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SIMULTANEOUS IDENTIFICATION OF UNKNOWN INITIAL TEMPERATURE AND HEAT SOURCE

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ABSTRACT. We investigate in this paper an ill-posed backward heat conduction problem of determining the unknown initial temperature and heat source from given observation at a fixed internal location and the solution value at terminal time. Unlike the classical single parameter identification problems, this ill-posed problem requires the determination of two independent unknown functions from scattered measurement of noisy data. Proof on the uniqueness of the solution is obtained by transforming the original heat conduction equation into an operator equation of the first kind. A new algorithm for the construction of the solution to the backward problem is derived by using the Landweber iteration method for the solution of the corresponding conjugate operator equation. Numerical verification on the efficiency and accuracy of the proposed algorithm is performed by solving several benchmark examples. The proposed method is readily extendable to solve more general multi-parameter identification problems.

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1. **Introduction**

In the modelling of physical phenomena, heat conduction problems are commonly encountered in many branches of engineering and sciences. For real industrial application, there is a need to determine some thermo-physical properties of a heat conducting body from given measurements of initial temperature value, diffusion coefficient, source term and/or boundary conditions. These measurements, however, are in general very difficult to obtain and hence some kinds of indirect methods are proposed as inverse heat conduction problems [3–5, 7–11]. It is well known that these kinds of inverse heat conduction problems are ill-posed in the sense that a small noise in the given data can induce enormous error in the approximation of the solution.

Inverse problems for determining single parameter in heat conduction equations have been well studied in the literatures (see, for instance [6, 15, 23, 26, 30, 34]). It is

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well-known that the inverse problem of identifying the unknown initial temperature from final observation data is severely ill-posed in the Hadamard sense that any arbitrarily small changes in the input data may lead to arbitrarily enormous changes in the solution (see [17, 25]). For inverse problem of identifying the unknown heat source, on the other hand, the level of ill-posedness is not too severe. It has been proven that the problem can be transformed to a numerical differentiation problem (see [29]). The inverse initial value problems for parabolic equations have attracted the attention of many authors (see [6, 23]), whilst the inverse source problems are interested to many engineers (see [2, 3, 18, 19]). To obtain stable solutions for these ill-posed problems, proper regularization techniques are necessary. For instance, the authors in [6] proposed a quasi Tikhonov regularization method to treat the two-dimensional backward heat conduction problem by fundamental solutions.

The identification of unknown heat source for heat conduction equation with variable coefficients has been studied in [18] by using the boundary element method. A similar problem is also discussed in [30] using the optimal control method, in which the global uniqueness and stability of the minimizer are proven. Numerical reconstruction for the case of time-dependent heat source is obtained by using the Landweber iteration in [32]. In [16], the method of fundamental solutions **(MFS)** is firstly proposed to tackle the inverse heat conduction problem and later applied in [24,29] to handle the inverse source problem. Refer [31,33] for the case of space and time dependent heat source for simple 1D heat conduction equation with constant coefficient.

In this paper, we investigate an ill-posed backward heat conduction problem of simultaneously identifying the unknown initial temperature and heat source term as stated in the following:

Problem P Consider an initial-boundary value problem of heat conduction equation:

$$\begin{cases} u_t - \Delta u = f(t), & (x,t) \in Q = (0,l) \times (0,T], \\ u_x|_{x=0} = u_x|_{x=l} = 0, & t \in (0,T], \\ u|_{t=0} = \phi(x), & x \in (0,l), \end{cases} \tag{1.1}$$

where the initial temperature value $\phi(x)$ and the time-dependent heat source term $f(t)$ are unknown. Given the following two additional conditions

$$u(x,T) = \psi(x), \quad x \in [0,l], \tag{1.2}$$

and

$$u(x_0,t) = g(t), \quad t \in [0,T], \tag{1.3}$$

where $x_0 \in [0,l]$ is a fixed point and the functions g, ψ satisfy the admissible condition

$$g(T) = \psi(x_0), \quad g(0) = \phi(x_0).$$

The inverse problem **P** is to determine the unknown functions ϕ, f and u satisfying (1.1)–(1.3). As mentioned above, this problem is coupled by a severely ill-posed problem and a mildly ill-posed one.

In [28], the inverse problem of simultaneously reconstructing the initial value $\phi(x)$ and the heat radiative coefficient $p(x)$ in the following heat conduction equation:

$$\begin{cases} u_t - \Delta u + p(x)u = 0, & (x,t) \in Q = \Omega \times (0,T], \\ u|_{\partial\Omega} = 0, & t \in (0,T], \\ u|_{t=0} = \phi(x), & x \in \Omega, \end{cases}$$

has been studied, in which the measurements of the temperature are given at a fixed time and a subregion of the physical domain. The uniqueness of this inverse problem is proven by using the Carleman-type estimation and the numerical solution is obtained by the finite element method. In [20, 27], the inverse problem of simultaneously identifying two spatial dependent coefficients $f(x)$ and $\phi(x)$ in the following heat conduction equation

$$\begin{cases} u_t - \Delta u = f(x), & (x,t) \in Q = \Omega \times (0,T], \\ u|_{\partial\Omega} = 0, & t \in (0,T], \\ u|_{t=0} = \phi(x), & x \in \Omega, \end{cases}$$

is studied, in which the additional conditions are given at two different terminal observation time $t = T_1$ and $t = T_2$, respectively. The numerical results are obtained by the boundary element method (**BEM**) and the method of fundamental solutions (**MFS**), respectively.

In this following sections, we first show that the inverse problem **P** can be divided into an inverse initial value problem and an inverse source problem. Although the solutions to these two problems can be obtained iteratively, the numerical computations involve solving a numerical differential problem for noisy input data, which is severely ill-conditioned and unstable. To tackle this stability problem, we first transform the original heat conduction equation into an operator equation of the first kind and propose a new algorithm to reconstruct the unknown initial temperature and heat source simultaneously by adopting the Landweber-iteration method (see [12,21]). Unlike the classical single-parameter identification problems, which is rather difficult to obtain the exact formation of the conjugate operator, we successfully reconstruct the unknown values from using the δ-function and the operator decomposition method in solving the corresponding conjugate operator equation.

This paper is organized as follows. In Section 2, the uniqueness of the solution for the inverse problem **P** is proven. In Section 3, the inverse problem **P** is transformed into an operator equation of first kind and the specific form of the conjugate operator is resolved. In Section 4, a new numerical algorithm based on the Landweber iteration method is devised to simultaneously reconstruct the approximated solutions of the unknown initial temperature and heat source. Some benchmark numerical examples are given in the last section to demonstrate the validity and effectiveness of the proposed method.

