Dynamic Systems and Applications 25 (2016) 29-38

SPECTRAL ANALYSIS OF A MATRIX-VALUED QUANTUM-DIFFERENCE OPERATOR

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ABSTRACT. The aim of this work is to find a polynomial-type Jost solution of a self adjoint matrix-valued q-difference equation of second order and investigate the spectral properties of the operator L generated by this q-difference expression by using asymptotic behavior of the Jost solution.

AMS (MOS) Subject Classification. 39A05, 39A70, 47A05.

This paper is dedicated to the 25th Anniversary of the journal

Dynamic Systems and Applications

and to its founder and Editor-in-Chief

Professor Masilamani Sambandham

1. INTRODUCTION

In recent years, quantum calculus which appeared as a connection between mathematics and physics has led to rapid development of the theory of q-difference equations. It arose interest due to high demand of mathematics that models quantum competing. The study of q-difference equations has become an important area of research because of the fact that such equations occur in a variety of real-world problems. Quantum difference operators have an interest role due to their applications in several mathematical areas such as number theory, orthogonal polynomials, mathematical control theories, combinatorics, basic hyper-geometric functions and other sciences of quantum theory such as mechanics and the theory of relativity. Several problems of q-difference equations have been treated by various authors [13, 15, 16]. q-analogues of some well-known definitions and theorems of ordinary calculus have been given in [12]. It has been shown that quantum calculus is a subfield of a more general mathematical field of time scales calculus. Some useful generalizations and important results were given for dynamic equations on arbitrary time scales, which

Received December 18, 2015

contain q-difference equations as a special case in [7, 8]. Also, spectral analysis of non-selfadjoint q-difference equations has been investigated in [1, 2, 5, 6] using the analytical properties of the Jost solutions. But spectral analysis of matrix-valued q-difference equations including a polynomial type Jost solution has not been investigated yet. For related results concerning standard Jacobi matrix theory, we refer the reader to [9, 10].

Hereafter, we are concerned with a specific time scale called the q-time scale defined as follows:

$$q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0\},$$

where $q > 1$ and $\mathbb{N}_0$ denotes the set of nonnegative integers. Let us introduce the Hilbert space $\ell_2(q^{\mathbb{N}}, \mathbb{C}^m)$ consisting of all vector sequences

$$y = \left\{y(t) \in \mathbb{C}^m : t \in q^{\mathbb{N}}\right\}$$

such that $\sum_{t\in q^{\mathbb{N}}} \|y(t)\|^2_{\mathbb{C}^m}\mu(t) < \infty$ with the inner product

$$\langle y, z\rangle_q := \sum_{t\in q^{\mathbb{N}}} (y(t), z(t))_{\mathbb{C}^m}\, \mu(t),$$

where $\mathbb{C}^m$ is m-dimensional ($m < \infty$) Euclidean space and $\mu(t) = (q-1)t$ for all $t \in q^{\mathbb{N}}$. $\|\cdot\|_{\mathbb{C}^m}$ and $(\cdot,\cdot)_{\mathbb{C}^m}$ denote the norm and inner product in $\mathbb{C}^m$, respectively. Furthermore, we denote by L the q-difference operator generated in $\ell_2(q^{\mathbb{N}}, \mathbb{C}^m)$ by the q-difference expression

$$(ly)(t) := qA(t)y(qt) + B(t)y(t) + A\left(\frac{t}{q}\right) y\left(\frac{t}{q}\right), \quad t \in q^{\mathbb{N}},$$

and the boundary condition $y(1) = 0$, where $A(t)$, $t \in q^{\mathbb{N}_0}$ and $B(t)$, $t \in q^{\mathbb{N}}$ are linear operators (matrices) acting in $\mathbb{C}^m$. Throughout the paper, we will assume that $A(t)$ is invertible, $A(t) = A^*(t)$ for all $t \in q^{\mathbb{N}_0}$ and $B(t) = B^*(t)$ for all $t \in q^{\mathbb{N}}$, where $*$ denotes the adjoint operator. Note that we can also define the operator L using the infinite matrix

