Dynamic Systems and Applications 25 (2016) 47-60

MONOTONICITY AND CONVEXITY FOR NABLA FRACTIONAL q-DIFFERENCES

JIA BAOGUO[a], LYNN ERBE[b], AND ALLAN PETERSON[b]

[a]School of Mathematics and Computational Science, Sun Yat-Sen University
Guangzhou, China, 510275
mcsjbg@mail.sysu.edu.cn
[b]Department of Mathematics, University of Nebraska-Lincoln, Lincoln,
NE 68588-0130, U.S.A.
lerbe2@math.unl.edu
apeterson1@math.unl.edu

ABSTRACT. In this paper, we examine the relation between monotonicity and convexity for nabla fractional q-differences. In particular we prove that

Theorem A. Assume $f : q^{\mathbb{N}_0} \to \mathbb{R}$, $\nabla_q^\nu f(t) \geq 0$ for each $t \in q^{\mathbb{N}_0}$, with $1 < \nu < 2$, then $\nabla_q f(t) \geq 0$ for $t \in q^{\mathbb{N}_1}$.

Theorem B. Assume $f : q^{\mathbb{N}_0} \to \mathbb{R}$, $\nabla_q^\nu f(t) \geq 0$ for each $t \in q^{\mathbb{N}_1}$, with $2 < \nu < 3$, then $\nabla_q^2 f(t) \geq 0$ for $t \in q^{\mathbb{N}_2}$.

This shows that, in some sense, the positivity of the νth order q-fractional difference has a strong connection to the monotonicity and convexity of $f(t)$.

AMS (MOS) Subject Classification. 26A33, 26A48, 39A70, 39A99.

1. Introduction

Discrete fractional calculus has generated much interest in recent years. Some of the work has employed the forward or delta difference. We refer the readers to [4], [8], [3], for example, and more recently [17], [14]. Probably more work has been developed for the backward or nabla difference and we refer the readers to the papers [9],[15]. There has been some work (see [5] and [8]) to develop relations between the forward and backward fractional operators, Δ^ν and ∇^ν and fractional calculus on time scales.

The study of fractional calculus in discrete settings has been initiated in [1], [2] and [12]. while the papers [1] and [2] present the introduction to fractional q-derivatives and q-integrals, the paper [12] discusses basics of fractional difference calculus.

*This work is supported by the National Natural Science Foundation of China (No. 11271380) and Guangdong Province Key Laboratory of Computational Science

Corresponding author: apeterson1@math.unl.edu

Received January 13, 2016 1056-2176 $15.00

In [13], [16], [18] and [19], the authors discussed the relation between the positivity of the ν-th order fractional difference and the monotonicity and convexity of $f(t)$. We intend to investigate the corresponding results in the q-fractional nabla difference case.

In this paper, we are concerned with the the relation between the positivity of the νth order nabla q-fractional difference and the monotonicity and convexity of $f(t)$. The main results are the following theorems.

Theorem A. Assume $f : q^{\mathbb{N}_0} \to \mathbb{R}$, $\nabla_q^{\nu} f(t) \geq 0$ for each $t \in q^{\mathbb{N}_0}$, with $1 < \nu < 2$, then $\nabla_q f(t) \geq 0$ for $t \in q^{\mathbb{N}_1}$.

Theorem B. Assume $f : q^{\mathbb{N}_0} \to \mathbb{R}$, $\nabla_q^{\nu} f(t) \geq 0$ for each $t \in q^{\mathbb{N}_1}$, with $2 < \nu < 3$, then $\nabla_q^2 f(t) \geq 0$ for $t \in q^{\mathbb{N}_2}$.

This shows that, in some sense, the positivity of the νth order q-fractional difference has a strong connection to the convexity of $f(t)$.

2. Basic Lemmas

First we introduce some notation used in the quantum calculus (q-calculus) (see [10]). For any real number α and $q > 0, q \neq 1$ we set $[\alpha]_q := \frac{q^{\alpha}-1}{q-1}$. Then we have the q-analogy of $n!$ in the form $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$ for $n = 1, 2, \ldots$, whereas for $n = 0$ we put $[0]_q!; = 1$. If $q = 1$, then $[\alpha]_1 := \alpha$ and $[n]_1!$ becomes the standard factorial. Further, the q-binomial coefficient is introduced by use of relations

$$\begin{bmatrix} \alpha \\ 0 \end{bmatrix}_q := 1,$$

$$\begin{bmatrix} \alpha \\ n \end{bmatrix}_q := \frac{[\alpha]_q[\alpha-1]_q \cdots [\alpha-n+1]_q}{[n]_q!},$$

where $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}_0$. The extension of the q-binomial coefficient to non-integer value n is allowed via the Γ_q function defined for $0 < q < 1$ as

$$\Gamma_q(t) := \frac{(q,q)_\infty (1-q)^{1-t}}{(q^t,q)_\infty}, \tag{2.1}$$

where $(a,q)_\infty = \prod_{j=0}^{\infty}(1-aq^j)$ and $t \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$. It is easy to check that Γ_q satisfies the functional relation $\Gamma_q(t+1) = [t]_q \Gamma_q(t)$. The q-analogue of the power function is introduced as

$$(t-s)_q^{(\alpha)} := t^{\alpha} \frac{(\frac{s}{t}, q)_\infty}{(q^{\alpha}\frac{s}{t}, q)_\infty}, \quad t \neq 0, 0 < q < 1, \alpha \in \mathbb{R}. \tag{2.2}$$

For $\alpha = n$, a positive integer, this expression reduces to

$$(t-s)_q^{(n)} = t^n \prod_{j=0}^{n-1}(1 - q^j \frac{s}{t}).$$

The following definitions appear in [10].

