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POSITIVE SOLUTIONS FOR NONLOCAL SEMIPOSITONE FIRST ORDER BOUNDARY VALUE PROBLEMS

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ABSTRACT. We establish the existence of positive solutions for a nonlinear nonlocal first order boundary value problem by applying a variety of fixed point theorems. Emphasis is put on the fact that the nonlinear terms f_i may take negative values.

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1. INTRODUCTION

Boundary value problems with nonlocal boundary conditions arise in a variety of different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity and plasma physics can be reduced to nonlocal problems. They include two, three and multi-point boundary value problems as special cases and also they have attracted the attention of many authors [3, 6-10, 15] such as Gallardo [5], Karakostas and Tsamatos [12] and the references therein.

While Nieto and Rodriguez-Lopez [14] considered the first order problem

$$y'(t) + r(t)y(t) = f(t), \quad t \in [0,1],$$
$$\lambda y(t) = \sum_{j=1}^{n} \gamma_j y(t_j),$$

Anderson [1] studied the following problem (if $\mathbb{T} = \mathbb{R}$)

$$y'(t) + r(t)y(t) = \lambda f(t, y(t)), \quad t \in [0,1],$$
$$y(0) = y(1) + \sum_{j=1}^{n} \gamma_j y(t_j),$$

where the nonlinear term $f(t, y(t))$ is allowed to take on negative values.

Zhao [16] investigated the problem (if $\mathbb{T} = \mathbb{R}$)

$$y'(t) + r(t)y(t) = \lambda f(t, y(t)), \quad t \in [0,1],$$

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$$y(0) = g(y(1)),$$

where g denotes a nonlinear term.

In 2013, Anderson [2] interested in the first order boundary value problem given by

$$y'(t) - r(t)y(t) = \sum_{i=1}^{m} f_i(t, y(t)), \quad t \in [0,1],$$
$$\lambda y(0) = y(1) + \sum_{j=1}^{n} \Lambda_j(\tau_j, y(\tau_j)), \quad \tau_j \in [0,1],$$

where nonlinear continuous functions f_i and Λ_j are all nonnegative.

In [9], using Krasnosel'ski`ı's fixed point theorem, Goodrich studied the existence of a positive solution to the first-order problem given by (if $\mathbb{T} = \mathbb{R}$)

$$y'(t) + p(t)y(t) = \lambda f(t, y(t)), \quad t \in (a,b),$$
$$y(a) = y(b) + \int_{\tau_1}^{\tau_2} F(s, y(s))ds,$$

where $\tau_1, \tau_2 \in [a,b]_{\mathbb{T}}$ with $\tau_1 < \tau_2$, p and F are nonnegative functions and the nonlinearity f can be negative for some values of t and y.

In recent paper [4], Çetin and Topal concerned with the existence and iteration of positive solutions for the nonlinear nonlocal first order multipoint problem

$$y'(t) + p(t)y(t) = \lambda \sum_{i=1}^{n} f_i(t, y(t)), \quad t \in [0,1], \tag{1.1}$$

$$y(0) = y(1) + \sum_{j=1}^{m} g_j(t_j, y(t_j)), \quad t_j \in [0,1], \tag{1.2}$$

with $\lambda = 1$.

Motivated by the above works, in this paper, we will interested in the existence of at least one, two and three positive solutions to the semipositone first-order nonlinear boundary value problem (1.1)–(1.2), where the continuous function $f_i : [0,1] \times [0,\infty) \to \mathbb{R}$ is semipositone, i.e., $f_i(t,y)$ needn't be positive for all $t \in [0,1]$ and all $y \geq 0$, $p : [0,1] \to [0,\infty)$ is continuous and does not vanish identically and the nonlinear functions $g_j : [0,1] \times [0,\infty) \to [0,\infty)$ satisfy

$$0 \leq v_j^*(t,y)y \leq g_j(t,y) \leq v_j(t,y)y, \quad t \in [0,1]$$

for some positive continuous (possibly nonlinear) functions $v_j, v_j^* : [0,1] \times [0,\infty) \to [0,\infty)$.

2. PRELIMINARIES

In this section we collect some preliminary results that will be used in main results which give the existence of positive solutions of the main problem (1.1)–(1.2). We initially construct the Green function for the linear first order boundary value problem

$$y'(t) + p(t)y(t) = 1, \quad t \in [0,1], \tag{2.1}$$

$$y(0) = y(1). \tag{2.2}$$

The techniques here are standard when establishing the Green function given in [4].

Lemma 2.1. *The unique solution $w(t)$ of the problem* (2.1)–(2.2)

$$w(t) = \int_0^1 G(t,s)ds,$$

where

$$G(t,s) = \frac{\exp(-\int_s^t p(\xi)d\xi)}{1-\exp(-\int_0^1 p(\xi)d\xi)} \begin{cases} 1, & s < t; \\ \exp(-\int_0^t p(\xi)d\xi), & s \geq t \end{cases} \tag{2.3}$$

satisfies $w(t) \leq \frac{1+\exp(\int_0^1 p(\xi)d\xi)}{1-\exp(-\int_0^1 p(\xi)d\xi)}$.