$$J = \begin{pmatrix} B(q) & qA(q) & 0 & 0 & 0 & \cdots \\ A(q) & B(q^2) & qA(q^2) & 0 & 0 & \cdots \\ 0 & A(q^2) & B(q^3) & qA(q^3) & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where 0 is the zero operator in $\mathbb{C}^m$. It is clear that L is self adjoint in $\ell_2(q^{\mathbb{N}}, \mathbb{C}^m)$. Related to the operator L, we will consider the matrix q-difference equation of second order

$$qA(t)y(qt) + B(t)y(t) + A\left(\frac{t}{q}\right) y\left(\frac{t}{q}\right) = \lambda y(t), \quad t \in q^{\mathbb{N}}. \tag{1.1}$$

The paper is organized as follows: In Section 2, we find a polynomial-type Jost solution of (1.1) and investigate analytical properties and asymptotic behavior of this solution. In Section 3, we get that the continuous spectrum of L is $[-2\sqrt{q}, 2\sqrt{q}]$, and L has at most finitely many simple real eigenvalues. The purpose of this paper is to extend some results for matrix difference equation obtained in [4] to the case of q-difference equations.

2. JOST SOLUTION OF $(ly)(t) = \lambda y(t)$

We assume that the matrix sequences $\{A(t)\}$ and $\{B(t)\}$, $t \in q^{\mathbb{N}}$ satisfy

$$\sum_{t \in q^{\mathbb{N}}} (\|I - A(t)\| + \|B(t)\|) < \infty, \tag{2.1}$$

where $\|\cdot\|$ denotes the matrix norm in $\mathbb{C}^m$ and I is the identity matrix. Let $E(\cdot, z)$ denote the bounded matrix solution of the equation

$$qA(t)y(qt) + B(t)y(t) + A\left(\frac{t}{q}\right) y\left(\frac{t}{q}\right) = \sqrt{q}(z + z^{-1})y(t), \quad t \in q^{\mathbb{N}}, \tag{2.2}$$

satisfying the condition

$$\lim_{t\to\infty} E(t, z) z^{-\frac{\ln t}{\ln q}} \sqrt{\mu(t)} = I, \quad z \in D_0 := \{z \in \mathbb{C} : |z| = 1\}. \tag{2.3}$$

The solution $E(\cdot, z)$ is called the Jost solution of (2.2). The following theorem establishes the existence of the Jost solution under the condition (2.1).

Theorem 2.1. *Assume* (2.1). *Let the solution* $E(\cdot, z)$ *be the Jost solution of* (2.2). *Then* $E(\cdot, z)$ *exists and is given by*

$$E(t, z) = \frac{z^{\frac{\ln t}{\ln q}}}{\sqrt{\mu(t)}} I + \sum_{s \in [qt,\infty)\cap q^{\mathbb{N}}} \sqrt{\frac{s}{qt}} \quad \frac{z^{\frac{\ln s - \ln t}{\ln q}} - z^{\frac{\ln t - \ln s}{\ln q}}}{z - z^{-1}} H(s), \tag{2.4}$$

where

$$H(s) := \left[I - A\left(\frac{s}{q}\right)\right] E\left(\frac{s}{q}, z\right) - B(s)E(s, z) + q[I - A(s)]E(qs, z).$$

Proof. From (2.2), we get that

$$E\left(\frac{t}{q}\right) + qE(qt) - \sqrt{q}(z + z^{-1})E(t) = H(t). \tag{2.5}$$