Definition 2.1 (Nabla Fractional Sum)**.** Let $f : q^{\mathbb{N}_0} \to \mathbb{R}$ be given and $q > 1$, $\nu > 0$. Then

$$\nabla_{q,\rho(1)}^{-\nu} f(t) := \frac{1}{\Gamma_{q^{-1}}(\nu)} \int_{\rho(1)}^{t} (t - q^{-1}\tau)_{q^{-1}}^{(\nu-1)} f(\tau) \nabla_q \tau, \tag{2.3}$$

for $t \in q^{\mathbb{N}_0}$, where $\rho(1) = q^{-1}$ and by convention $\nabla_{q,\rho(1)}^{-\nu} f(\rho(1)) = 0$.

Definition 2.2. Let $\nu \in \mathbb{R}^+$, $f : q^{\mathbb{N}_0} \to \mathbb{R}$ and let $t \in q^{\mathbb{N}_0}$. Then we define the nabla q-fractional derivative of f at t by

$$(\nabla_{q,\rho(1)}^{\nu} f)(t) := (\nabla_q^m \nabla_{q,\rho(1)}^{-(m-\nu)} f)(t),$$

where $m \in \mathbb{N}_1$ satisfies $m - 1 < \nu < m$.

Lemma 2.3. *For $q > 1$,*

1. *The nabla q-derivative of the q-factorial function $(t - s)_{q^{-1}}^{(\alpha)}$ with respect to t is*

$$_t\nabla_q (t - s)_{q^{-1}}^{(\alpha)} = \frac{1 - q^{-\alpha}}{1 - q^{-1}} (t - s)_{q^{-1}}^{(\alpha-1)},$$

2. *The nabla q-derivative of the q-factorial function $(t - s)_{q^{-1}}^{(\alpha)}$ with respect to s is*

$$_s\nabla_q (t - s)_{q^{-1}}^{(\alpha)} = -\frac{1 - q^{-\alpha}}{1 - q^{-1}} (t - q^{-1}s)_{q^{-1}}^{(\alpha-1)},$$

where $\alpha \in \mathbb{R}$.

Proof. (i)

$$\begin{aligned}
_t\nabla_q (t - s)_{q^{-1}}^{(\alpha)} &= \frac{(t - s)_{q^{-1}}^{(\alpha)} - (q^{-1}t - s)_{q^{-1}}^{(\alpha)}}{t - q^{-1}t} \\
&= \frac{t^{\alpha} \frac{(s/t, q^{-1})_\infty}{(q^{-\alpha}s/t, q^{-1})_\infty} - q^{-\alpha} t^{\alpha} \frac{(qs/t, q^{-1})_\infty}{(q^{-\alpha+1}s/t, q^{-1})_\infty}}{t - q^{-1}t} \\
&= \frac{t^{\alpha} \prod_{n=0}^{\infty} \frac{1 - q^{-n}s/t}{1 - q^{-\alpha-n}s/t} - q^{-\alpha} t^{\alpha} \prod_{n=0}^{\infty} \frac{1 - q^{1-n}s/t}{1 - q^{1-\alpha-n}s/t}}{t - q^{-1}t} \\
&= \frac{t^{\alpha} \prod_{n=0}^{\infty} \frac{1 - q^{-n}s/t}{1 - q^{-\alpha-n}s/t} \left[1 - \frac{q^{-\alpha}(1 - qs/t)}{1 - q^{1-\alpha}s/t}\right]}{t - q^{-1}t} \\
&= \frac{t^{\alpha-1} \prod_{n=0}^{\infty} \frac{1 - q^{-n}s/t}{1 - q^{-\alpha-n}s/t} \left[\frac{1 - q^{-\alpha}}{1 - q^{1-\alpha}s/t}\right]}{1 - q^{-1}} \\
&= \frac{1 - q^{-\alpha}}{1 - q^{-1}} t^{\alpha-1} \prod_{n=0}^{\infty} \frac{1 - q^{-n}s/t}{1 - q^{-(\alpha-1)-n}s/t} \\
&= \frac{1 - q^{-\alpha}}{1 - q^{-1}} \cdot \frac{t^{\alpha-1}(s/t, q^{-1})_\infty}{(q^{-(\alpha-1)}s/t, q^{-1})_\infty} \\
&= \frac{1 - q^{-\alpha}}{1 - q^{-1}} (t - s)_{q^{-1}}^{(\alpha-1)}.
\end{aligned}$$