Proof. Set

$$\begin{aligned} w(t) &= \int_0^1 G(t,s)ds \\ &= \int_0^t \frac{\exp(-\int_s^t p(\xi)d\xi)}{1-\exp(-\int_0^1 p(\xi)d\xi)}ds + \int_t^1 \frac{\exp(-\int_s^t p(\xi)d\xi)\exp(-\int_0^1 p(\xi)d\xi)}{1-\exp(-\int_0^1 p(\xi)d\xi)}ds \\ &\leq \int_0^t \frac{1}{1-\exp(-\int_0^1 p(\xi)d\xi)}ds + \int_t^1 \frac{\exp(\int_s^t p(\xi)d\xi)\exp(-\int_0^1 p(\xi)d\xi)}{1-\exp(-\int_0^1 p(\xi)d\xi)}ds \\ &\leq \frac{1}{1-\exp(-\int_0^1 p(\xi)d\xi)}\left(t + \exp(\int_0^1 p(\xi)d\xi)\right) \\ &\leq \frac{1+\exp(\int_0^1 p(\xi)d\xi)}{1-\exp(-\int_0^1 p(\xi)d\xi)}. \end{aligned}$$

Lemma 2.2 ([4])**.** *The Green function $G(t,s)$ which is given by* (2.3) *satisfies*

$$\exp\left(-\int_0^1 p(\xi)d\xi\right)G(s,s) \leq G(t,s) \leq \exp\left(\int_0^1 p(\xi)d\xi\right)G(s,s).$$

Lemma 2.3. *The function $y(t)$ is a solution of the problem* (1.1)–(1.2) *if and only if*

$$y(t) = \lambda \sum_{i=1}^n \int_0^1 G(t,s)f_i(s,y(s))ds + \frac{\exp(-\int_0^t p(\xi)d\xi)}{1-\exp(-\int_0^1 p(\xi)d\xi)}\sum_{j=1}^m g_j(t_j,y(t_j))$$

where $G(t,s)$ is defined by (2.3)

In the proof of Theorem 2.2 in [2], the similar result has been given. Therefore, we don't restate the proof here. Also the solution of the problem (1.1)–(1.2) is given in [4] for $\lambda = 1$.

The following theorems play an important role to prove our main results.

Theorem 2.4 ([11]). *Let $E = (E, \|\cdot\|)$ be a Banach space, and let $P \subset E$ be a cone in B. Assume Ω_1, Ω_2 are bounded open subsets of E with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$, and let $S : P \cap (\overline{\Omega_2} \backslash \Omega_1) \to P$ be a continuous and completely continuous operator such that, either*

(a) $\|Su\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$, *and* $\|Su\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$, *or*
(b) $\|Su\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$, *and* $\|Su\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$.

Then S has a fixed point in $P \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Theorem 2.5 ([13]). *Let P be a cone in Banach space E. Let α, β and γ be three increasing, nonnegative and continuous functionals on P, satisfying for some $c > 0$ and $M > 0$ such that $\gamma(x) \leq \beta(x) \leq \alpha(x)$ and $\|x\| \leq M\gamma(x)$, for all $x \in \overline{P(\gamma, c)}$ Suppose there exists a completely continuous operator $T : \overline{P(\gamma, c)} \to P$ and $0 < a < b < c$ such that*

(i) $\gamma(Tx) < c$, *for all* $x \in \partial P(\gamma, c)$;
(ii) $\beta(Tx) > b$, *for all* $x \in \partial P(\beta, b)$;
(iii) $P(\alpha, a) \neq \emptyset$ *and* $\alpha(Tx) < a$ *for all* $x \in \partial P(\alpha, a)$.

Then T has at least three positive solutions $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ satisfying

$$0 < \alpha(x_1) < a < \alpha(x_2), \quad \beta(x_2) < b < \beta(x_3), \quad \gamma(x_3) < c.$$

3. MAIN RESULTS

In this section, we will give the existence results of positive solutions for the problem (1.1)–(1.2).

Let E denote the Banach space $\mathcal{C}[0, 1]$ with the norm $\|y\| = \max_{t\in[0,1]} |y(t)|$. Define the cone $P \subset E$ by

$$P = \{y \in E : y(t) \geq \gamma\|y\|, \quad t \in [0, 1]\}$$

where $\gamma = \frac{1-\exp(-\int_0^1 p(\xi)d\xi)}{1+\exp(\int_0^1 p(\xi)d\xi)} \exp\left(-\int_0^1 p(\xi)d\xi\right)$ and we can easily see that $0 < \gamma < 1$.

The main results of this paper are following:

Theorem 3.1. *Assume that the following conditions are satisfied:*

(C_1) *There exists a constant $M > 0$ such that $f_i(t, y) \geq -M$ for all $(t, y) \in [0, 1] \times [0, \infty)$,*
(C_2) $\lim_{y\to\infty} \frac{f_i(t,y)}{y} = \infty$ *for* $t \in [0, 1]$.

Then there exists a positive number λ^* such that the problem (1.1)–(1.2) *has at least one positive solution for* $0 < \lambda < \lambda^*$.

Proof. Let $x(t) = \lambda M w(t)$, where $w(t)$ is the unique solution of the boundary value problem (2.1)–(2.2).

