirisak [1]. Quenching phenomena in $\mathbb{R}^N$ without a concentrated source was studied by Dai and Zeng [3] for the source $\alpha(1+|x|)^\beta/(1-u)$.

Since for given α, R and β, the term $\alpha(1+R)^\beta$ is a constant, it follows from Theorem 3.1 of Chan and Tragoonsirisak [2] that we have the following result.

Theorem 1.2. *For $N \leq 2$, u always blows up in a finite time.*

Henceforth, we consider $N \geq 3$. In Section 2, we give a formula for the critical value α^* such that u exists globally for $\alpha \leq \alpha^*$ and blows up in a finite time for $\alpha > \alpha^*$. Effects of R and β on the blow-up problem are investigated. We study the case $f(u) = u^p$ where $p > 1$ in Section 3. We prove that the solution exists globally when the initial value $M(0)$ from (1.3) is small enough and the solution blows up in a finite time when $M(0)$ is large enough. The effect of the exponent p is also studied.

2. EFFECTS OF α, R AND β

The following result follows from Theorem 4.2 of Chan and Tragoonsirisak [2].

Theorem 2.1. *If $u(x,t) \le C$ for some positive constant C, then $u(x,t)$ converges from below to a solution $U(x) = \lim_{t\to\infty} u(x,t)$ of the nonlinear integral equation,*

$$U(x) = \alpha\,(1+R)^{\beta}\int_{\partial B} G(x-\xi) f(U(\xi))dS_{\xi},$$

where

$$G(x) = \frac{\Gamma\left(\frac{N}{2}+1\right)}{N(N-2)\,\pi^{N/2}}\cdot\frac{1}{|x|^{N-2}}.$$

The next result follows from Theorems 4.3 to 4.4 of Chan and Tragoonsirisak [2].

Theorem 2.2. *There exists a unique*

$$\alpha^* = \frac{(N-2)\pi^{(N-3)/2}}{R(1+R)^{\beta}\,\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi\right)}\cdot\sup_{M(0)<s<\infty}\left(\frac{s}{f(s)}\right),$$

where for $N=3$, $\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi = 1$, such that u exists globally for $\alpha \le \alpha^*$, and u blows up in a finite time for $\alpha > \alpha^*$. If $f(0) = 0$, then

$$\alpha^* = \frac{(N-2)\,\pi^{(N-3)/2}}{R(1+R)^{\beta}\,\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi\right)}\cdot\left(\frac{M(0)}{f(M(0))}\right).$$

Let $\sup_{0<s<\infty}(s/f(s))$ occur at $s = \tilde{s} \in (0,\infty)$. If $f(0) > 0$, then

$$\alpha^* = \begin{cases} \frac{(N-2)\pi^{(N-3)/2}}{R(1+R)^{\beta}\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi\right)}\cdot\left(\frac{\tilde{s}}{f(\tilde{s})}\right) & \text{if } M(0) < \tilde{s}, \\ \frac{(N-2)\pi^{(N-3)/2}}{R(1+R)^{\beta}\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi\right)}\cdot\left(\frac{M(0)}{f(M(0))}\right) & \text{if } M(0) \ge \tilde{s}. \end{cases}$$

We study the effects of R and β.

Lemma 2.3. (i) *If*

$$R(1+R)^{\beta} \le \frac{(N-2)\,\pi^{(N-3)/2}}{\alpha\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi\right)}\cdot\sup_{M(0)<s<\infty}\left(\frac{s}{f(s)}\right), \tag{2.1}$$

then u exists globally.

(ii) *If*

$$R(1+R)^{\beta} > \frac{(N-2)\pi^{(N-3)/2}}{\alpha\;\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi\right)}\cdot\sup_{M(0)<s<\infty}\left(\frac{s}{f(s)}\right), \tag{2.2}$$

then u blows up in a finite time.

Proof. (i) (2.1) is equivalent to $\alpha \le \alpha^*$. By Theorem 2.2, u exists globally.