2. Uniqueness

Assume that $\phi(x)$ is a given function which is consistent with the homogeneous Neumann boundary condition and satisfies

$$\phi(x) \in C^{3,\alpha}(0,l) \tag{2.1}$$

and $f(t)$ is a given function which satisfies

$$f(t) \in C^{\frac{\alpha}{2}}(0,T), \tag{2.2}$$

where α is a constant in $(0,1)$. From the well-known Schauder's theory for parabolic equations (see [13, 22]), we know that there exists a unique solution, $u(x,t) \in C^{3+\alpha,1+\frac{\alpha}{2}}(\bar{Q})$, to the direct problem (1.1).

Therefore, we assume that the additional observation data $\psi(x)$ and $g(t)$ satisfy the following conditions:

$$\psi(x) \in C^{3,\alpha}(0,l) \tag{2.3}$$

and

$$g(t) \in C^{1+\frac{\alpha}{2}}(0,T). \tag{2.4}$$

Theorem 2.1. *Assume that* (ϕ_i, f_i), $i = 1,2$ *are solutions of the inverse problem* $\boldsymbol{P}$ *corresponding to the additional observation data* (ψ_i, g_i), $i = 1,2$ *which satisfy* (2.3)–(2.4). *If*

$$(\psi_1, g_1) = (\psi_2, g_2), \tag{2.5}$$

then we have

$$(\phi_1, f_1) = (\phi_2, f_2), \quad (x,t) \in \bar{Q}.$$

Proof. It is clear that (ϕ_1, f_1) satisfies the following equation:

$$\begin{cases} u_{1t} - \Delta u_1 = f_1(t), & (x,t) \in Q, \\ u_{1x}|_{x=0} = u_{1x}|_{x=l} = 0, \\ u_1|_{t=0} = \phi_1(x), \end{cases} \tag{2.6}$$

and

$$u_1(x,T) = \psi_1(x), \quad u_1(x_0,t) = g_1(t). \tag{2.7}$$

For (ϕ_2, f_2), we have

$$(2.8)\qquad \begin{cases} u_{1t} - \Delta u_2 = f_2(t), & (x,t) \in Q, \\ u_{2x}|_{x=0} = u_{2x}|_{x=l} = 0, \\ u_2|_{t=0} = \phi_2(x), \end{cases}$$

and

$$(2.9)\qquad u_2(x,T) = \psi_2(x), \quad u_2(x_0,t) = g_2(t).$$

Let

$$F(t) = f_1(t) - f_2(t), \quad \Phi(x) = \phi_1(x) - \phi_2(x), \quad U(x,t) = u_1(x,t) - u_2(x,t).$$

From (2.6)–(2.9) and the condition (2.5), we have

$$(2.10)\qquad \begin{cases} U_t - \Delta U = F(t), & (x,t) \in Q, \\ U_x|_{x=0} = U_x|_{x=l} = 0, \\ U|_{t=0} = \Phi(x), \end{cases}$$

and

$$(2.11)\qquad U(x,T) = 0, \quad U(x_0,t) = 0.$$

To obtain the uniqueness result, we need to illustrate that equations (2.10) and (2.11) has only trivial solution as follow:

Differentiating U with x and denoting by $V = U_x$, then V satisfies the following equation:

$$\begin{cases} V_t - \Delta V = 0, & (x,t) \in Q, \\ V|_{x=0} = V|_{x=l} = 0, \\ V|_{t=0} = \Phi'(x), \end{cases}$$

and

$$V(x,T) = 0.$$

By the standard theory of backward problem for parabolic equations (see [17]), we know that

$$\Phi'(x) \equiv 0,$$

i.e.,

$$(2.12)\qquad \Phi(x) = C,$$

where C is a constant. Since $U(x_0,0) = \Phi(x_0) = 0$, from (2.12) we have

$$\phi_1(x) - \phi_2(x) = \Phi(x) \equiv 0.$$

Therefore, $U(x,t)$ satisfies the following equation:

$$\begin{cases} U_t - \Delta U = F(t), & (x,t) \in Q, \\ U_x|_{x=0} = U_x|_{x=l} = 0, \\ U|_{t=0} = 0, \end{cases}$$

and

$$U(x_0, t) = 0.$$

By the standard theory of inverse source problem for parabolic equations (see [32]), we have

$$F(t) = f_1(t) - f_2(t) = 0, \quad t \in [0, T].$$

This completes the proof of Theorem 2.1. □

We remark here that the regularity conditions (2.1) and (2.2) are required to prove the uniqueness of the solution. It is not necessary in the numerical computation.

3. Operator equation of the first kind

Denote the function space:

$$H^1_\omega(0, l) = \{u \mid u \in H^1(0, l), u_x|_{x=0} = u_x|_{x=l} = 0\},$$

where H^1 is the normal Sobolev space (see [1]).

Assume that

$$f(t) \in L^2(0, T) \text{ and } \phi(x) \in L^2(0, l).$$

From the standard parabolic theory (see [13, 22]), there exists a unique weak solution $u(x, t)$ to the forward problem (1.1) with the following regularities:

$$u(x, t) \in L^2\left(0, T; H^1_\omega(0, l)\right) \bigcap L^\infty\left(0, T; L^2(0, l)\right).$$

Let K be the Parameter-to-Data mapping:

$$K: \ L^2(0, T) \times L^2(0, l) \ \rightarrow \ L^2(0, T) \times L^2(0, l)$$

$$K \begin{pmatrix} f \\ \phi \end{pmatrix} = \begin{pmatrix} g \\ \psi \end{pmatrix}. \tag{3.1}$$

Let $G(x - \xi, t - \tau)$ be the solution of the following equation:

$$\begin{cases} G_t - \Delta G = \delta(x - \xi, t - \tau), & (x, t) \in Q, \\ G_x|_{x=0} = G_x|_{x=l} = 0, \\ G|_{t=0} = 0. \end{cases}$$

It can be observed that $G(x-\xi, t-\tau)$ is the Green's function of the operator $\partial_t - \Delta$ with homogeneous Neumann boundary condition. By the well-known Green's formulation, the solution of (1.1) has the following form:

$$u(x, t) = \int_0^t \int_0^l G(x - \xi, t - \tau) f(\tau) d\xi d\tau + \int_0^l G(x - \xi, t)\phi(\xi) d\xi. \tag{3.2}$$

From (3.2), the operator K can be rewritten as

$$K = \begin{pmatrix} P_1 & P_2 \\ Q_1 & Q_2 \end{pmatrix}. \tag{3.3}$$

Here, P_1 is defined as

$$P_1 f = u_1(x_0, t) = g_1(t), \tag{3.4}$$

where $u_1(x,t)$ satisfies the following equation:

$$\begin{cases} u_{1t} - \Delta u_1 = f(t), & (x,t) \in Q, \\ u_{1x}|_{x=0} = u_{1x}|_{x=l} = 0, \\ u_1|_{t=0} = 0; \end{cases} \tag{3.5}$$