Since $\frac{z^{\frac{\ln t}{\ln q}}}{\sqrt{\mu(t)}} I$ and $\frac{z^{-\frac{\ln t}{\ln q}}}{\sqrt{\mu(t)}} I$ are linearly independent solutions of the homogeneous equation

$$E\left(\frac{t}{q}\right) + qE(qt) - \sqrt{q}(z + z^{-1})E(t) = 0,$$

we obtain the general solution of (2.5) as

$$\begin{aligned} E(t, z) &= \frac{z^{\frac{\ln t}{\ln q}}}{\sqrt{\mu(t)}} \alpha + \frac{z^{-\frac{\ln t}{\ln q}}}{\sqrt{\mu(t)}} \beta \\ &\quad + \sum_{s \in [qt,\infty)\cap q^{\mathbb{N}}} \sqrt{\frac{\mu(s)}{q}} \frac{1}{\sqrt{\mu(t)}} \quad \frac{z^{\frac{\ln s - \ln t}{\ln q}} - z^{\frac{\ln t - \ln s}{\ln q}}}{z - z^{-1}} H(s), \end{aligned} \tag{2.6}$$

by using the method of variation of parameters, where α and β are constants in $\mathbb{C}^m$. Using (2.3) and (2.6), we find $\alpha = I$ and $\beta = 0$. This shows $E(t,z)$ satisfies (2.4). Since the series in (2.4) is convergent under assumption (2.1), $E(\cdot,z)$ exists. ☐

Theorem 2.2. *Assume* (2.1). *Then the Jost solution* $E(\cdot,z)$, *has a representation*

$$E(t,z) = T(t)\frac{z^{\frac{\ln t}{\ln q}}}{\sqrt{\mu(t)}}\left(I + \sum_{r\in q^{\mathbb{N}}} K(t,r) z^{\frac{\ln r}{\ln q}}\right), \quad t \in q^{\mathbb{N}_0}, \tag{2.7}$$

where $z \in D_0$, $T(t)$ *and* $K(t,r)$ *are expressed in terms of* $\{A(t)\}$ *and* $\{B(t)\}$.

Proof. Substituting $E(\cdot,z)$ defined by (2.7) into (2.2), we obtain

$$\begin{aligned}
&qA(t)T(qt)\frac{z^{\frac{\ln qt}{\ln q}}}{\sqrt{\mu(qt)}} + qA(t)T(qt)\frac{z^{\frac{\ln qt}{\ln q}}}{\sqrt{\mu(qt)}}\sum_{r\in q^{\mathbb{N}}} K(t,r) z^{\frac{\ln r}{\ln q}} \\
&\quad + A\left(\frac{t}{q}\right)T\left(\frac{t}{q}\right)\frac{z^{\frac{\ln \frac{t}{q}}{\ln q}}}{\sqrt{\mu(\frac{t}{q})}} + A\left(\frac{t}{q}\right)T\left(\frac{t}{q}\right)\frac{z^{\frac{\ln \frac{t}{q}}{\ln q}}}{\sqrt{\mu(\frac{t}{q})}}\sum_{r\in q^{\mathbb{N}}} K\left(\frac{t}{q},r\right) z^{\frac{\ln r}{\ln q}} \\
&= \sqrt{q}(z+z^{-1})\alpha(t)\frac{z^{\frac{\ln t}{\ln q}}}{\sqrt{\mu(t)}} + \sqrt{q}(z+z^{-1})\alpha(t)\frac{z^{\frac{\ln t}{\ln q}}}{\sqrt{\mu(t)}}\sum_{r\in q^{\mathbb{N}}} K(t,r) z^{\frac{\ln r}{\ln q}}
\end{aligned}$$

and

$$\begin{aligned}
T(t) &= \prod_{p\in[t,\infty)\cap q^{\mathbb{N}}} [A(p)]^{-1}, \\
K(t,q) &= -\frac{1}{\sqrt{q}}\sum_{p\in[qt,\infty)\cap q^{\mathbb{N}}} T^{-1}(p)B(p)T(p), \\
K(t,q^2) &= -\frac{1}{\sqrt{q}}\sum_{p\in[qt,\infty)\cap q^{\mathbb{N}}} T^{-1}(p)B(p)T(p)K(p,q) \\
&\quad + \sum_{p\in[qt,\infty)\cap q^{\mathbb{N}}} T^{-1}(p)\left[I - A^2(p)\right]T(p), \\
K(t,rq^2) &= K(qt,r) + \sum_{p\in[qt,\infty)\cap q^{\mathbb{N}}} T^{-1}(p)\left[I - A^2(p)\right]T(p)K(qp,r) \\
&\quad - \frac{1}{\sqrt{q}}\sum_{p\in[qt,\infty)\cap q^{\mathbb{N}}} T^{-1}(p)B(p)T(p)K(p,qr),
\end{aligned}$$

for $r \in q^{\mathbb{N}}$ and $t \in q^{\mathbb{N}_0}$. Due to the condition (2.1) the infinite product and the series in $K(t,r)$ are absolutely convergent. ☐