We shall show that the following boundary value problem

$$u'(t) + p(t)u(t) = \lambda \sum_{i=1}^{n} F_i(t, u_x(t)), \quad t \in [0,1], \tag{3.1}$$

$$u(0) = u(1) + \sum_{j=1}^{m} g_j(t_j, u_x(t_j)), \quad t_j \in [0,1], \tag{3.2}$$

has at least one positive solution where we'll take $F_i(t, u_x(t)) := f_i(t, u_x) + M$ and $u_x(t) := \max\{u(t) - x(t), 0\}$.

We define the operator $T : P \to E$ with

$$Tu(t) := \lambda \int_0^1 G(t,s) \sum_{i=1}^{n} F_i(s, u_x(s))ds + \frac{\exp(-\int_0^t p(\xi)d\xi)}{1 - \exp(-\int_0^1 p(\xi)d\xi)} \sum_{j=1}^{m} g_j(t_j, u_x(t_j)).$$

It is well known that the existence of positive solutions to the problem (3.1)–(3.2) is equivalent to the existence of fixed points of the operator T in our cone P.

First, it is obvious that T is completely continuous by a standard application of the Arzela-Ascoli Theorem.

Also, for any $u \in P$, we get

$$\begin{aligned}
Tu(t) &= \lambda \int_0^t G(t,s) \sum_{i=1}^{n} F_i(s, u_x(s))ds + \lambda \int_t^1 G(t,s) \sum_{i=1}^{n} F_i(s, u_x(s))ds \\
&\quad + \frac{\exp(-\int_0^t p(\xi)d\xi)}{1 - \exp(-\int_0^1 p(\xi)d\xi)} \sum_{j=1}^{m} g_j(t_j, u_x(t_j)) \\
&\geq \lambda \int_0^t \exp\left(\int_0^1 p(\xi)d\xi\right) G(s,s) \sum_{i=1}^{n} F_i(s, u_x(s))ds \\
&\quad + \lambda \int_t^1 G(s,s) \sum_{i=1}^{n} F_i(s, u_x(s))ds + \frac{\exp(-\int_0^1 p(\xi)d\xi)}{1 - \exp(-\int_0^1 p(\xi)d\xi)} \sum_{j=1}^{m} g_j(t_j, u_x(t_j)) \\
&\geq \lambda \exp\left(-\int_0^1 p(\xi)d\xi\right) \int_0^t \exp\left(\int_0^1 p(\xi)d\xi\right) G(s,s) \sum_{i=1}^{n} F_i(s, u_x(s))ds \\
&\quad + \lambda \exp\left(-\int_0^1 p(\xi)d\xi\right) \int_t^1 \exp\left(\int_0^1 p(\xi)d\xi\right) G(s,s) \sum_{i=1}^{n} F_i(s, u_x(s))ds \\
&\quad + \frac{\exp(-\int_0^1 p(\xi)d\xi)}{1 - \exp(-\int_0^1 p(\xi)d\xi)} \sum_{j=1}^{m} g_j(t_j, u_x(t_j))
\end{aligned}$$

$$
\begin{aligned}
&= \exp\left(-\int_0^1 p(\xi)d\xi\right)\left\{\lambda\int_0^1 \exp\left(\int_0^1 p(\xi)d\xi\right)G(s,s)\sum_{i=1}^n F_i(s,u_x(s))ds\right.\\
&\quad \left.+\frac{1}{1-\exp(-\int_0^1 p(\xi)d\xi)}\sum_{j=1}^m g_j(t_j,u_x(t_j))\right\}\\
&\geq \exp\left(-\int_0^1 p(\xi)d\xi\right)\frac{1-\exp(-\int_0^1 p(\xi)d\xi)}{1+\exp(\int_0^1 p(\xi)d\xi)}\|Tu\| = \gamma\|Tu\|.
\end{aligned}
$$

So, we get $T(P) \subseteq P$.

For each $r>0$, set $\beta_j = \max_{(t,y)\in[0,1]\times[0,r]} v_j(t,y)$. Let define $M_i = \max\{f_i(t,y)+M : (t,y)\in[0,1]\times[0,r]\}$ and choose

$$
\lambda^* = \min\left\{\frac{r(1-\exp(-\int_0^1 p(\xi)d\xi)) - r\sum_{j=1}^m \beta_j}{\sum_{i=1}^n M_i}, \frac{R\gamma^2}{2M}\right\}.
$$

Let $\Omega_r = \{u\in E : \|u\| < r\}$. Then for any $u\in P\cap\partial\Omega_r$ we have $0\leq u_x(t)\leq u(t)\leq \|u\| = r$ and

$$
\begin{aligned}
\|Tu\| &\leq \lambda\int_0^1 \exp(\int_0^1 p(\xi)d\xi)G(s,s)\sum_{i=1}^n F_i(s,u_x(s))ds\\
&\quad +\frac{1}{1-\exp(-\int_0^1 p(\xi)d\xi)}\sum_{j=1}^m g_j(t_j,u_x(t_j))\\
&\leq \lambda\int_0^1 \frac{1}{1-\exp(-\int_0^1 p(\xi)d\xi)}\sum_{i=1}^n M_i ds + \frac{\|u_x\|\sum_{j=1}^m \beta_j}{1-\exp(-\int_0^1 p(\xi)d\xi)}\\
&\leq \frac{1}{1-\exp(-\int_0^1 p(\xi)d\xi)}\left\{\lambda\sum_{i=1}^n M_i + \|u_x\|\sum_{j=1}^m \beta_j\right\}\\
&\leq \frac{1}{1-\exp(-\int_0^1 p(\xi)d\xi)}\left\{\lambda^*\sum_{i=1}^n M_i + r\sum_{j=1}^m \beta_j\right\} \leq r = \|u\|.
\end{aligned}
$$

Therefore, we get $\|Tu\|\leq\|u\|$, for $u\in P\cap\partial\Omega_r$.