(ii) Since (2.2) is equivalent to $\alpha > \alpha^*$, it follows from Theorem 2.2 that u blows up in a finite time. □

Let $\varphi(R) = R(1+R)^{\beta}$. We have

$$\varphi'(R) = (1+R)^{\beta-1}\left[1+(\beta+1)R\right]. \tag{2.3}$$

Theorem 2.4. *For a given α, if $\beta > -1$, then there exists a unique R^* such that u exists globally for $R \le R^*$ and blows up in a finite time for $R > R^*$.*

Proof. Using (2.3), we have for $\beta > -1$,

$$\begin{cases} \varphi'(R) > 0 \text{ for } R > 0, \\ \varphi(0) = 0, \\ \lim_{R\to\infty}\varphi(R) = \infty. \end{cases} \tag{2.4}$$

By solving

$$R(1+R)^{\beta} = \frac{(N-2)\pi^{(N-3)/2}}{\alpha\,\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi\right)}\cdot\sup_{M(0)<s<\infty}\left(\frac{s}{f(s)}\right)$$

for R, it follows from (2.4) that there exists exactly one solution, denoted by R^*. The theorem then follows from Lemma 2.3. □

Theorem 2.5. *For $\beta = -1$,*

(i) *if*

$$\alpha \le \frac{(N-2)\pi^{(N-3)/2}}{\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi\right)}\cdot\sup_{M(0)<s<\infty}\left(\frac{s}{f(s)}\right), \tag{2.5}$$

then u exists globally for any R.

(ii) *if*

$$\alpha > \frac{(N-2)\pi^{(N-3)/2}}{\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi\right)}\cdot\sup_{M(0)<s<\infty}\left(\frac{s}{f(s)}\right), \tag{2.6}$$

then there exists a unique R^* such that u exists globally for $R \le R^*$ and blows up in a finite time for $R > R^*$.

Proof. (i) It follows from (2.3) that for $\beta = -1$,

$$\begin{cases} \varphi'(R) > 0 \text{ for } R \ge 0, \\ \varphi(0) = 0, \\ \lim_{R\to\infty}\varphi(R) = 1. \end{cases} \tag{2.7}$$

Then, (2.5) is equivalent to $\alpha < \alpha^*$. By Theorem 2.2, u exists globally for any R.

(ii) From (2.6),

$$\frac{(N-2)\pi^{(N-3)/2}}{\alpha\ \Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^\pi \sin^i\varphi d\varphi\right)}\cdot\sup_{M(0)<s<\infty}\left(\frac{s}{f\left(s\right)}\right)<1.$$

By solving

$$\frac{R}{1+R}=\frac{(N-2)\pi^{(N-3)/2}}{\alpha\ \Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^\pi \sin^i\varphi d\varphi\right)}\cdot\sup_{M(0)<s<\infty}\left(\frac{s}{f\left(s\right)}\right)$$

for R, it follows from (2.7) that there exists only one solution denoted by R^*. The result then follows from Lemma 2.3. □

Theorem 2.6. *For $\beta<-1$,*

(i) *if*

$$\alpha\leq\frac{(N-2)\pi^{(N-3)/2}\left(-\beta-1\right)}{\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^\pi \sin^i\varphi d\varphi\right)\left(\frac{\beta}{\beta+1}\right)^\beta}\cdot\sup_{M(0)<s<\infty}\left(\frac{s}{f\left(s\right)}\right),\tag{2.8}$$

then u exists globally for any R.

(ii) *if*

$$\alpha>\frac{(N-2)\pi^{(N-3)/2}\left(-\beta-1\right)}{\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^\pi \sin^i\varphi d\varphi\right)\left(\frac{\beta}{\beta+1}\right)^\beta}\cdot\sup_{M(0)<s<\infty}\left(\frac{s}{f\left(s\right)}\right),\tag{2.9}$$

then there exist R^{**} and R^{***} such that u exists globally for $R\leq R^{**}$ or $R\geq R^{***}$, and blows up in a finite time for $R^{**}<R<R^{***}$.

Proof. (i) From (2.3), we have for $\beta<-1$,

$$\begin{cases}\varphi\left(R\right)\ \text{attains its maximum at}\ R=-\frac{1}{\beta+1},\\ \varphi'\left(R\right)>0\ \text{for}\ R<-\frac{1}{\beta+1},\\ \varphi'\left(R\right)<0\ \text{for}\ R>-\frac{1}{\beta+1}.\end{cases}\tag{2.10}$$

(2.8) is equivalent to $\alpha\leq\alpha^*$. By Theorem 2.2, u exists globally.