In the following, we will assume that the matrix sequences $\{A(t)\}$ and $\{B(t)\}$, $t \in q^{\mathbb{N}}$ satisfy

$$\sum_{t\in q^{\mathbb{N}}} \frac{\ln t}{\ln q}\left(\|I - A(t)\| + \|B(t)\|\right) < \infty. \tag{2.8}$$

Theorem 2.3. *Under the condition* (2.8)*, the Jost solution* $E(\cdot, z)$ *has an analytic continuation from* D_0 *to* $\{z \in \mathbb{C} : |z| < 1\} \setminus \{0\}$*.*

Proof. By using induction and the equalities for $K(t, r)$ given in Theorem 2.2, we obtain

$$\|K(t,r)\| \leq C \sum_{p \in \left[tq^{\lfloor \frac{\ln r}{2 \ln q} \rfloor}, \infty\right) \cap q^{\mathbb{N}}} \left(\|I - A(p)\| + \|B(p)\|\right), \tag{2.9}$$

where $\lfloor \frac{\ln r}{2 \ln q} \rfloor$ is the integer part of $\frac{\ln r}{2 \ln q}$ and $C > 0$ is a constant. From (2.9), we get that the series in (2.7) is absolutely convergent in $D := \{z \in \mathbb{C} : |z| \leq 1\} \setminus \{0\}$. This shows that $E(\cdot, z)$ has an analytic continuation from D_0 to $\{z \in \mathbb{C} : |z| < 1\} \setminus \{0\}$. □

Note that (2.8) implies (2.1). For the inequality (2.9), condition (2.1) is enough, but for analytic continuation, we need condition (2.8).

Theorem 2.4. *Under the condition* (2.8)*, the Jost solution satisfies the following asymptotic equation for* $z \in D$*:*

$$E(t,z) = \frac{z^{\frac{\ln t}{\ln q}}}{\sqrt{\mu(t)}} \left(I + o(1)\right), \quad t \to \infty. \tag{2.10}$$

Proof. Using the definition of $T(t)$, (2.8), and (2.9), we can write

$$\lim_{t \to \infty} T(t) = I \tag{2.11}$$

and

$$\sum_{r \in q^{\mathbb{N}}} K(t,r) z^{\frac{\ln r}{\ln q}} = \mathrm{o}(1), \quad z \in D, \quad t \to \infty. \tag{2.12}$$

From (2.7), (2.11), and (2.12), we get (2.10). □

3. CONTINUOUS AND DISCRETE SPECTRUM OF L

Let L_0 and L_1 denote the q-difference operators generated in $\ell_2(q^{\mathbb{N}}, \mathbb{C}^m)$ by the q-difference expressions

$$(l_0 y)(t) = q y(qt) + y\left(\frac{t}{q}\right)$$

and

$$(l_1 y)(t) = q\left[A(t) - I\right] y(qt) + B(t) y(t) + \left[A\left(\frac{t}{q}\right) - I\right] y\left(\frac{t}{q}\right)$$

with the boundary condition $y(1) = 0$, respectively. It is clear that $L = L_0 + L_1$.