We set $\alpha_j = \min_{(t,y)\in[0,1]\times[\frac{\gamma R}{2},R]} v_j^*(t,y)$ for $R>r$. Let K_i be positive real numbers such that

$$
\frac{\gamma}{\exp(\int_0^1 p(\xi)d\xi)-1}\left\{\frac{\lambda}{2}\sum_{i=1}^n K_i + \sum_{j=1}^m \alpha_j\right\}\geq 1.
$$

In view of (C_2), there exists a constant $N>0$ such that for all $z\geq N$ and $t\in[0,1]$, $F_i(t,z) = f_i(t,z)+M\geq K_i z$.

Now set $R = r+\frac{2N}{\gamma}$ and $\Omega_R = \{u\in E : \|u\|\leq R\}$. We shall prove that $\|Tu\|\geq\|u\|$ for $u\in P\cap\partial\Omega_R$.

Let $u \in P \cap \partial\Omega_R$, then $0 \le u_x(t) \le u(t) \le \|u\| = R$. From Lemma 2.1 and the fact that $u \in P$, we get

$$u(t) - x(t) \ge u(t) - \frac{\lambda M}{\gamma} \ge \gamma\|u\| - \frac{\lambda^* M}{\gamma} \ge \gamma R - \frac{\lambda^* M}{\gamma} \ge \gamma R - \frac{\gamma R}{2} = \frac{\gamma R}{2} > 0,$$

and also $u(t) - x(t) \ge \frac{\gamma R}{2} > N$. Thus, we have

$$F_i(t, u_x(t)) = F_i(t, (u-x)(t)) \ge K_i(u(t) - x(t)) \ge K_i \frac{\gamma R}{2} > 0.$$

So we obtain

$$\begin{aligned}
\|Tu\| &= \lambda \int_0^1 \exp\left(\int_0^1 p(\xi)d\xi\right) G(s,s) \sum_{i=1}^n F_i(s, u_x(s))ds \\
&\quad + \frac{\exp(-\int_0^1 p(\xi)d\xi)}{1 - \exp(-\int_0^1 p(\xi)d\xi)} \sum_{j=1}^m g_j(t_j, u_x(t_j)) \\
&\ge \lambda \int_0^1 \exp\left(\int_0^1 p(\xi)d\xi\right) \frac{\exp(-\int_0^s p(\xi)d\xi)\exp(-\int_0^1 p(\xi)d\xi)}{1 - \exp(-\int_0^1 p(\xi)d\xi)} \sum_{i=1}^n F_i(s, u_x(s))ds \\
&\quad + \frac{\exp(-\int_0^1 p(\xi)d\xi)}{1 - \exp(-\int_0^1 p(\xi)d\xi)} \sum_{j=1}^m g_j(t_j, u_x(t_j)) \\
&\ge \lambda \int_0^1 \frac{\exp(-\int_0^s p(\xi)d\xi)}{1 - \exp(-\int_0^1 p(\xi)d\xi)} \sum_{i=1}^n \frac{K_i}{2}\gamma R ds \\
&\quad + \frac{\exp(-\int_0^1 p(\xi)d\xi)}{1 - \exp(-\int_0^1 p(\xi)d\xi)} \sum_{j=1}^m v_j^*(t_j, u_x(t_j))u_x(t_j) \\
&\ge \frac{\exp(-\int_0^1 p(\xi)d\xi)}{1 - \exp(-\int_0^1 p(\xi)d\xi)} \left\{\lambda \frac{\gamma R}{2} \sum_{i=1}^n K_i + R \sum_{j=1}^m \alpha_j\right\} \\
&\ge \frac{1}{\exp(\int_0^1 p(\xi)d\xi) - 1} \gamma R \left\{\frac{\lambda}{2} \sum_{i=1}^n K_i + \sum_{j=1}^m \alpha_j\right\} \ge R = \|u\|.
\end{aligned}$$

Therefore we get $\|Tu\| \ge \|u\|$ for $u \in P \cap \partial\Omega_R$. Consequently, it follows from Theorem 2.1, T has a fixed point $u \in P$ such that $r \le \|u\| \le R$.