P_2 is defined as

$$P_2 \phi = u_2(x_0, t) = g_2(t), \tag{3.6}$$

where $u_2(x,t)$ satisfies the following equation:

$$\begin{cases} u_{2t} - \Delta u_2 = 0, & (x,t) \in Q, \\ u_{2x}|_{x=0} = u_{2x}|_{x=l} = 0, \\ u_2|_{t=0} = \phi(x); \end{cases} \tag{3.7}$$

Q_1 is defined as

$$Q_1 f = u_1(x, T) = \psi_1(x); \tag{3.8}$$

and Q_2 is defined as

$$Q_2 \phi = u_2(x, T) = \psi_2(x). \tag{3.9}$$

Combining (3.1), (3.3)-(3.9), we have

$$P_1 f + P_2 \phi = g_1 + g_2 = g,$$

and

$$Q_1 f + Q_2 \phi = \psi_1 + \psi_2 = \psi.$$

From (3.1) the conjugate operator K^* of K can be written as

$$K^* = \begin{pmatrix} P_1^* & Q_1^* \\ P_2^* & Q_2^* \end{pmatrix}. \tag{3.10}$$

In the following we will derive the specific form of the operators $P_1^*,\ P_2^*,\ Q_1^*$ and Q_2^*. We first need the following several lemmas.

Lemma 3.1. *For any given $h_1(t) \in L^2(0,T)$, let $v_1(x_0, \cdot) = P_1^* h$, where P_1^* is the conjugate operator of P_1. Then v_1 satisfies the following backward parabolic equation:*

$$\begin{cases} -v_{1t} - \Delta v_1 = h(t), & (x,t) \in Q, \\ v_{1x}|_{x=0} = v_{1x}|_{x=l} = 0, \\ v_1|_{t=T} = 0. \end{cases} \tag{3.11}$$

The proof can be found in [32].

Lemma 3.2. *For any given $\xi_1(x) \in L^2(0,l)$, let $v_2(\cdot,0) = Q_2^*\xi_1$, where Q_2^* is the conjugate operator of Q_2. Then v_2 satisfies the following backward parabolic equation:*

$$\begin{cases} -v_{2t} - \Delta v_2 = 0, & (x,t) \in Q, \\ v_{2x}|_{x=0} = v_{2x}|_{x=l} = 0, & \\ v_2|_{t=T} = \xi_1(x). & \end{cases} \tag{3.12}$$

Proof. Let L be the differential operator:

$$Lu_2 = u_{2t} - \Delta u_2,$$

and L^* denotes the adjoint operator of L:

$$L^* v_2 = -v_{2t} - \Delta v_2.$$

From (3.7) and (3.12) we have

$$\begin{aligned} 0 &= \int_0^T \int_0^l (v_2 L u_2 - u_2 L^* v_2) dx dt \\ &= \int_0^T \int_0^l (v_2 u_{2t} + u_2 v_{2t}) dx dt + \int_0^T \int_0^l (u_2 \Delta v_2 - v_2 \Delta u_2) dx dt \\ &= \int_0^l u_2 v_2 \Big|_{t=0}^{t=T} dx + \int_0^T (u_2 v_{2x} - v_2 u_{2x}) \Big|_{x=0}^{x=l} dt \\ &= \int_0^l u_2(x,T) v_2(x,T) dx - \int_0^l u_2(x,0) v_2(x,0) dx, \end{aligned}$$

i.e.,

$$\langle Q_2 \phi, \xi_1 \rangle = \langle \phi, v_2(x,0) \rangle.$$

This completes the proof of Lemma 3.2. □

Lemma 3.3. *For any given $\xi_2(x) \in L^2(0,l)$, let $v_3(\cdot) = Q_1^*\xi_2$, where Q_1^* is the conjugate operator of Q_1. Then we have*

$$v_3(t) = \int_0^l \widetilde{v_3}(x,t) dx,$$

where $\widetilde{v_3}$ satisfies the following parabolic equation:

$$\begin{cases} -\widetilde{v_3}_t - \Delta \widetilde{v_3} = 0, & (x,t) \in Q, \\ \widetilde{v_3}_x|_{x=0} = \widetilde{v_3}_x|_{x=l} = 0, & \\ \widetilde{v_3}|_{t=T} = \xi_2(x). & \end{cases} \tag{3.13}$$

Proof. From (3.5) and (3.13) we have

$$\begin{aligned} \int_0^T \int_0^l (\widetilde{v_3} L u_1 - u_1 L^* \widetilde{v_3}) dx dt &= \int_0^l u_1 \widetilde{v_3} \Big|_{t=0}^{t=T} dx \\ &= \int_0^l u_1(x,T) \widetilde{v_3}(x,T) dx \\ &= \langle Q_1 f, \ \xi_2 \rangle. \end{aligned} \tag{3.14}$$

From (3.5) and the definition of v_3, the left-hand-side of (3.14) can be rewritten as

$$\begin{aligned}\int_0^T \int_0^l \widetilde{v_3} f(t) dx dt &= \int_0^T f(t) \left(\int_0^l \widetilde{v_3} dx \right) dt \\ &= \langle f,\ v_3(\cdot) \rangle \\ &= \langle Q_1 f,\ \xi_2 \rangle.\end{aligned}$$

This completes the proof of Lemma 3.3. □

Lemma 3.4. *For any given $h_2(t) \in L^2(0,T)$, let $v_4(\cdot, 0) = P_2^* h_2$, where P_2^* is the conjugate operator of P_2. Then v_4 satisfies the following backward parabolic equation:*

$$\begin{cases} -v_{4t} - \Delta v_4 = h_2(t)\delta(x - x_0), & (x,t) \in Q, \\ v_{4x}|_{x=0} = v_{4x}|_{x=l} = 0, \\ v_4|_{t=T} = 0, \end{cases} \tag{3.15}$$

where δ is the Delta-function concentrated at $x = x_0$.

Proof. From (3.6), (3.7) and (3.15) we have

$$\begin{aligned}\int_0^T \int_0^l (v_4 L u_2 - u_2 L^* v_4) dx dt &= \int_0^l u_2 v_4 \Big|_{t=0}^{t=T} dx \\ &= -\int_0^l u_2(x,0) v_4(x,0) dx \\ &= -\langle \phi(x),\ v_4(x,0) \rangle.\end{aligned} \tag{3.16}$$

Moreover, the left-hand-side of (3.16) can be rewritten as

$$\begin{aligned}-\int_0^T \int_0^l u_2(x,t)\delta(x - x_0) h_2(t) dx dt &= -\int_0^T u_2(x_0,t) h_2(t) dt \\ &= -\langle P_2 \phi,\ h_2 \rangle.\end{aligned} \tag{3.17}$$

The proof can be obtained from (3.16) and (3.17). □

We then have the following theorem.