(ii)

$$
\begin{aligned}
{}_s\nabla_q(t-s)^{(\alpha)}_{q^{-1}} &= \frac{(t-s)^{(\alpha)}_{q^{-1}} - (t-q^{-1}s)^{(\alpha)}_{q^{-1}}}{s-q^{-1}s} \\
&= \frac{t^{\alpha}\frac{(s/t,q^{-1})_\infty}{(q^{-\alpha}s/t,q^{-1})_\infty} - t^{\alpha}\frac{(q^{-1}s/t,q^{-1})_\infty}{(q^{-\alpha-1}s/t,q^{-1})_\infty}}{s-q^{-1}s} \\
&= \frac{t^{\alpha}\prod_{n=0}^{\infty}\frac{1-q^{-n}s/t}{1-q^{-\alpha-n}s/t} - t^{\alpha}\prod_{n=0}^{\infty}\frac{1-q^{-1-n}s/t}{1-q^{-\alpha-1-n}s/t}}{s-q^{-1}s} \\
&= \frac{t^{\alpha}\prod_{n=1}^{\infty}\frac{1-q^{-n}s/t}{1-q^{-\alpha-n}s/t}\left[\frac{1-s/t}{1-q^{-\alpha}s/t}-1\right]}{s-q^{-1}s} \\
&= \frac{t^{\alpha-1}\prod_{j=0}^{\infty}\frac{1-q^{-j-1}s/t}{1-q^{-\alpha-j-1}s/t}\left[\frac{-1+q^{-\alpha}}{1-q^{-\alpha}s/t}\right]}{1-q^{-1}} \\
&= -\frac{1-q^{-\alpha}}{1-q^{-1}}t^{\alpha-1}\prod_{j=0}^{\infty}\frac{1-q^{-j-1}s/t}{1-q^{-(\alpha-1)-j-1}s/t} \\
&= -\frac{1-q^{-\alpha}}{1-q^{-1}}\cdot\frac{t^{\alpha-1}(q^{-1}s/t,q^{-1})_\infty}{(q^{-(\alpha-1)}q^{-1}s/t,q^{-1})_\infty} \\
&= -\frac{1-q^{-\alpha}}{1-q^{-1}}(t-q^{-1}s)^{(\alpha-1)}_{q^{-1}}.
\end{aligned}
$$

□

The following lemma appears in [9].

Lemma 2.4 (Leibniz Rule)**.** *Assume* $f: q^{\mathbb{N}_1}\times q^{\mathbb{N}_1}\to\mathbb{R}$*. Then*

$$
{}_t\nabla_q\Big[\int_1^t f(t,s)\nabla_q s\Big] = \int_1^t {}_t\nabla_q f(t,s)\nabla_q s + f(q^{-1}t,t)
$$

for $t\in q^{\mathbb{N}_1}$*.*

3. **Monotonicity**

Theorem 3.1. *Assume* $f: q^{\mathbb{N}_0}\to\mathbb{R}$*,* $\nabla_q^{\nu}f(t)\geq 0$ *for each* $t\in q^{\mathbb{N}_0}$*, with* $1<\nu<2$*, then* $\nabla_q f(t)\geq 0$ *for* $t\in q^{\mathbb{N}_1}$*.*

Proof. Using Definition 2.1, Lemma 2.4 and Lemma 2.3 (i), we have (using $(q^{-1}t-q^{-1}t)^{(-\nu)}_{q^{-1}}=0$)

$$
\begin{aligned}
(3.1)\qquad & \nabla^{\nu}_{q,\rho(1)}x(t) \\
&= \nabla_q^2\nabla^{-(2-\nu)}_{q,\rho(1)}x(t) \\
&= \frac{1}{\Gamma_{q^{-1}}(2-\nu)}\nabla_q^2\int_{\rho(1)}^t (t-q^{-1}s)^{(1-\nu)}_{q^{-1}}x(s)\nabla_q s \\
&\overset{L2.3,L2.4}{=} \frac{1}{\Gamma_{q^{-1}}(2-\nu)}\cdot\frac{1-q^{\nu-1}}{1-q^{-1}}\nabla_q\int_{\rho(1)}^t (t-q^{-1}s)^{(-\nu)}_{q^{-1}}x(s)\nabla_q s
\end{aligned}
$$

$$\stackrel{L2.3,L2.4}{=} \frac{1}{\Gamma_{q^{-1}}(2-\nu)} \cdot \frac{(1-q^{\nu-1})(1-q^{\nu})}{(1-q^{-1})^2} \int_{\rho(1)}^{t} (t-q^{-1}s)_{q^{-1}}^{(-\nu-1)} x(s) \nabla_q s \tag{3.2}$$

$$= \frac{1}{\Gamma_{q^{-1}}(2-\nu)} \cdot \frac{(1-q^{\nu-1})(1-q^{\nu})}{(1-q^{-1})^2} \Big[\int_{\rho(1)}^{1} (t-q^{-1}s)_{q^{-1}}^{(-\nu-1)} x(s) \nabla_q s + \int_{1}^{t} (t-q^{-1}s)_{q^{-1}}^{(-\nu-1)} x(s) \nabla_q s \Big]$$

$$= \frac{(1-q^{\nu-1})(1-q^{\nu})}{\Gamma_{q^{-1}}(2-\nu)(1-q^{-1})^2} (t-q^{-1})_{q^{-1}}^{(-\nu-1)} x(1)(1-q^{-1}) + \frac{1}{\Gamma_{q^{-1}}(2-\nu)} \cdot \frac{(1-q^{\nu-1})(1-q^{\nu})}{(1-q^{-1})^2} \int_{1}^{t} (t-q^{-1}s)_{q^{-1}}^{(-\nu-1)} x(s) \nabla_q s$$

where we use

$$\int_{\rho(1)}^{1} (t-q^{-1}s)_{q^{-1}}^{(-\nu-1)} x(s) \nabla_q s = (t-q^{-1})_{q^{-1}}^{(-\nu-1)} x(1)(1-q^{-1}).$$