Moreover, $u(t) \ge \gamma\|u\| \ge \gamma r \ge 2M\lambda\gamma^{-1}$. Hence for $t \in [0,1]$, $y(t) = u(t) - x(t) \ge \frac{2M\lambda}{\gamma} - \frac{\lambda M}{\gamma} = \frac{\lambda M}{\gamma} > 0$ and it can be easily seen that y is a solution of the problem (1.1)–(1.2). □

Theorem 3.2. *Assume that the condition* (C_1) *and the following condition are satisfied:*

(C_3) $\lim_{y\to\infty} \frac{f_i(t,y)}{y} = \infty$ *and* $\lim_{y\to 0} \frac{f_i(t,y)}{y} = \infty$ *for* $t \in [0,1]$.

Then there exists a positive number λ^* *such that the problem* (1.1)–(1.2) *has at least two positive solutions for* $0 < \lambda < \lambda^*$.

Proof. By the similar proof of Theorem 1, we can easily obtain that $\|Tu\| \leq \|u\|$ for $u \in P \cap \partial\Omega_{r_1}$ for each $r_1 > 0$ and $\lambda^* > \min\left\{\frac{r_1(1-\exp(-\int_0^1 p(\xi)d\xi))-r_1\sum_{j=1}^m \beta_j^*}{\sum_{i=1}^n M_i}, \frac{r_2\gamma^2}{2M}\right\}$ with $\beta_j^* = \max_{(t,y)\in[0,1]\times[0,r_1]} v_j(t,y)$.

If $\lim_{y\to 0}\frac{f_i(t,y)}{y} = \infty$, there exists a positive number $r_2 < r_1$ such that $f_i(t,y) \geq \eta_i y$ for $0 \leq y \leq r_2$, where $\eta_i > 0$ is chosen such that

$$\frac{\gamma}{\exp(\int_0^1 p(\xi)d\xi) - 1}\left\{\frac{\lambda}{2}\sum_{i=1}^n \eta_i + \sum_{j=1}^m \alpha_j^*\right\} \geq 1$$

with $\alpha_j^* = \min_{(t,y)\in[0,1]\times[\frac{\gamma r_2}{2},r_2]} v_j^*(t,y)$. Then $F_i(t,u_x) \geq \eta_i u_x$ for $u_x \in \Omega_{r_2}$, $t \in [0,1]$. So, we get $\|Tu\| \geq \frac{\gamma r_2}{\exp(\int_0^1 p(\xi)d\xi)-1}\left\{\frac{\lambda}{2}\sum_{i=1}^n \eta_i + \sum_{j=1}^m \alpha_j^*\right\} \geq r_2 = \|u\|$ for $u \in P\cap\partial\Omega_{r_2}$.

Thus T has a fixed point u_1 in $\Omega_{r_1}\setminus\overline{\Omega}_{r_2}$ for $0 < \lambda < \lambda^*$ and it can be easily seen that $y_1(t) := u_1(t) - x(t)$ is a positive solution of the problem (1.1)–(1.2) such that $r_2 \leq \|y_1\| \leq r_1$.

If $\lim_{y\to\infty}\frac{f_i(t,y)}{y} = \infty$, there exists an $H > 0$ such that $f_i(t,y) \geq \zeta_i y$ for $y \geq H$, where $\zeta_i > 0$ is chosen such that

$$\frac{\gamma}{\exp(\int_0^1 p(\xi)d\xi) - 1}\left\{\frac{\lambda}{2}\sum_{i=1}^n \zeta_i + \sum_{j=1}^m \alpha_j^{**}\right\} \geq 1$$

with $\alpha_j^{**} = \min_{(t,y)\in[0,1]\times[\frac{\gamma r_3}{2},r_3]} v_j^*(t,y)$. Then $F_i(t,u_x) \geq \zeta_i u_x$ for $z \geq H$.

Let $r_3 = \max\{2r_1, \frac{H}{\gamma}\}$. If $u \in \partial\Omega_{r_3}$, then $\min_{t\in[0,1]} u(t) \geq \gamma\|u\| \geq H$ and $r_3 = \|u\| \geq u_x(t) \geq (u-x)(t) \geq \frac{\gamma r_3}{2}$. Hence $F_i(t,u_x) \geq \zeta_i(u-x)(t) \geq \zeta_i\frac{\gamma r_3}{2}$, similarly in Theorem 3.1. Since

$$\begin{aligned}
\|Tu\| &= \lambda\int_0^1 \exp\left(\int_0^1 p(\xi)d\xi\right) G(s,s)\sum_{i=1}^n F_i(s,u_x(s))ds \\
&\quad + \frac{\exp(-\int_0^1 p(\xi)d\xi)}{1-\exp(-\int_0^1 p(\xi)d\xi)}\sum_{j=1}^m g_j(t_j,u_x(t_j)) \\
&\geq \frac{\lambda}{1-\exp(-\int_0^1 p(\xi)d\xi)}\int_0^1 \exp\left(-\int_0^1 p(\xi)d\xi\right)\sum_{i=1}^n \zeta_i\frac{1}{2}\gamma r_3 ds \\
&\quad + \frac{\exp(-\int_0^1 p(\xi)d\xi)}{1-\exp(-\int_0^1 p(\xi)d\xi)}\sum_{j=1}^m v_j\|u_x\| \\
&\geq \frac{1}{\exp(\int_0^1 p(\xi)d\xi) - 1}\gamma r_3\left\{\frac{\lambda}{2}\sum_{i=1}^n \zeta_i + \sum_{j=1}^m \alpha_j^{**}\right\} \geq r_3 = \|u\|,
\end{aligned}$$

we get $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_{r_3}$.