Theorem 3.5. *The conjugate operator K^* of K is given by* (3.10)*, where the specific forms of the operators P_1^*, P_2^*, Q_1^* and Q_2^* are given in Lemmas* 3.1–3.4.

4. **Convergence**

In this section, we use the Landweber iteration method to deal with the operator equation (3.1).

Since equation (3.1) can be rewritten in the following form:

$$\begin{pmatrix} f \\ \phi \end{pmatrix} = (I - aK^*K) \begin{pmatrix} f \\ \phi \end{pmatrix} + aK^* \begin{pmatrix} g \\ \psi \end{pmatrix} \tag{4.1}$$

for some $a > 0$, we use the iteration method to compute the solution of (4.1), i.e.,

$$\begin{pmatrix} f_0 \\ \phi_0 \end{pmatrix} = \begin{pmatrix} f_{initial} \\ \phi_{initial} \end{pmatrix},$$

$$\begin{pmatrix} f_m \\ \phi_m \end{pmatrix} = (I - aK^*K)\begin{pmatrix} f_{m-1} \\ \phi_{m-1} \end{pmatrix} + aK^*\begin{pmatrix} g \\ \psi \end{pmatrix}, \quad m = 1, 2, 3, \ldots. \tag{4.2}$$

From the definition of K and (4.2) we have

$$\begin{aligned}\begin{pmatrix} f_m \\ \phi_m \end{pmatrix} &= \begin{pmatrix} f_{m-1} \\ \phi_{m-1} \end{pmatrix} - aK^*\left(K\begin{pmatrix} f_{m-1} \\ \phi_{m-1} \end{pmatrix} - \begin{pmatrix} g \\ \psi \end{pmatrix}\right) \\ &= \begin{pmatrix} f_{m-1} \\ \phi_{m-1} \end{pmatrix} - aK^*\begin{pmatrix} u_{m-1}(x_0, t) - g(t) \\ u_{m-1}(x, T) - \psi(x) \end{pmatrix},\end{aligned}$$

where u_{m-1} is the solution of (1.1) with $\begin{pmatrix} f \\ \phi \end{pmatrix} = \begin{pmatrix} f_{m-1} \\ \phi_{m-1} \end{pmatrix}$.

The procedure of the iteration algorithm can be stated as follows:

(Step 1). Choose an initial value for the iteration $\begin{pmatrix} f \\ \phi \end{pmatrix} = \begin{pmatrix} f_0 \\ \phi_0 \end{pmatrix}$. For simplicity, we can choose $\begin{pmatrix} f_0(t) \\ \phi_0(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $t \in (0, T)$, $x \in (0, l)$.

(Step 2). Solve the initial-boundary value problem (1.1) to obtain the solution $u_0(x, t)$, where $\begin{pmatrix} f \\ \phi \end{pmatrix} = \begin{pmatrix} f_0 \\ \phi_0 \end{pmatrix}$.

(Step 3). Let

$$\overrightarrow{V_0} = \begin{pmatrix} v_1(x_0, t) + v_3(x_0, t) \\ v_2(x, 0) + v_4(x, 0) \end{pmatrix},$$

where v_1 is the solution of (3.11) with $h_1 = u_0(x_0, \cdot) - g(\cdot)$, v_2 is the solution of (3.12) with $\xi_1 = u_0(\cdot, T) - \psi(\cdot)$, $v_3 = \int_0^l \widetilde{v_3}(x, t)dx$, and $\widetilde{v_3}$ is the solution of (3.13) with $\xi_2 = u_0(\cdot, T) - \psi(\cdot)$, and v_4 is the solution of (3.15) with $h_2 = u_0(x_0, \cdot) - g(\cdot)$.

(Step 4). Let

$$\begin{pmatrix} f_1(t) \\ \phi_1(x) \end{pmatrix} = \begin{pmatrix} f_0(t) \\ \phi_0(x) \end{pmatrix} - a\overrightarrow{V_0},$$

and $u_1(x, t)$ be the solution of (1.1) with $\begin{pmatrix} f \\ \phi \end{pmatrix} = \begin{pmatrix} f_1 \\ \phi_1 \end{pmatrix}$.

(Step 5). Select two arbitrarily small positive constants ε_1 and ε_2 as error bounds. Compute

$$\begin{pmatrix} \|u_1(x_0, t) - g(t)\| \\ \|u_1(x, T) - \psi(x)\| \end{pmatrix}$$

and compare it with $\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$.

If

$$\|u_1(x_0,t) - g(t)\| < \varepsilon_1, \text{ and } \|u_1(x,T) - \psi(x)\| < \varepsilon_2,$$

then stop the iteration and take $\begin{pmatrix} f \\ \phi \end{pmatrix} = \begin{pmatrix} f_1 \\ \phi_1 \end{pmatrix}$;

If

$$\|u_1(x_0,t) - g(t)\| \geq \varepsilon_1, \text{ or } \|u_1(x,T) - \psi(x)\| \geq \varepsilon_2,$$

then go back to **(Step 3)**. Let $\begin{pmatrix} f_1 \\ \phi_1 \end{pmatrix}$ be a new initial value and repeat the iteration until the error bounds are reached.

Remark 4.1. In **(Step 3)** there is a δ-function in (3.15). It is well known that the function $\delta(x - x_0)$ has singularity at $x = x_0$. To avoid this singularity, in the numerical procedure, the delta function is approximated by the Gaussian function:

$$\delta(x - x_0) \approx \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{(x-x_0)^2}{2\sigma^2}}, \quad \sigma > 0,$$

where the parameter σ is taken to be relatively small.

By the standard theory of the Landweber iteration (see [12, 21]), we have the following convergence results.

Theorem 4.1. *Let $g \in L^2(0,T)$, $\psi \in H^1(0,l)$ and u, f and ϕ be the unique solution to the inverse problem $\boldsymbol{P}$ according to Theorem 2.1. Assume that a satisfies $0 < a < 1/\|K\|^2$ and u_m, f_m and ϕ_m be the mth approximation in the above iterative procedure, we have*

$$\lim_{m\to\infty} \|f - f_m\|_{L^2(0,T)} = 0 \text{ and } \lim_{m\to\infty} \|\phi - \phi_m\|_{L^2(0,l)} = 0$$

for every initial function $f_0 \in L^2(0,T)$ and $\phi_0 \in L^2(0,l)$.

Since the proposed iterative procedure is a regularization method, it works with inexact data. Assume that the exact solutions f and ϕ are attainable, i.e., there exist functions $f \in L^2(0,T)$ and $\phi \in L^2(0,l)$ such that

$$u(x_0,t;f,\phi) = g(t), \quad u(x,T;f,\phi) = \psi(x)$$

and an upper bound δ for the noise level

$$\|g^\delta - g\| \leq \delta, \quad \|\psi^\delta - \psi\| \leq \delta$$

of the observation is known *a-priori*. Given the noise level δ, we can use the discrepancy principle [12] to obtain a stopping criterion for the iterative algorithm.