Lemma 3.1. *The operator* L_0 *is self adjoint.*

Proof. For all $y \in \ell_2(q^{\mathbb{N}}, \mathbb{C}^m)$, we can write

$$\|L_0 y\|_q \leq 2\sqrt{q}\|y\|_q,$$

so L_0 is bounded in the Hilbert space $\ell_2(q^{\mathbb{N}}, \mathbb{C}^m)$. Since

$$\begin{aligned}\langle l_0 y, z\rangle_q &= \sum_{t\in q^{\mathbb{N}}} \mu(t)\left(l_0 y(t), z(t)\right)_{\mathbb{C}^m} \\ &= \sum_{t\in q^{\mathbb{N}}} \mu(t)(z(t))^* \left(q y(qt) + y\left(\frac{t}{q}\right)\right) \\ &= \sum_{t\in q^{\mathbb{N}}} \mu(t)\left(q z(qt) + z\left(\frac{t}{q}\right)\right)^* y(t) = \langle y, l_0 z\rangle_q,\end{aligned}$$

L_0 is a symmetric operator in $l_2(q^{\mathbb{N}}, \mathbb{C}^m)$, i.e., L_0 is self adjoint. □

It is easy to see that L_0 has no eigenvalues, so the spectrum of the operator L_0 consists only of its continuous spectrum on the real axis. Since L_0 is self adjoint and bounded in the Hilbert space $\ell_2(q^{\mathbb{N}}, \mathbb{C}^m)$, its spectrum is on the real axis and in the interval $[-n, N]$, where

$$n = \inf_{\|u\|_q=1} \langle L_0 u, u\rangle_q, \quad N = \sup_{\|u\|_q=1} \langle L_0 u, u\rangle_q$$

for all $u \in \ell_2(q^{\mathbb{N}}, \mathbb{C}^m)$ [14]. First, we find that $n = -2\sqrt{q}$ and $N = 2\sqrt{q}$. Then we get $\lambda = N + r \in \rho(L_0)$ and $\lambda = n - r \in \rho(L_0)$, where $r > 0$, $\rho(L_0)$ is resolvent set of the operator L_0, i.e., $\rho(L_0) = \mathbb{C}\backslash\sigma(L_0)$ and $\sigma(L_0)$ is the spectrum of the operator L_0. Finally, we obtain that $\pm 2\sqrt{q} \in \sigma(L_0)$. So we can write

$$\sigma(L_0) = \sigma_{\mathrm{c}}(L_0) = [-2\sqrt{q}, 2\sqrt{q}].$$

Another way of finding the continuous spectrum of the operator L_0 is given in [1].

Theorem 3.2. *Assume* (2.8). *Then* $\sigma_{\mathrm{c}}(L) = [-2\sqrt{q}, 2\sqrt{q}]$.

Proof. From (2.8), it can be easily seen that L_1 is a compact operator in $\ell_2(q^{\mathbb{N}}, \mathbb{C}^m)$. Since $L = L_0 + L_1$ and $L_0 = L_0^*$, we get

$$\sigma_{\mathrm{c}}(L_0) = \sigma_{\mathrm{c}}(L) = [-2\sqrt{q}, 2\sqrt{q}]$$

from Weyl's theorem of a compact perturbation [11, p. 13]. □

Since the operator L is self adjoint, the eigenvalues of L are real. Let $a(z) := \det E(1, z)$ for all $z \in D$, where $E(1, z)$ denotes the Jost function of L. From the definition of the eigenvalues, we obtain that

$$\sigma_{\mathrm{d}}(L) = \left\{\lambda \in \mathbb{R} : \ \lambda = \sqrt{q}(z + z^{-1}), \ z \in (-1, 0) \cup (0, 1), \ a(z) = 0\right\},$$

where $\sigma_{\mathrm{d}}(L)$ denotes the set of all eigenvalues of L. $\sigma_{\mathrm{d}}(L)$ and $\sigma_{\mathrm{c}}(L)$ are separated sets, so we can write

$$\sigma_{\mathrm{d}}(L) \subset (-\infty, -2\sqrt{q}) \cup (2\sqrt{q}, \infty).$$

Definition 3.3. The multiplicity of a zero of the function $a(z)$ is called the multiplicity of the corresponding eigenvalue of L.