Then it follows from Theorem 2.1, T has a fixed point $u_2 \in P$ in $\Omega_{r_3}\setminus\overline{\Omega}_{r_1}$ for $\lambda < \lambda^*$ and it can be easily seen that $y_2(t) := u_2(t) - x(t)$ is a positive solution of the problem (1.1)–(1.2) such that $r_1 \leq \|y_2\| \leq r_3$.

Consequently, (1.1)–(1.2) has two positive solutions for $0 < \lambda < \lambda^*$ such that

$$r_2 \le \|y_1\| \le r_1 \le \|y_2\| \le r_3.$$

□

Now, we will give the existence of at least three positive solutions of the problem (1.1)–(1.2).

For notational convenience, we denote k and K by

$$k := \lambda n \left(\frac{\exp(\int_0^1 p(\xi)d\xi) - 1}{\exp(-\int_0^1 p(\xi)d\xi)} - \gamma \sum_{j=1}^{m} \alpha_j^{***} \right)^{-1},$$

$$K := \lambda n \left(1 - \exp\left(-\int_0^1 p(\xi)d\xi \right) - \frac{1}{\gamma} \sum_{j=1}^{m} \beta_j^{**} \right)^{-1},$$

where $\alpha_j^{***} = \min_{(t,y)\in[0,1]\times[\gamma b, \frac{b}{\gamma}]} v_j^*(t,y)$ and $\beta_j^{**} = \max_{(t,y)\in[0,1]\times[0,\frac{c}{\gamma}]} v_j(t,y)$ for $0 < b < c$.

Let the increasing, nonnegative and continuous functionals ϕ, φ and ψ be defined respectively on the cone P by,

$$\phi(y) := \max_{t\in[0,1]} y(t) = y(t_0), \;\; \varphi(y) := \min_{t\in[0,1]} y(t) = y(t_1) \;\; \text{and} \;\; \psi(y) := \min_{t\in[0,1]} y(t) = y(t_1).$$

We see that, for each $y \in P$, $\psi(y) = \varphi(y) \le \phi(y)$. In addition, for each $y \in P$, we know that $\|y\| \le \frac{y(t_1)}{\gamma}$. That is $\|y\| \le \frac{\psi(y)}{\gamma}$ for all $y \in P$.

Theorem 3.3. *Suppose that there exist constants $0 < a < \gamma(1-\gamma)b < b < c$ such that*

(C_4) $f_i(t,y) < \frac{c}{K} - M$, *for* $t \in [0,1]$, $y \in [0, \frac{c}{\gamma}]$,
(C_5) $f_i(t,y) > \frac{b}{k} - M$, *for* $t \in [0,1]$, $y \in [\gamma b, \frac{b}{\gamma}]$,
(C_6) $f_i(t,y) < \frac{a}{K} - M$, *for* $t \in [0,1]$, $y \in [0, a]$.

Then the problem (1.1)–(1.2) *has at least three positive solutions* y_1, y_2, y_3 *and there exist* $0 < d < a$ *such that*

$$d \le \phi(y_1) < a < \phi(y_2) + \lambda M\|\omega\|, \quad \varphi(y_2) < b < \varphi(y_3), \quad \psi(y_3) < c$$

for $0 < \lambda < \frac{d}{M}$.

Proof. Firstly, we consider the modified problem (1.6)–(1.7). It is well known that the existence of positive solutions to the boundary value problem (1.6)–(1.7) is equivalent to the existence of fixed points of the operator

$$Tu(t) := \lambda \int_0^1 G(t,s) \sum_{i=1}^{n} F_i(s, u_x(s))ds + \frac{\exp(-\int_0^t p(\xi)d\xi)}{1 - \exp(-\int_0^1 p(\xi)d\xi)} \sum_{j=1}^{m} g_j(t_j, u_x(t_j)),$$

where $G(t,s)$ is given by (2.5).

To make use of property (i) of Theorem 2.2, we choose $u \in \partial P(\psi, c)$. Then $\psi(u) := \min_{t\in[0,1]} u(t) = u(t_1) = c$. If we are recalling that $\|u\| \leq \frac{\psi(u)}{\gamma} \leq \frac{c}{\gamma}$, we have $0 \leq u_x(t) \leq u(t) \leq \|u\| \leq \frac{c}{\gamma}$ for all $t \in [0, 1]$.