Theorem 4.2. *Let $r > 1$ and $\|g^\delta\|$, $\|\psi^\delta\| \geq r\delta$ and $f_{m,\delta}$ and $\phi_{m,\delta}$ be the mth approximation defined in (4.1) with (g, ψ) replaced by (g^δ, ψ^δ), for some $0 < a < 1/\|K\|^2$. Then we have*

$$\lim_{m\to\infty} \|u_{m,\delta}(x_0, \cdot) - g^\delta\| = 0, \quad \lim_{m\to\infty} \|u_{m,\delta}(\cdot, T) - \psi^\delta\| = 0$$

for every $\delta > 0$, which implies the following stopping rule: one can stop the iterative procedure at the $m(\delta)$th iteration, where $m(\delta) \in \mathbb{N}$ is the smallest integer with

$$\min\left\{\|u_{m,\delta}(x_0, \cdot) - g^\delta\|, \quad \|u_{m,\delta}(\cdot, T) - \psi^\delta\|\right\} \leq r\delta.$$

Moreover, for $m(\delta)$ we have

$$\delta^2 m(\delta) \to 0, \text{ as } \delta \to 0.$$

The proof of Theorem 4.2 can be found in [12, 21].

5. Numerical examples

We construct in this section three numerical experiments to verify the stability and effectiveness of the proposed algorithm. In all these experiments, the values of the basic parameters are:

$$l = 0.5, \quad T = 0.2, \quad x_0 = 0.25, \quad a = 1.$$

In the numerical computations for the solutions of the direct problems, the standard finite difference method is applied, where the spatial step size Δx and the time step size Δt are taken as:

$$\Delta x = \Delta t = 0.01.$$

We use the symbols σ_ϕ and σ_f to denote the stopping parameters in the iteration procedure, i.e.,

$$\sigma_\phi = \|u(x, T; f, \phi) - \psi^\delta(x)\|_{L^2(0,l)},$$
$$\sigma_f = \|u(x_0, t; f, \phi) - g^\delta(t)\|_{L^2(0,T)},$$

and the symbols $\mathcal{E}_\phi$ and $\mathcal{E}_f$ to denote the relative L^2-norm error between the exact solutions $\phi(x)$, $f(t)$ and the numerical reconstructed solutions $\tilde{\phi}(x)$, $\tilde{f}(t)$, i.e.,

$$\mathcal{E}_\phi = \|\tilde{\phi}(x) - \phi(x)\|/\|\phi(x)\|,$$
$$\mathcal{E}_f = \|\tilde{f}(t) - f(t)\|/\|f(t)\|.$$

For general heat source term $f(x, t)$, it is not difficult to construct an analytic solution $u(x, t)$ of heat conduction equations which satisfies the given initial value $\phi(x)$ and heat source $f(x, t)$. In fact, one can specify an arbitrary function $u(x, t)$ which satisfies the homogeneous Neumann condition and substitute it into (1.1) to obtain the exact initial temperature $\phi(x)$ and heat source $f(x, t)$. However, in the inverse problem **P**, there contains two independent unknown functions $\phi(x)$ and $f(t)$ such that the time-dependent heat source function f is independent of the spatial variable x in which the above analytic construction method does not work. Therefore, in the following numerical computations, the observation data $\psi(x)$ and $g(t)$ are given by

$$\psi(x) = u(x, T; \phi, f), \quad g(t) = u(x_0, t; \phi, f),$$

where $u(x, t)$ is the numerical approximation to the solution of (1.1) with given input data $\phi(x)$ and $f(t)$.

Example 1. In the first numerical experiment, we consider

$$\phi(x) = \frac{1}{10} \sin^2(2\pi x), \quad x \in [0, l],$$
$$f(t) = 100, \quad t \in (0, T],$$

in which the heat source is a constant function and the initial temperature value is a quadratic sine function.

The reconstructed solutions $\phi(x)$ and $f(t)$ with exact input data are shown in Fig. 1. The initial guess $f_0(t)$ is taken to be zero. Noticing that the exact solution is 100, we can observe from Fig. 1 that although the initial guess is no good, the iterative algorithm converges stably and efficiently to a satisfactory reconstructed solution. The initial temperature value $\phi(x)$ is also well reconstructed as seen from Fig. 1 that the curves of the numerical approximation and the exact solution coincide almost everywhere except some small interval close to the boundary.

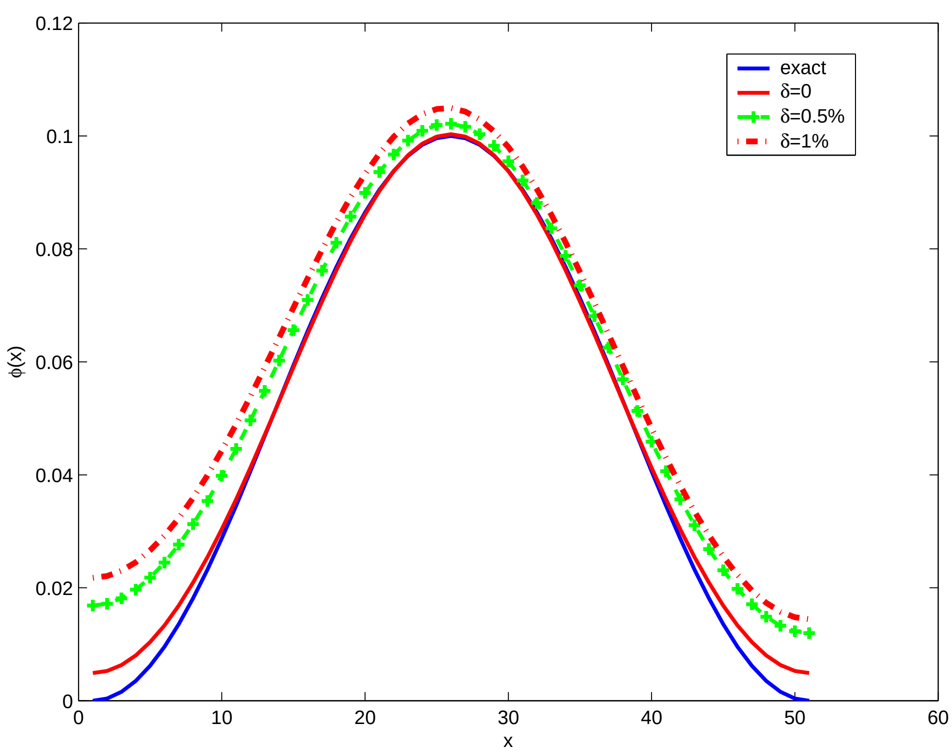

Fig. 1. Reconstruction of the initial value with noisy data for Example 1.