Theorem 3.4. *Assume* (2.8)*. Then the operator L has at most finitely many simple real eigenvalues.*

Proof. Let $J_1^{(k)}$ and $J_2^{(k)}$ denote the matrices

$$(J_1^{(k)})_{ij} = \begin{cases} B(q^i) & \text{if} \quad 1 \leq i = j \leq k \\ q(A(q^i) - I) & \text{if} \quad j = i+1, \quad 1 \leq i \leq k \\ A(q^{i-1}) - I & \text{if} \quad j = i-1, \quad 2 \leq i \leq k \\ 0 & \text{otherwise,} \end{cases}$$

$$(J_2^{(k)})_{ij} = \begin{cases} B(q^i) & \text{if} \quad i = j \geq k+1 \\ qA(q^i) & \text{if} \quad j = i+1, \quad i \geq k+1 \\ A(q^{i-1}) & \text{if} \quad j = i-1, \quad i \geq k+1 \\ qI & \text{if} \quad j = i+1, \quad 1 \leq i < k+1 \\ I & \text{if} \quad j = i-1, \quad 2 \leq i < k+1 \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $J = J_1^{(k)} + J_2^{(k)}$. We denote by $L_1^{(k)}$ and $L_2^{(k)}$ self adjoint operators generated in $\ell_2(q^{\mathbb{N}}, \mathbb{C}^m)$ by $J_1^{(k)}$ and $J_2^{(k)}$, respectively. Thus, we can write $L = L_1^{(k)} + L_2^{(k)}$, and for sufficiently large k the operator $L_2^{(k)}$ has no eigenvalues. Because under the condition (2.8), $A(q^k) \to I$, $B(q^k) \to 0$ for $k \to \infty$, and the Jost solution of the equation

$$L_2^{(k)} y = \sqrt{q}(z + z^{-1}) y$$

satisfies

$$E^{(k)}(t, z) = \frac{z^{\frac{\ln t}{\ln q}}}{\sqrt{\mu(t)}} \left(I + \mathrm{o}(1)\right), \quad t \in q^{\mathbb{N}_0}. \tag{3.1}$$

From the equation (3.1), we obtain $E^{(k)}(1, z) = \frac{I}{\sqrt{q-1}} + \mathrm{o}(1)$, i.e.,

$$\det E^{(k)}(1, z) \neq 0,$$

and this gives that $L_2^{(k)}$ has no eigenvalues. Also $L_2^{(k)}$ has continuous spectrum filling the segment $[-2\sqrt{q}, 2\sqrt{q}]$ (see Theorem 3.2). On the other hand, the operator $L_1^{(k)}$ is a finite-dimensional self adjoint operator. From the theorem of decomposition of linear operators [11, Theorem 12], we get that L has at most finitely many real eigenvalues. If we obtain that the zeros of the function $a(z)$ are simple, we will complete the proof.

Let z_0 be one of the zeros of $a(z)$. Now we show that

$$\frac{\mathrm{d}}{\mathrm{d}z}a(z)\mid_{z_0}\neq 0.$$

Hence $a(z_0) = \det E(1, z_0) = 0$, there is a non-zero vector u such that $E(1, z_0)u = 0$ (see [3]). As we know, $E(t, z)$ is the Jost solution of (2.2), i.e.,

$$qA(t)E(qt, z) + B(t)E(t, z) + A\left(\frac{t}{q}\right)E\left(\frac{t}{q}, z\right) = \sqrt{q}(z + z^{-1})E(t, z). \tag{3.2}$$

Differentiating (3.2) with respect to z, we have

$$\begin{aligned} &qA(t)\frac{\mathrm{d}}{\mathrm{d}z}E(qt, z) + B(t)\frac{\mathrm{d}}{\mathrm{d}z}E(t, z) + A\left(\frac{t}{q}\right)\frac{\mathrm{d}}{\mathrm{d}z}E\left(\frac{t}{q}, z\right) \\ &\quad = \sqrt{q}(z + z^{-1})\frac{\mathrm{d}}{\mathrm{d}z}E(t, z) + \sqrt{q}(1 - z^{-2})E(t, z). \end{aligned} \tag{3.3}$$