(ii) From (2.9),

$$\frac{(N-2)\pi^{(N-3)/2}}{\alpha\ \Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^\pi \sin^i\varphi d\varphi\right)}\cdot\sup_{M(0)<s<\infty}\left(\frac{s}{f\left(s\right)}\right)<\varphi\left(-\frac{1}{\beta+1}\right).$$

By solving

$$\left(1+R\right)^\beta R=\frac{(N-2)\pi^{(N-3)/2}}{\alpha\ \Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^\pi \sin^i\varphi d\varphi\right)}\cdot\sup_{M(0)<s<\infty}\left(\frac{s}{f\left(s\right)}\right)$$

for R, it follows from (2.10) that there exist two solutions. Let us denote the solution less than $(-\beta-1)^{-1}$ by R^{**}, and the one larger than $(-\beta-1)^{-1}$ by R^{***}. The theorem then follows from Lemma 2.3. □

3. EFFECTS OF THE INITIAL CONDITION $M(0)$ AND $f(u)$

From now on, let $f(u) = u^p$ where $p > 1$. We study the effect of the initial value $M(0)$ (in (1.3)) on the boundary of the ball. For convenience, let $M = M(0)$.

Lemma 3.1. (i) *If*

$$M^{p-1} \leq \frac{(N-2)\,\pi^{(N-3)/2}}{\alpha R\,(1+R)^{\beta}\,\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi\right)}, \tag{3.1}$$

then u exists globally.

(ii) *If*

$$M^{p-1} > \frac{(N-2)\pi^{(N-3)/2}}{\alpha R\,(1+R)^{\beta}\,\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi\right)}, \tag{3.2}$$

then u blows up in a finite time.

Proof. Since (3.1) is equivalent to $\alpha \leq \alpha^*$, and (3.2) is equivalent to $\alpha > \alpha^*$, the lemma follows from Theorem 2.2. □

Let $\mu(M) = M^{p-1}$. We have

$$\mu'(M) = (p-1)\,M^{p-2}. \tag{3.3}$$

Theorem 3.2. *There exists a unique M^* such that u exists globally for $M \leq M^*$ and blows up in a finite time for $M > M^*$.*

Proof. From (3.3), $\mu'(M)$ is positive for $p > 1$ and $M > 0$. By solving

$$M^{p-1} = \frac{(N-2)\pi^{(N-3)/2}}{\alpha R\,(1+R)^{\beta}\,\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi\right)} \tag{3.4}$$

for M, there exists exactly one solution, denoted by M^*:

$$M^* = \left[\frac{(N-2)\pi^{(N-3)/2}}{\alpha R\,(1+R)^{\beta}\,\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi\right)}\right]^{\frac{1}{p-1}}.$$

□

Let $\omega(p) = M^{p-1}$. We have

$$\omega'(p) = M^{p-1}\ln M. \tag{3.5}$$

Theorem 3.3. *If $M > 1$, then there exists a unique p^* such that u exists globally for $p \leq p^*$ and blows up in a finite time for $p > p^*$.*

Proof. From (3.5), we have $\omega'(p) > 0$ for $M > 1$. It follows from Lemma 3.1 that there exists exactly one solution, denoted by p^*, such that u exists globally for $p \leq p^*$ and blows up in a finite time for $p > p^*$. By solving (3.4) for p, we have

(3.6)
$$p^* = 1 + \frac{\ln\left[(N-2)\pi^{(N-3)/2}\right] - \ln\left[\alpha R(1+R)^\beta \Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^\pi \sin^i\varphi d\varphi\right)\right]}{\ln M}.$$

□

Theorem 3.4. *If $M < 1$, then there exists a unique p^* such that u exists globally for $p \geq p^*$ and blows up in a finite time for $p < p^*$.*

Proof. From (3.5), we have $\omega'(p) < 0$ for $M < 1$. It follows from Lemma 3.1 that there exists exactly one solution, denoted by p^*, such that u exists globally for $p \geq p^*$ and blows up in a finite time for $p < p^*$. By solving (3.4) for p, we have (3.6). □

For the case $M = 1$, the results do not depend on the exponent p. From Lemma 3.1, we have the following results.

Corollary 3.5. *For $M = 1$,*

(i) *if*
$$\alpha \leq \frac{(N-2)\pi^{(N-3)/2}}{R(1+R)^\beta \Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^\pi \sin^i\varphi d\varphi\right)},$$
then u exists globally for any p.

(ii) *if*
$$\alpha > \frac{(N-2)\pi^{(N-3)/2}}{R(1+R)^\beta \Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^\pi \sin^i\varphi d\varphi\right)},$$
then u blows up in a finite time for any p.

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