From Lemma 2.3 (ii), integrating by parts, we have that (using $(q^{-1}t - q^{-1}t)_{q^{-1}}^{(-\nu)} = 0$) for $t = q^k$, $k \geq 1$

$$\frac{1}{\Gamma_{q^{-1}}(2-\nu)} \cdot \frac{(1-q^{\nu-1})(1-q^{\nu})}{(1-q^{-1})^2} \int_{1}^{t} (t-q^{-1}s)_{q^{-1}}^{(-\nu-1)} x(s) \nabla_q s \tag{3.3}$$

$$= -\frac{1}{\Gamma_{q^{-1}}(2-\nu)} \frac{(1-q^{\nu-1})}{(1-q^{-1})} \int_{1}^{t} \Big[{}_s\nabla_q (t-s)_{q^{-1}}^{(-\nu)} \Big] x(s) \nabla_q s$$

$$= -\frac{1}{\Gamma_{q^{-1}}(2-\nu)} \frac{(1-q^{\nu-1})}{(1-q^{-1})} \Big\{ \Big[(t-s)_{q^{-1}}^{(-\nu)} x(s) \Big]_1^t - \int_{1}^{t} (t-q^{-1}s)_{q^{-1}}^{(-\nu)} \nabla_q x(s) \nabla_q s \Big\}$$

$$= \frac{1}{\Gamma_{q^{-1}}(2-\nu)} \frac{(1-q^{\nu-1})}{(1-q^{-1})} \Big[(t-1)_{q^{-1}}^{(-\nu)} x(1) + \int_{1}^{t} (t-q^{-1}s)_{q^{-1}}^{(-\nu)} \nabla_q x(s) \nabla_q s \Big]$$

$$= \frac{1}{\Gamma_{q^{-1}}(2-\nu)} \frac{(1-q^{\nu-1})}{(1-q^{-1})} \Big[(q^k-1)_{q^{-1}}^{(-\nu)} x(1) + \sum_{i=1}^{k-1} (q^k - q^{-1}q^i)_{q^{-1}}^{(-\nu)} \nabla_q x(q^i)(q^i - q^{i-1}) + (q^k - q^{-1}q^k)_{q^{-1}}^{(-\nu)} \nabla_q x(q^k)(q^k - q^{k-1}) \Big],$$

when $k = 1$, by our convention the above sum is zero. From (3.1) and (3.3), we get that

$$0 \leq \nabla_{q,\rho(1)}^{\nu} x(t) \tag{3.4}$$

$$= \frac{1-q^{\nu-1}}{\Gamma_{q^{-1}}(2-\nu)(1-q^{-1})} \Big[(1-q^{\nu})(q^k - q^{-1})_{q^{-1}}^{(-\nu-1)} + (q^k-1)_{q^{-1}}^{(-\nu)} \Big] x(1)$$

$$+ \frac{1-q^{\nu-1}}{\Gamma_{q^{-1}}(2-\nu)(1-q^{-1})} \Big[\sum_{i=1}^{k-1} (q^k - q^{-1}q^i)_{q^{-1}}^{(-\nu)} \nabla_q x(q^i)(q^i - q^{i-1})$$

$$+ (q^k - q^{-1}q^k)_{q^{-1}}^{(-\nu)} \nabla_q x(q^k)(q^k - q^{k-1})\Big].$$

Therefore we have (using $1 - q^{\nu-1} < 0$)

$$\begin{aligned}
&- (q^k - q^{-1}q^k)_{q^{-1}}^{(-\nu)} \nabla_q x(q^k)(q^k - q^{k-1}) \\
&\geq \Big[(1-q^\nu)(q^k - q^{-1})_{q^{-1}}^{(-\nu-1)} + (q^k - 1)_{q^{-1}}^{(-\nu)}\Big] x(1) \\
&\quad + \sum_{i=1}^{k-1} (q^k - q^{-1}q^i)_{q^{-1}}^{(-\nu)} \nabla_q x(q^i)(q^i - q^{i-1}).
\end{aligned} \tag{3.5}$$

Note that $\Gamma_{q^{-1}}(2-\nu) > 0$ for $1 < \nu < 2$ and

$$\begin{aligned}
(q^k - q^{-1}q^k)_{q^{-1}}^{(-\nu)} &= q^{-k\nu} \frac{(q^{-1}, q^{-1})_\infty}{(q^\nu q^{-1}, q^{-1})_\infty} \\
&= q^{-k\nu} \frac{\prod_{j=0}^\infty (1 - q^{-1}q^{-j})}{\prod_{j=0}^\infty (1 - q^{\nu-1}q^{-j})} \\
&< 0,
\end{aligned} \tag{3.6}$$

and for $1 \leq i \leq k-1$, we have

$$\begin{aligned}
(q^k - q^{-1}q^i)_{q^{-1}}^{(-\nu)} &= q^{-k\nu} \frac{(q^{i-1-k}, q^{-1})_\infty}{(q^\nu q^{i-1-k}, q^{-1})_\infty} \\
&= q^{-k\nu} \frac{\prod_{j=0}^\infty (1 - q^{i-1-k}q^{-j})}{\prod_{j=0}^\infty (1 - q^{\nu+i-1-k}q^{-j})} \\
&> 0.
\end{aligned} \tag{3.7}$$