If we take the conjugate complex of (3.3), then we get

$$\begin{aligned} &q\left[\frac{\mathrm{d}}{\mathrm{d}z}E(qt, z)\right]^* A^*(t) + \left[\frac{\mathrm{d}}{\mathrm{d}z}E(t, z)\right]^* B^*(t) + \left[\frac{\mathrm{d}}{\mathrm{d}z}E\left(\frac{t}{q}, z\right)\right]^* A^*\left(\frac{t}{q}\right) \\ &\quad = \sqrt{q}(\overline{z + z^{-1}})\left[\frac{\mathrm{d}}{\mathrm{d}z}E(t, z)\right]^* + \sqrt{q}(\overline{1 - z^{-2}})E^*(t, z). \end{aligned} \tag{3.4}$$

Multiplying (3.2) by $\left[\frac{\mathrm{d}}{\mathrm{d}z}E(t, z)\right]^*$ on the left-hand side and (3.4) by $E(t, z)$ on the right-hand side, then subtracting the resulting equations, we obtain

$$\begin{aligned} &q\left[\frac{\mathrm{d}}{\mathrm{d}z}E(t, z)\right]^* A(t)E(qt, z) - q\left[\frac{\mathrm{d}}{\mathrm{d}z}E(qt, z)\right]^* A(t)E(t, z) \\ &\quad + \left[\frac{\mathrm{d}}{\mathrm{d}z}E(t, z)\right]^* A\left(\frac{t}{q}\right)E\left(\frac{t}{q}, z\right) - \left[\frac{\mathrm{d}}{\mathrm{d}z}E\left(\frac{t}{q}, z\right)\right]^* A\left(\frac{t}{q}\right)E(t, z) \\ &= \sqrt{q}\left[z + z^{-1} - (\overline{z + z^{-1}})\right]G(t, z) - \sqrt{q}(\overline{1 - z^{-2}})F(t, z), \end{aligned} \tag{3.5}$$

where $G(t, z) = \left[\frac{\mathrm{d}}{\mathrm{d}z}E(t, z)\right]^* E(t, z)$ and $F(t, z) = E^*(t, z)E(t, z)$. From (3.5), we get

$$\begin{aligned} &\left[\frac{\mathrm{d}}{\mathrm{d}z}E(q, z)\right]^* A(1)E(1, z) - \left[\frac{\mathrm{d}}{\mathrm{d}z}E(1, z)\right]^* A(1)E(q, z) \\ &= \sqrt{q}\left[(z + z^{-1}) - (\overline{z + z^{-1}})\right]\sum_{t\in q^{\mathbb{N}}} q^{\frac{\ln t}{\ln q}-1}G(t, z) \\ &\quad - \sqrt{q}(\overline{1 - z^{-2}})\sum_{t\in q^{\mathbb{N}}} q^{\frac{\ln t}{\ln q}-1}F(t, z). \end{aligned} \tag{3.6}$$

If we write (3.6) for $z = z_0$, and $u \in \ell_2(q^{\mathbb{N}}, \mathbb{C}^m)$, $u \neq 0$, then we obtain

$$\left\langle A(1)E(q,z)u, \frac{\mathrm{d}}{\mathrm{d}z}E(1,z_0)u \right\rangle = \sqrt{q}(1 - z_0^{-2}) \sum_{t \in q^{\mathbb{N}}} q^{\frac{\ln t}{\ln q} - 1} \|E(t,z_0)u\|^2. \tag{3.7}$$

Since $z_0 \neq 0$ and $\|E(t,z_0)u\|^2 \neq 0$ for all t, we get

$$\left\langle A(1)E(q,z)u, \frac{\mathrm{d}}{\mathrm{d}z}E(1,z_0)u \right\rangle \neq 0 \tag{3.8}$$

by using (3.7). From (3.8), we obtain that $\frac{\mathrm{d}}{\mathrm{d}z}E(1,z) \mid_{z_0} u \neq 0$, that is, all zeros of $a(z)$ are simple. □

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