From (C_4), we get

$$\begin{aligned}
\psi(Tu) &= (Tu)(t_1) = \lambda \int_0^1 G(t,s) \sum_{i=1}^n F_i(s, u_x(s))ds \\
&\quad + \frac{\exp(-\int_0^{t_1} p(\xi)d\xi)}{1-\exp(-\int_0^1 p(\xi)d\xi)} \sum_{j=1}^m g_j(t_j, u_x(t_j)) \\
&\leq \lambda \int_0^1 \exp\left(\int_0^1 p(\xi)d\xi\right) G(s,s) \sum_{i=1}^n F_i(s, u_x(s))ds + \frac{\|u_x\| \sum_{j=1}^m \beta_j^{**}}{1-\exp(-\int_0^1 p(\xi)d\xi)} \\
&\leq \frac{\lambda}{1-\exp(-\int_0^1 p(\xi)d\xi)} \int_0^1 \sum_{i=1}^n (f_i + M)ds + \frac{\|u_x\| \sum_{j=1}^m \beta_j^{**}}{1-\exp(-\int_0^1 p(\xi)d\xi)} \\
&\leq \frac{1}{1-\exp(-\int_0^1 p(\xi)d\xi)} \left\{ \lambda \sum_{i=1}^n (\frac{c}{K} - M + M) + \frac{c}{\gamma} \sum_{j=1}^m \beta_j^{**} \right\} \\
&= \frac{1}{1-\exp(-\int_0^1 p(\xi)d\xi)} \left\{ \lambda \frac{c}{K} n + \frac{c}{\gamma} \sum_{j=1}^m \beta_j^{**} \right\} \\
&= \frac{c}{1-\exp(-\int_0^1 p(\xi)d\xi)} \left\{ \frac{\lambda n}{K} + \frac{1}{\gamma} \sum_{j=1}^m \beta_j^{**} \right\} = c,
\end{aligned}$$

where $K = \lambda n \left(1 - \exp\left(-\int_0^1 p(\xi)d\xi\right) - \frac{1}{\gamma}\sum_{j=1}^m \beta_j\right)^{-1}$. Then condition (i) of Theorem 2.2 holds.

Secondly, we show that (ii) of Theorem 2.2 is fulfilled. For this, we select $u \in \partial P(\varphi, b)$. Then $\varphi(u) = \min_{t\in[0,1]} u(t) = y(t_1) = b$. This means $u(t) \geq b$ for all $t \in [0, 1]$. Since $u \in P$ and $\lambda < \frac{d}{M} < \frac{\gamma(1-\gamma)b}{M}$, we have

$$\begin{aligned}
\|u\| \geq u(t) \geq u_x(t) \geq (u-x)(t) &\geq b - \lambda M \omega(t) \geq b - \lambda M \|\omega\| \\
&\geq b - \frac{\lambda M}{\gamma} \geq b - \frac{\gamma(1-\gamma)b}{M}\frac{M}{\gamma} = \gamma b.
\end{aligned}$$

Note that $\|u\| \leq \frac{\varphi(u)}{\gamma} \leq \frac{b}{\gamma}$. So we have $\gamma b \leq u_x(t) \leq \frac{b}{\gamma}$ for all $t \in [0, 1]$. Therefore

$$\begin{aligned}
\varphi(Tu) &= (Tu)(t_1) = \lambda \int_0^1 G(t_1,s) \sum_{i=1}^n F_i(s, u_x(s))ds \\
&\quad + \frac{\exp(-\int_0^{t_1} p(\xi)d\xi)}{1-\exp(-\int_0^1 p(\xi)d\xi)} \sum_{j=1}^m g_j(t_j, u_x(t_j)) \\
&\geq \lambda \int_0^1 \exp\left(-\int_0^1 p(\xi)d\xi\right) G(s,s) \sum_{i=1}^n (f_i + M)ds
\end{aligned}$$

$$
\begin{aligned}
& +\frac{\exp(-\int_0^1 p(\xi)d\xi)}{1-\exp(-\int_0^1 p(\xi)d\xi)}\sum_{j=1}^m \alpha_j^{***}\|u_x\| \\
\geq\ & \lambda \exp\left(-\int_0^1 p(\xi)d\xi\right)\frac{\exp(-\int_0^1 p(\xi)d\xi)}{1-\exp(-\int_0^1 p(\xi)d\xi)}\sum_{i=1}^n (f_i+M) \\
& +\frac{\exp(-\int_0^1 p(\xi)d\xi)}{1-\exp(-\int_0^1 p(\xi)d\xi)}\sum_{j=1}^m \alpha_j^{***}\gamma b \\
\geq\ & \frac{\exp(-\int_0^1 p(\xi)d\xi)}{1-\exp(-\int_0^1 p(\xi)d\xi)}\left\{\lambda \exp\left(-\int_0^1 p(\xi)d\xi\right)\sum_{i=1}^n \frac{b}{k}+\gamma b\sum_{j=1}^m \alpha_j^{***}\right\} \\
=\ & \frac{\exp(-\int_0^1 p(\xi)d\xi)}{1-\exp(-\int_0^1 p(\xi)d\xi)}\left\{\lambda \exp\left(-\int_0^1 p(\xi)d\xi\right)\frac{b}{k}n+\gamma b\sum_{j=1}^m \alpha_j^{***}\right\} \\
\geq\ & \frac{\exp(-2\int_0^1 p(\xi)d\xi)}{1-\exp(-\int_0^1 p(\xi)d\xi)} b\left\{\frac{\lambda n}{k}+\gamma\sum_{j=1}^m \alpha_j^{***}\right\}=b,
\end{aligned}
$$

where $k=\lambda n\left(\frac{\exp(\int_0^1 p(\xi)d\xi)-1}{\exp(-\int_0^1 p(\xi)d\xi)}-\gamma\sum_{j=1}^m \alpha_j^{***}\right)^{-1}$. Then condition (ii) of Theorem 2.2 holds.