For $k \geq 1$, we have

$$\begin{aligned}
&(1-q^\nu)(q^k - q^{-1})_{q^{-1}}^{(-\nu-1)} + (q^k - 1)_{q^{-1}}^{(-\nu)} \\
&= (1-q^\nu) q^{-k\nu-k} \frac{(q^{-1-k}, q^{-1})_\infty}{(q^{\nu+1}q^{-1-k}, q^{-1})_\infty} + q^{-k\nu} \frac{(q^{-k}, q^{-1})_\infty}{(q^\nu q^{-k}, q^{-1})_\infty} \\
&= (1-q^\nu) q^{-k\nu-k} \frac{\prod_{j=0}^\infty (1 - q^{-1-k}q^{-j})}{\prod_{j=0}^\infty (1 - q^{\nu-k}q^{-j})} + q^{-k\nu} \frac{\prod_{j=0}^\infty (1 - q^{-k}q^{-j})}{\prod_{j=0}^\infty (1 - q^{\nu-k}q^{-j})} \\
&= q^{-k\nu} \frac{\prod_{j=0}^\infty (1 - q^{-1-k}q^{-j})}{\prod_{j=0}^\infty (1 - q^{\nu-k}q^{-j})} \Big[(1-q^\nu)q^{-k} + (1 - q^{-k})\Big] \\
&= q^{-k\nu} \frac{\prod_{j=0}^\infty (1 - q^{-1-k}q^{-j})}{\prod_{j=0}^\infty (1 - q^{\nu-k}q^{-j})} (1 - q^{\nu-k}) \\
&= q^{-k\nu} \frac{\prod_{j=0}^\infty (1 - q^{-1-k}q^{-j})}{\prod_{j=1}^\infty (1 - q^{\nu-k}q^{-j})} \\
&\geq 0,
\end{aligned} \tag{3.8}$$

where we use $1 < \nu < 2$. When $t = 1$, from (3.1) and (3.2), we have

$$0 \leq \nabla_{q,\rho(1)}^\nu x(t)|_{t=1} \tag{3.9}$$

$$= \frac{(1-q^{\nu-1})(1-q^{\nu})}{\Gamma_{q^{-1}}(2-\nu)(1-q^{-1})^2}\int_{\rho(1)}^{1}(1-q^{-1}s)_{q^{-1}}^{(-\nu-1)}x(s)\nabla_q s$$
$$= \frac{(1-q^{\nu-1})(1-q^{\nu})}{\Gamma_{q^{-1}}(2-\nu)(1-q^{-1})}(1-q^{-1})_{q^{-1}}^{(-\nu-1)}x(1).$$

Using $1<\nu<2$, we have

$$(1-q^{-1})_{q^{-1}}^{(-\nu-1)} = \frac{(q^{-1},q^{-1})_\infty}{(q^{\nu},q^{-1})_\infty} = \frac{\prod_{j=0}^{\infty}(1-q^{-1}q^{-j})}{\prod_{j=0}^{\infty}(1-q^{\nu}q^{-j})} > 0. \tag{3.10}$$

From (3.9) and (3.10), we get that

$$x(1) \geq 0. \tag{3.11}$$

In the following, we will prove that $\nabla_q x(q^k) \geq 0$ for $k \geq 1$ by the principle of strong induction.

When $k=1$, from (3.5), we get that

$$-(q-1)_{q^{-1}}^{(-\nu)}\nabla_q x(q)(q-1) \geq \Big[(1-q^{\nu})(q-q^{-1})_{q^{-1}}^{(-\nu-1)} + (q-1)_{q^{-1}}^{(-\nu)}\Big]x(1). \tag{3.12}$$

Therefore from (3.6), (3.8) and (3.11), we get that $\nabla_q x(q) \geq 0$.

Suppose $k \geq 1$ and that $\nabla_q x(q^i) \geq 0$, for $i = 1,2,3,\ldots,k-1$. Then from (3.5) and (3.6), we have $\nabla_q x(q^k) \geq 0 \geq 0$, so this completes the proof.

□

4. **Convexity**

Theorem 4.1. *Assume $f : q^{\mathbb{N}_0} \to \mathbb{R}$, $\nabla_q^{\nu} f(t) \geq 0$ for each $t \in q^{\mathbb{N}_1}$, with $2<\nu<3$, then $\nabla_q^2 f(t) \geq 0$ for $t \in q^{\mathbb{N}_2}$.*