We also consider the case of noisy input data to test the stability of our algorithm. The noisy data are generated in the following form

$$\psi^\delta(x) = \psi(x)[1 + \delta \times \text{random}(x)], \quad g^\delta(t) = g(t)[1 + \delta \times \text{random}(t)] \tag{5.1}$$

with $\delta = 0.5\%$ and $\delta = 1\%$. The reconstruction results are displayed in Fig. 1 and Fig. 2 in which satisfactory approximation is obtained even under the case of noisy data. We can see from Fig. 1 that the inverse backward problem is very sensitive to the data errors whilst the inverse source problem is only mildly affected. For exact observation data, the accuracy for the initial temperature value can achieve to 3.87%, whilst for inexact case with noise level $\delta = 0.5\%$, the accuracy only decreases to 13.99%. The iteration number k, stopping parameter σ, and relative L^2-norm error $\mathcal{E}$ for various noisy level δ are given in Tab. 1.

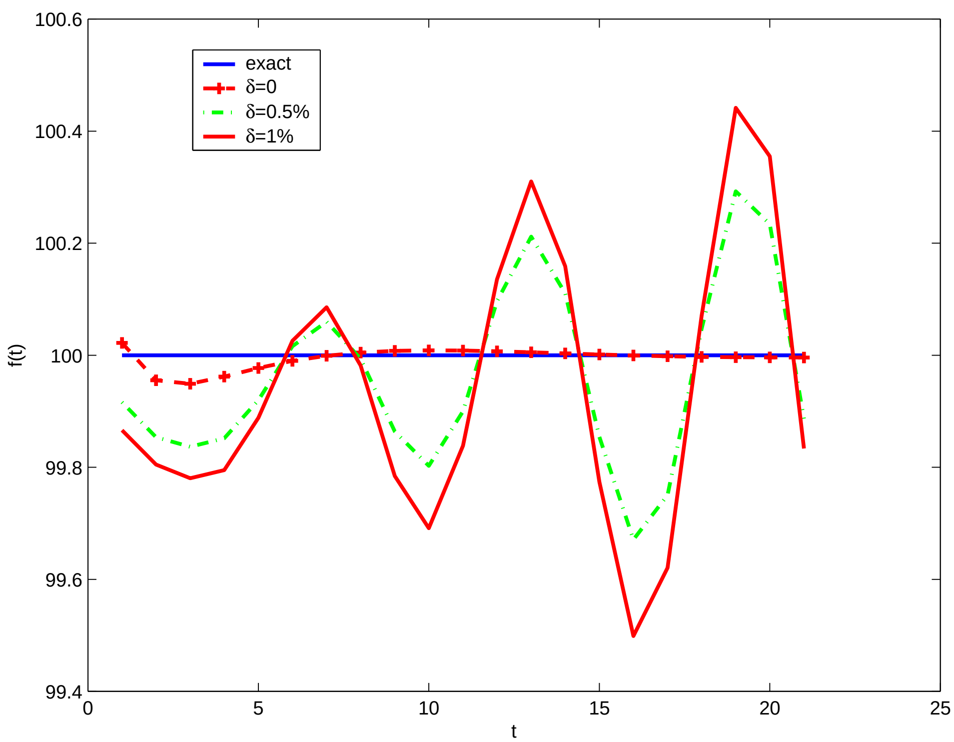

Fig. 2. Reconstruction of the heat source with noisy data for Example 1.

Table 1. The values of k, $\mathcal{E}_\phi$, $\mathcal{E}_f$, σ_ϕ and σ_f with various noisy level δ for Example 1.

	$\delta = 0$	$\delta = 0.5\%$	$\delta = 1\%$
k	500	400	300
$\mathcal{E}_\phi$	3.87×10^{-2}	1.399×10^{-1}	1.907×10^{-1}
$\mathcal{E}_f$	2×10^{-4}	1.7×10^{-3}	2.5×10^{-3}
σ_ϕ	3.47×10^{-7}	5.1×10^{-2}	1.002×10^{-1}
σ_f	1×10^{-4}	1.81×10^{-2}	3.66×10^{-2}

Remark 5.1. To facilitate an easy check on the numerical results, the random functions in (5.1) are replaced by

$$\psi^\delta(x_j) = \psi(x_j)[1 + \delta \times \sin(x_j)], \quad g^\delta(t_j) = g(t_j)[1 + \delta \times \sin(t_j)].$$

Example 2. In the second numerical experiment, we take

$$\phi(x) = 0.255, \quad x \in [0, l],$$
$$f(t) = \begin{cases} 10t, & 0 \le t \le \frac{T}{2}, \\ 10(T - t), & \frac{T}{2} \le t \le T. \end{cases}$$

In this example, the initial value is a constant function and the heat source is a continuous but not differentiable function. It can be easily seen that there is a sharp point at $t = \frac{T}{2}$ which, in general, is very difficult to be reconstructed.

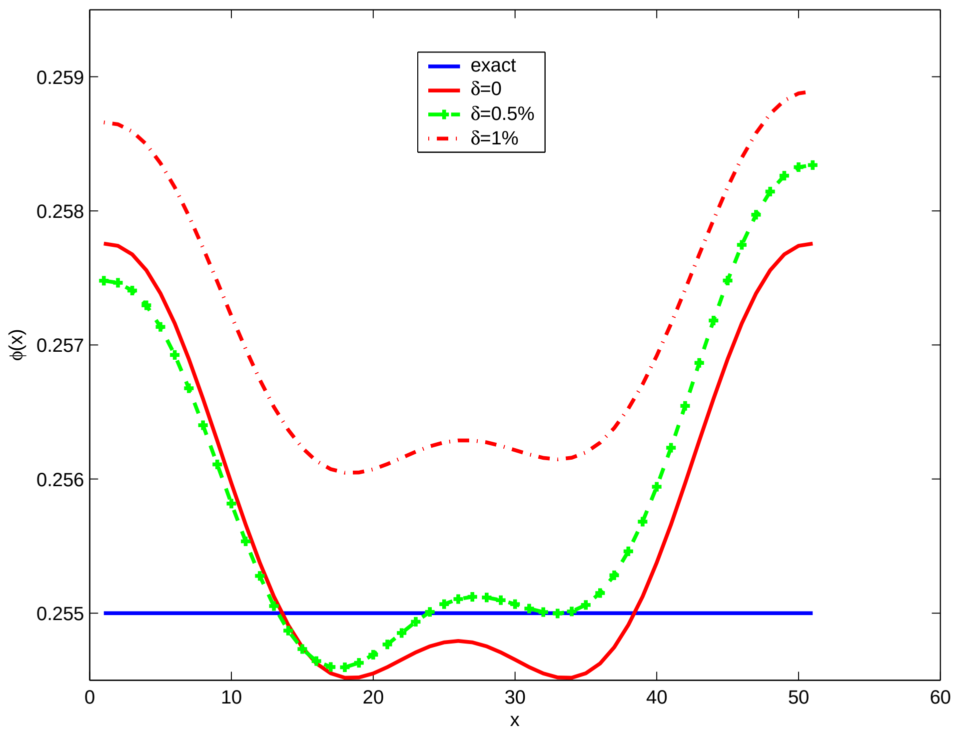

Fig. 3. Reconstruction of the initial value with noisy data for Example 2.