Finally, we verify that (iii) of Theorem 2.2 is also satisfied. We note that $u(t)=\frac{a}{2}$ for $t\in[0,1]$ is a member of $P(\phi,a)$ since $\phi(u)=\frac{a}{2}<a$. Therefore $P(\phi,a)\neq\varnothing$. Now let $u\in P(\phi,a)$. Then $\phi(u)=\max_{t\in[0,1]}u(t)=u(t_0)=a$. This means $0\leq u_x(t)\leq u(t)\leq a$, for all $t\in[0,1]$.

Then by condition (C_6) of this theorem and since

$$
\max_{(t,y)\in[0,1]\times[0,a]} v_j(t,y)\leq \max_{(t,y)\in[0,1]\times[0,\frac{c}{\gamma}]} v_j(t,y)=\beta_j^{**},
$$

we have

$$
\begin{aligned}
\phi(Tu)\ =\ & (Tu)(t_0)=\lambda\int_0^1 G(t_0,s)\sum_{i=1}^n F_i(s,u_x(s))ds \\
& +\frac{\exp(-\int_0^{t_0} p(\xi)d\xi)}{1-\exp(-\int_0^1 p(\xi)d\xi)}\sum_{j=1}^m g_j(t_j,u_x(t_j)) \\
\leq\ & \lambda\int_0^1 \exp\left(\int_0^1 p(\xi)d\xi\right)G(s,s)\sum_{i=1}^n F_i(s,u_x(s))ds+\frac{\|u_x\|\sum_{j=1}^m \beta_j^{**}}{1-\exp(-\int_0^1 p(\xi)d\xi)} \\
\leq\ & \frac{\lambda}{1-\exp(-\int_0^1 p(\xi)d\xi)}\int_0^1\sum_{i=1}^n (f_i+M)ds+\frac{\|u_x\|\sum_{j=1}^m \beta_j^{**}}{1-\exp(-\int_0^1 p(\xi)d\xi)} \\
\leq\ & \frac{1}{1-\exp(-\int_0^1 p(\xi)d\xi)}\left\{\lambda\sum_{i=1}^n \left(\frac{a}{K}-M+M\right)+\frac{a}{\gamma}\sum_{j=1}^m \beta_j^{**}\right\} \\
=\ & \frac{1}{1-\exp(-\int_0^1 p(\xi)d\xi)}\left\{\lambda\frac{a}{K}n+\frac{a}{\gamma}\sum_{j=1}^m \beta_j^{**}\right\}
\end{aligned}
$$

$$= \frac{a}{1-\exp(-\int_0^1 p(\xi)d\xi)}\left\{\frac{\lambda n}{K}+\frac{\sum_{j=1}^m \beta_j^{**}}{\gamma}\right\}=a,$$

where $K = \lambda n\left(1-\exp\left(-\int_0^1 p(\xi)d\xi\right)-\frac{1}{\gamma}\sum_{j=1}^m \beta_j^{**}\right)^{-1}$. Hence condition (iii) of Theorem 2.2 is satisfied.

Consequently it follows from Theorem 2.2 that T has at least three fixed points which are positive solutions u_1, u_2 and u_3 belonging to $\overline{P(\psi,c)}$ of (1.6)–(1.7) such that

$$0<\phi(u_1)<a<\phi(u_2), \quad \varphi(u_2)<b<\varphi(u_3), \quad \psi(u_3)<c,$$

and so there exists $d>0$ such that $d \le \max_{t\in[0,1]} u_1(t)=\phi(u_1)$.

Similarly in Theorem 3.1 and Theorem 3.2, we can easily show that $y_i(t) := u_i(t)-x(t)$, $(i=1,2,3)$ are the solutions of the problem (1.1)–(1.2).

Firstly, we shall show that $y_i(t)$ are positive. Since the solutions of the problem (1.6)–(1.7) satisfy the inequalities $d \le \phi(u_1) < a < \phi(u_2)$, $\varphi(u_2) < b < \varphi(u_3)$, $\psi(u_3) < c$ and the definitions of the functionals ϕ, φ and ψ, we obtain $\|u_i\| \ge d$ for $i=1,2,3$. Thus

$$\begin{aligned} y_i(t) &= u_i(t)-x(t)=u_i(t)-\lambda M\omega(t) \ge u_i(t)-\lambda M\frac{1}{\gamma} \\ &\ge u_i(t)-\lambda M\frac{u_i(t)}{\|u_i\|} \ge \left(1-\frac{\lambda M}{d}\right)u_i(t)>0, \end{aligned}$$

since $0<\lambda<\frac{d}{M}$. Also, $y_i(t)$ $(i=1,2,3)$ satisfy the problem (1.1)–(1.2) similarly in Theorem 3.1 and Theorem 3.2.

So, it follows that y_1, y_2 and y_3 are positive solutions of the problem (1.1)–(1.2). In addition, we get

$$d \le \phi(y_1) \le \phi(u_1) < a < \phi(u_2) < \phi(u_2)+\lambda M\|\omega\|,$$

$$\varphi(y_2) \le \varphi(u_2) < b \le \varphi(y_3) \le \varphi(u_3), \quad \text{and} \quad \psi(u_3)<c.$$

□

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