Proof. Using Definition 2.1, Lemma 2.4 and Lemma 2.3 (i), we have (using $(q^{-1}t - q^{-1}t)_{q^{-1}}^{(2-\nu)} = (q^{-1}t-q^{-1}t)_{q^{-1}}^{(1-\nu)} = (q^{-1}t-q^{-1}t)_{q^{-1}}^{(-\nu)} = 0$)

$$\begin{aligned}
&\nabla_{q,\rho(1)}^{\nu}x(t) \qquad (4.1)\\
&= \nabla_q^3 \nabla_{q,\rho(1)}^{-(3-\nu)}x(t)\\
&= \frac{1}{\Gamma_{q^{-1}}(3-\nu)}\nabla_q^3 \int_{\rho(1)}^{t}(t-q^{-1}s)_{q^{-1}}^{(2-\nu)}x(s)\nabla_q s\\
&\overset{L2.3,L2.4}{=} \frac{1}{\Gamma_{q^{-1}}(3-\nu)}\cdot\frac{1-q^{\nu-2}}{1-q^{-1}}\nabla_q^2\int_{\rho(1)}^{t}(t-q^{-1}s)_{q^{-1}}^{(1-\nu)}x(s)\nabla_q s\\
&\overset{L2.3,L2.4}{=} \frac{(1-q^{\nu-2})(1-q^{\nu-1})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^2}\nabla_q\int_{\rho(1)}^{t}(t-q^{-1}s)_{q^{-1}}^{(-\nu)}x(s)\nabla_q s\\
&\overset{L2.3,L2.4}{=} \frac{(1-q^{\nu-2})(1-q^{\nu-1})(1-q^{\nu})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^3}\int_{\rho(1)}^{t}(t-q^{-1}s)_{q^{-1}}^{(-\nu-1)}x(s)\nabla_q s \qquad (4.2)\\
&= \frac{(1-q^{\nu-2})(1-q^{\nu-1})(1-q^{\nu})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^3}\Big[\int_{\rho(1)}^{q}(t-q^{-1}s)_{q^{-1}}^{(-\nu-1)}x(s)\nabla_q s
\end{aligned}$$

$$+ \int_q^t (t - q^{-1}s)_{q^{-1}}^{(-\nu-1)} x(s) \nabla_q s \Big]$$

$$= \frac{(1-q^{\nu-2})(1-q^{\nu-1})(1-q^{\nu})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^3} \Big[(t-q^{-1})_{q^{-1}}^{(-\nu-1)} x(1)(1-q^{-1})$$

$$+ (t-1)_{q^{-1}}^{(-\nu-1)} x(q)(q-1) \Big]$$

$$+ \frac{(1-q^{\nu-2})(1-q^{\nu-1})(1-q^{\nu})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^3} \int_q^t (t-q^{-1}s)_{q^{-1}}^{(-\nu-1)} x(s) \nabla_q s$$

where we used

$$\int_{\rho(1)}^q (t-q^{-1}s)_{q^{-1}}^{(-\nu-1)} x(s) \nabla_q s = (t-q^{-1})_{q^{-1}}^{(-\nu-1)} x(1)(1-q^{-1}) + (t-1)_{q^{-1}}^{(-\nu-1)} x(q)(q-1).$$

From Lemma 2.3 (ii), integrating by parts twice, we have that (using $(q^{-1}t - q^{-1}t)_{q^{-1}}^{(-\nu)} = (t-t)_{q^{-1}}^{(-\nu+1)} = 0$) for $t = q^k$, $k \geq 2$

$$\frac{(1-q^{\nu-2})(1-q^{\nu-1})(1-q^{\nu})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^3} \int_q^t (t-q^{-1}s)_{q^{-1}}^{(-\nu-1)} x(s) \nabla_q s \tag{4.3}$$

$$= -\frac{(1-q^{\nu-2})(1-q^{\nu-1})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^2} \int_q^t \Big[{}_s\nabla_q (t-s)_{q^{-1}}^{(-\nu)} \Big] x(s) \nabla_q s$$

$$= -\frac{(1-q^{\nu-2})(1-q^{\nu-1})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^2} \Big\{ \Big[(t-s)_{q^{-1}}^{(-\nu)} x(s) \Big]_{s=q}^t$$

$$- \int_q^t (t-q^{-1}s)_{q^{-1}}^{(-\nu)} \nabla_q x(s) \nabla_q s \Big\}$$

$$= \frac{(1-q^{\nu-2})(1-q^{\nu-1})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^2} \Big[(t-q)_{q^{-1}}^{(-\nu)} x(q)$$

$$+ \int_q^t (t-q^{-1}s)_{q^{-1}}^{(-\nu)} \nabla_q x(s) \nabla_q s \Big]$$

$$= \frac{(1-q^{\nu-2})(1-q^{\nu-1})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^2} (q^k-q)_{q^{-1}}^{(-\nu)} x(q)$$

$$- \frac{1-q^{\nu-2}}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})} \int_q^t {}_s\nabla_q (t-s)_{q^{-1}}^{(-\nu+1)} \nabla_q x(s) \nabla_q s$$

$$= \frac{(1-q^{\nu-2})(1-q^{\nu-1})}{\Gamma_{q^{-1}}(2-\nu)(1-q^{-1})^2} (q^k-q)_{q^{-1}}^{(-\nu)} x(q)$$

$$- \frac{1-q^{\nu-2}}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})} \Big\{ \Big[(t-s)_{q^{-1}}^{(-\nu+1)} \nabla_q x(s) \Big]_{s=q}^t$$

$$- \int_q^t (t-q^{-1}s)_{q^{-1}}^{(-\nu+1)} \nabla_q^2 x(s) \nabla_q s \Big\}$$

$$= \frac{(1-q^{\nu-2})(1-q^{\nu-1})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^2} (q^k-q)_{q^{-1}}^{(-\nu)} x(q)$$