The exact solution and the recovery one are given in Fig. 3 and Fig. 4. The initial value is recovered more precise than the case given in Example 1. For the heat source, the reconstruction result is also satisfactory. We can observe from Fig. 4 that, after about 5000 iterations, the unknown heat source is recovered quite well except near the neighborhood of the cusp $x = 0.25$. In this experiment, due to the rather poor regularity, it is very difficult to exactly reconstruct the cuspidal property of the unknown coefficients. The iteration process converges slowly near the sharp point. To achieve a satisfactory result, we may need more iterations than that in Example 1.

Table 2. The values of k, $\mathcal{E}_\phi$, $\mathcal{E}_f$, σ_ϕ and σ_f with various noisy level δ for Example 2.

	$\delta = 0$	$\delta = 0.5\%$	$\delta = 1\%$
k	5000	4000	3000
$\mathcal{E}_\phi$	5.3×10^{-3}	8.5×10^{-3}	9.0×10^{-3}
$\mathcal{E}_f$	3.88×10^{-2}	5.38×10^{-2}	6.46×10^{-2}
σ_ϕ	3.54×10^{-9}	3.502×10^{-4}	7.154×10^{-4}
σ_f	1.034×10^{-4}	2.206×10^{-4}	3.562×10^{-4}

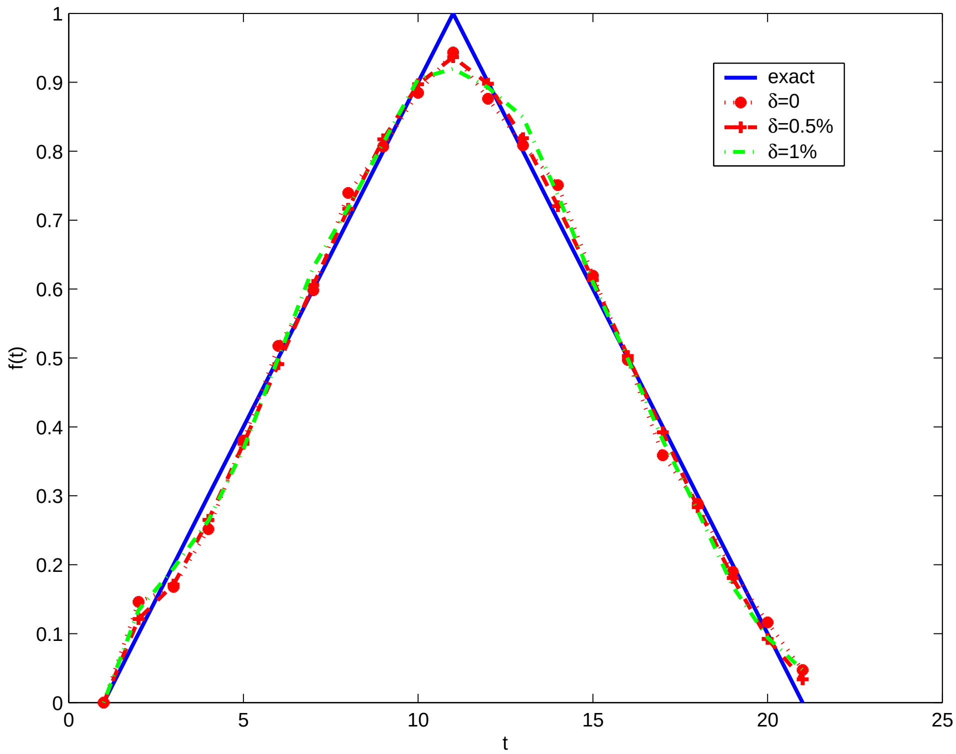

Fig. 4. Reconstruction of the heat source with noisy data for Example 2.

Likewise, the reconstructions of $\phi(x)$ and $f(t)$ from the noisy data $\psi^\delta(x)$ and $g^\delta(t)$ are also performed, where the noise level δ is taken as 0.5% and 1%, respectively. Some computation parameters k, $\mathcal{E}_\phi$, $\mathcal{E}_f$, σ_ϕ and σ_f for exact input data and noisy one respectively are given in Tab. 2.

Remark 5.2. In general, the main idea of regularization methods for inverse problems is to approximate the ill-posed solution by a sequence of regularized solutions which are smooth functions. From the numerical point of view, it is rather difficult to precisely reconstruct the discontinuous property of the unknown source function (the heat source coefficient in current model is continuous but its first derivative is discontinuous). This phenomenon is called over-smoothing in the theory of inverse problems. In such case, some efficient regularization methods such as total variation (**TV**) (see [12]) can be adopted.

Example 3. In the third numerical experiment, we consider a more complicated problem given by

$$\phi(x) = 1 + 100[x(l-x)]^2, \quad x \in [0, l],$$
$$f(t) = 100t(T-t), \quad t \in (0, T].$$

Being different from previous examples, the initial value and the heat source are all variable coefficients.