$$+ (1-q^{\nu-2}) \frac{(t-q)_{q^{-1}}^{(-\nu+1)} \nabla_q x(s)|_{s=q}}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})}$$

$$+\frac{(1-q^{\nu-2})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})}\sum_{i=2}^{k-1}(q^k-q^{-1}q^i)_{q^{-1}}^{(-\nu+1)}\nabla_q^2x(q^i)(q^i-q^{i-1})$$
$$+\frac{1-q^{\nu-2}}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})}(q^k-q^{-1}q^k)_{q^{-1}}^{(-\nu+1)}\nabla_q^2x(q^k)(q^k-q^{k-1}),$$

when $k=2$, by our convention the above sum is zero. From (4.1) and (4.3), we have

$$0\le\nabla_{q,\rho(1)}^{\nu}x(t) \tag{4.4}$$
$$=\frac{(1-q^{\nu-2})(1-q^{\nu-1})(1-q^{\nu})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^3}\Big[(q^k-q^{-1})_{q^{-1}}^{(-\nu-1)}x(1)(1-q^{-1})$$
$$+(q^k-1)_{q^{-1}}^{(-\nu-1)}x(q)(q-1)\Big] \tag{4.5}$$
$$+\frac{(1-q^{\nu-2})(1-q^{\nu-1})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^2}(q^k-q)_{q^{-1}}^{(-\nu)}x(q)$$
$$+(1-q^{\nu-2})\frac{(q^k-q)_{q^{-1}}^{(-\nu+1)}\nabla_qx(s)|_{s=q}}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})}$$
$$+\frac{1-q^{\nu-2}}{\Gamma_{q^{-1}}(2-\nu)(1-q^{-1})}\sum_{i=2}^{k-1}(q^k-q^{-1}q^i)_{q^{-1}}^{(-\nu+1)}\nabla_q^2x(q^i)(q^i-q^{i-1})$$
$$+\frac{(1-q^{\nu-2})}{\Gamma_{q^{-1}}(2-\nu)(1-q^{-1})}(q^k-q^{-1}q^k)_{q^{-1}}^{(-\nu+1)}\nabla_q^2x(q^k)(q^k-q^{k-1}).$$

Solving the above inequality, we get

$$\frac{1-q^{\nu-2}}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})}(q^k-q^{-1}q^k)_{q^{-1}}^{(-\nu+1)}\nabla_q^2x(q^k)(q^k-q^{k-1}) \tag{4.6}$$
$$\ge-\frac{(1-q^{\nu-2})(1-q^{\nu-1})(1-q^{\nu})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^3}\Big[(q^k-q^{-1})_{q^{-1}}^{(-\nu-1)}x(1)(1-q^{-1})$$
$$+(q^k-1)_{q^{-1}}^{(-\nu-1)}x(q)(q-1)\Big]$$
$$-\frac{(1-q^{\nu-2})(1-q^{\nu-1})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^2}(q^k-q)_{q^{-1}}^{(-\nu)}x(q)$$
$$-(1-q^{\nu-2})\frac{(q^k-q)_{q^{-1}}^{(-\nu+1)}\nabla_qx(s)|_{s=q}}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})}$$
$$-\frac{1-q^{\nu-2}}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})}\sum_{i=2}^{k-1}(q^k-q^{-1}q^i)_{q^{-1}}^{(-\nu+1)}\nabla_q^2x(q^i)(q^i-q^{i-1})$$
$$=-\frac{(1-q^{\nu-2})(1-q^{\nu-1})(1-q^{\nu})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^2}(q^k-q^{-1})_{q^{-1}}^{(-\nu-1)}x(1)$$
$$+\frac{q(1-q^{\nu-2})(q^k-q)_{q^{-1}}^{(-\nu+1)}}{\Gamma_{q^{-1}}(3-\nu)(q-1)^2}x(1)$$
$$-\frac{(1-q^{\nu-2})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})}\Big[\frac{(1-q^{\nu-1})(1-q^{\nu})q}{(1-q^{-1})}(q^k-1)_{q^{-1}}^{(-\nu-1)}$$

$$+\frac{(1-q^{\nu-1})}{(1-q^{-1})}(q^k-q)_{q^{-1}}^{(-\nu)}+\frac{(q^k-q)_{q^{-1}}^{(-\nu+1)}}{(q-1)}\Big]x(q)$$
$$-\sum_{i=2}^{k-1}(q^k-q^{i-1})_{q^{-1}}^{(-\nu+1)}\nabla_q^2x(q^i)(q^i-q^{i-1}).$$