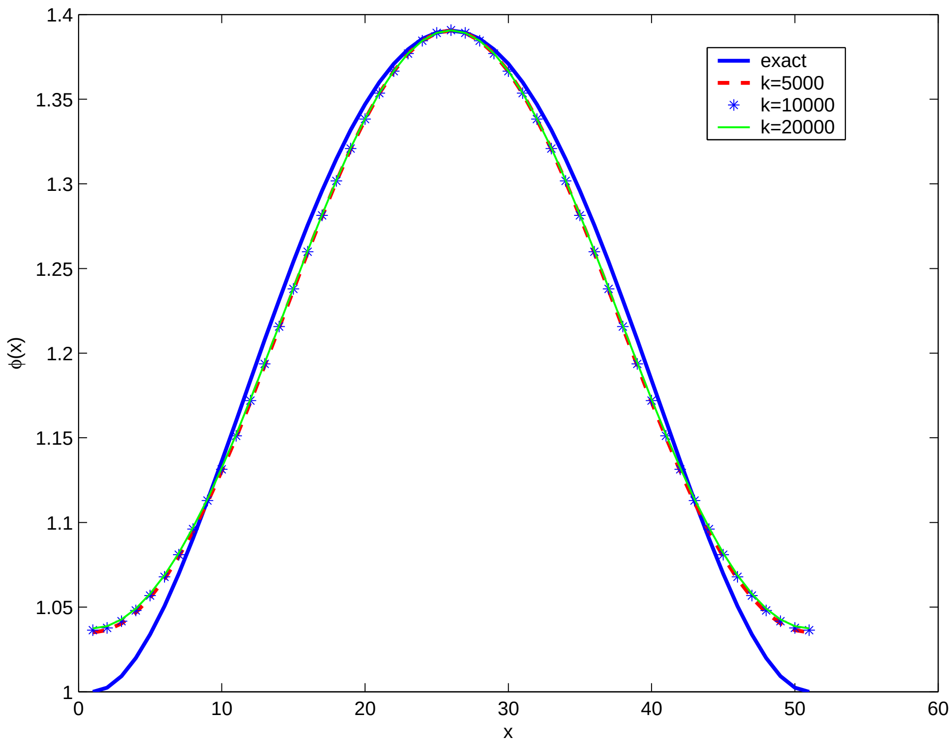

Fig. 5. Reconstruction of the initial value at different iteration steps for Example 3.

The reconstruction results for different iteration (denoted by k) are shown in Fig. 5 and Fig. 6. We can see that the shape of the unknown initial temperature value and heat source are recovered very well after 5000 iterations. After 5000 iterations, the iterative procedure converges slowly. The iteration number k, stopping parameter σ, and relative L^2-norm error $\mathcal{E}$ for various noisy level δ are given in Tab. 3. It can be seen that as $k = 5000$, the stopping parameters are $\sigma_\phi = 7.34 \times 10^{-12}$ and $\sigma_f = 2.188 \times 10^{-4}$. We observe here that as k increase to 20000, the error only decreases to 1.9115×10^{-13} and 5.58×10^{-5}.

Table 3. The values of k, $\mathcal{E}_\phi$, $\mathcal{E}_f$, σ_ϕ and σ_f for Example 3.

	$k = 5000$	$k = 10000$	$k = 20000$
$\mathcal{E}_\phi$	1.31×10^{-2}	1.33×10^{-2}	1.36×10^{-2}
$\mathcal{E}_f$	1.025×10^{-1}	9.46×10^{-2}	9.65×10^{-2}
σ_ϕ	7.34×10^{-12}	1.0384×10^{-12}	1.9115×10^{-13}
σ_f	2.188×10^{-4}	1.128×10^{-4}	5.58×10^{-5}

Remark 5.3. In this example, the noisy case is omitted mainly for the reason that the input data has to be computed from solving the forward problem. In fact, we can see from Tab. 3 that the relative L^2-norm error of $f(t)$ with $k = 10000$ is less than that of $k = 20000$, which in general the reverse case should occur. This phenomenon illustrates that the input data, which is obtained by the numerical solution of forward problem, contain unavoidable computation error and for the noisy case, the stopping criterion should be employed to terminate the iteration procedure.

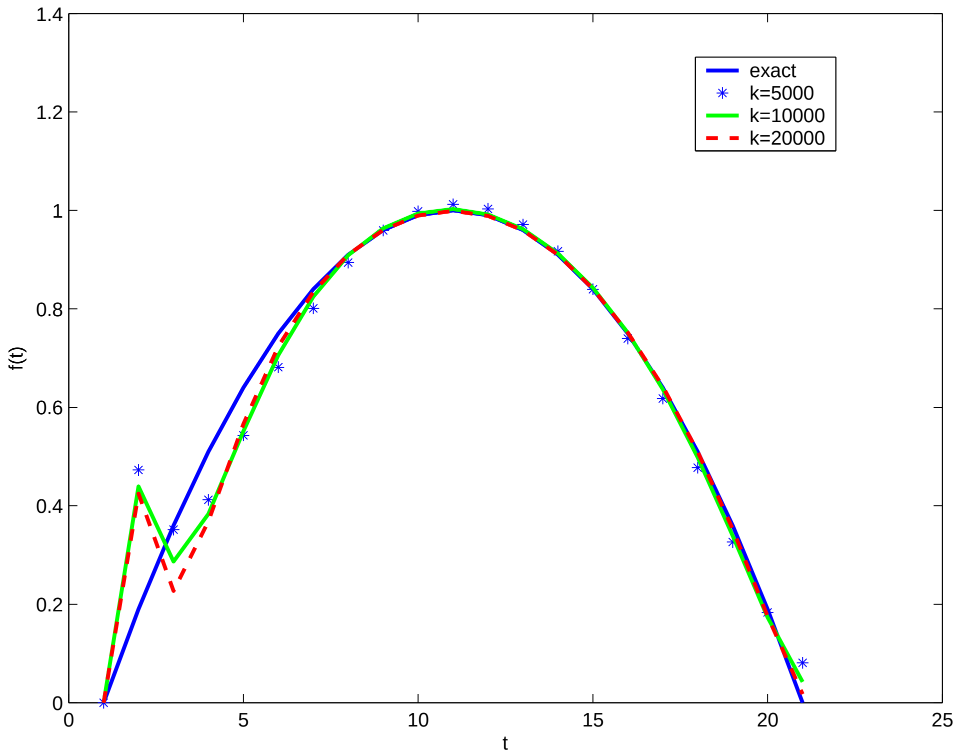

Fig. 6. Reconstruction of heat source at different iteration steps for Example 3.

6. **Concluding remarks**

In this paper, we consider an inverse problem of simultaneously reconstructing the initial temperature value ϕ and heat source f in the heat conduction equation (1.1) using two additional conditions specified at a fixed internal point and the terminal observation time. The uniqueness of the solution is proven and the inverse problem is transformed into an operator equation of first kind so that a new iterative algorithm on the basis of the Landweber iteration can be derived to stably obtain the numerical solution. As shown in the several numerical examples, the proposed method is quite effective for solving this kinds of inverse multi-parameter identification problems. As mentioned in Sec. 2, the inverse problem considered in this paper can be treated as two independent inverse coefficient problems, namely, an inverse source problem and an inverse backward problem. Furthermore, the transformation applies to the case of homogeneous Neumann condition. For other boundary condition cases, e.g., the

Dirichlet boundary condition

$$\begin{cases} u_t - \Delta u = f(t), & (x,t) \in Q, \\ u|_{x=0} = u|_{x=l} = 0, & \\ u|_{t=0} = \phi(x), & \end{cases}$$

the method mentioned above may not work because the boundary condition after differentiating with x is not known. Fortunately, with minor modification in **(Step 3)**, our proposed method is still applicable for these cases. It should be mentioned that with minor modification the conjugate gradient method (**CGM**) (see [14]) can also be applied. Finally, it is worth noting that the approach given in this paper is readily extendable to solve multi-dimensional inverse problems.

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