Note that (using $\Gamma_{q^{-1}}(3-\nu)>0$) for $k\geq 2$, we have

(4.7)

$$\begin{aligned}
&\frac{(1-q^{\nu-1})(1-q^{\nu})q}{(1-q^{-1})}(q^k-1)_{q^{-1}}^{(-\nu-1)}x(q)\\
&\quad+\frac{1-q^{\nu-1}}{(1-q^{-1})}(q^k-q)_{q^{-1}}^{(-\nu)}x(q)+\frac{(q^k-q)_{q^{-1}}^{(-\nu+1)}}{(q-1)}x(q)\\
&=\frac{q^{-k\nu}\prod_{j=0}^{\infty}(1-q^{-k-j})}{\prod_{j=0}^{\infty}(1-q^{\nu-k-j})}\\
&\quad\cdot\Big[\frac{(1-q^{\nu-1})(1-q^{\nu})q^{1-k}}{(1-q^{-1})(1-q^{\nu-k+1})}+\frac{(1-q^{\nu-1})(1-q^{1-k})}{(1-q^{-1})(1-q^{\nu-k+1})}+\frac{q^k-q}{(1-q^{-1})q}\Big]x(q)\\
&=-\frac{q^{-k\nu}\prod_{j=0}^{\infty}(1-q^{-k-j})}{(1-q^{-1})(1-q^{\nu-k+1})\prod_{j=0}^{\infty}(1-q^{\nu-k-j})}\\
&\quad\cdot\Big[(1-q^{\nu-1})(1-q^{\nu})q^{1-k}+(1-q^{\nu-1})(1-q^{1-k})+(q^{k-1}-1)(1-q^{\nu-k+1})\Big]x(q)\\
&=\frac{q^{-k\nu}(1-q^{\nu-k+1})(q^{k-1}-q^{\nu-1})\prod_{j=0}^{\infty}(1-q^{-k-j})}{(1-q^{-1})(1-q^{\nu-k+1})\prod_{j=0}^{\infty}(1-q^{\nu-k-j})}x(q)\\
&=\frac{q^{-k\nu}(q^{k-1}-q^{\nu-1})\prod_{j=0}^{\infty}(1-q^{-k-j})}{(1-q^{-1})\prod_{j=0}^{\infty}(1-q^{\nu-k-j})}x(q)\\
&\geq 0.
\end{aligned}$$

From (4.1) and (4.2), for $t=1$, we have

$$\begin{aligned}
(4.8)\qquad 0&\leq\nabla_{q,\rho(1)}^{\nu}x(t)|_{t=1}\\
&=\frac{(1-q^{\nu-2})(1-q^{\nu-1})(1-q^{\nu})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^3}(1-q^{-1})_{q^{-1}}^{(-\nu-1)}x(1)(1-q^{-1})\\
&=\frac{(1-q^{\nu-2})(1-q^{\nu-1})(1-q^{\nu})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^2}\frac{\prod_{j=0}^{\infty}(1-q^{-1-j})}{\prod_{j=0}^{\infty}(1-q^{\nu-j})}x(1).
\end{aligned}$$

From (4.1) and (4.2), for $t=q$, we have

(4.9)

$$\begin{aligned}
&0\leq\nabla_{q,\rho(1)}^{\nu}x(t)|_{t=q}\\
&\;=\frac{(1-q^{\nu-2})(1-q^{\nu-1})(1-q^{\nu})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^3}\Big[(q-q^{-1})_{q^{-1}}^{(-\nu-1)}x(1)(1-q^{-1})\\
&\quad+(q-1)_{q^{-1}}^{(-\nu-1)}x(q)(q-1)\Big]
\end{aligned}$$

$$= \frac{(1-q^{\nu-2})(1-q^{\nu-1})(1-q^{\nu})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^2} \cdot \frac{q^{-\nu-1}\prod_{j=0}^{\infty}(1-q^{-2-j})}{\prod_{j=0}^{\infty}(1-q^{\nu-1-j})} \Big[x(1) + \frac{q-1}{1-q^{\nu}}x(q)\Big].$$

From (4.9), we get

$$x(1) + \frac{q-1}{1-q^{\nu}}x(q) \le 0. \tag{4.10}$$

So from (4.8) and (4.10), we get that

$$x(q) \ge \frac{q^{\nu}-1}{q-1}x(1) \ge 0. \tag{4.11}$$

From (4.6), (4.7) and (4.11), we get that for $k \ge 2$

(4.12)

$$\begin{aligned}
&- \frac{(1-q^{\nu-2})(1-q^{\nu-1})(1-q^{\nu})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^2}(q^k-q^{-1})_{q^{-1}}^{(-\nu-1)}x(1) \\
&+ \frac{q(1-q^{\nu-2})(q^k-q)_{q^{-1}}^{(-\nu+1)}}{\Gamma_{q^{-1}}(3-\nu)(q-1)^2}x(1) \\
&- \frac{(1-q^{\nu-2})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})}\Big[\frac{(1-q^{\nu-1})(1-q^{\nu})q}{(1-q^{-1})}(q^k-1)_{q^{-1}}^{(-\nu-1)} \\
&+ \frac{(1-q^{\nu-1})}{(1-q^{-1})}(q^k-q)_{q^{-1}}^{(-\nu)} + \frac{(q^k-q)_{q^{-1}}^{(-\nu+1)}}{(q-1)}\Big]x(q) \\
\overset{(4.7)(4.11)}{\ge}\ & - \frac{(1-q^{\nu-2})(1-q^{\nu-1})(1-q^{\nu})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^2}\frac{q^{-k\nu-k}\prod_{j=0}^{\infty}(1-q^{-1-k-j})}{\prod_{j=0}^{\infty}(1-q^{\nu-k-j})}x(1) \\
&+ \frac{q(1-q^{\nu-2})}{\Gamma_{q^{-1}}(3-\nu)(q-1)^2}\frac{q^{-k\nu+k}\prod_{j=0}^{\infty}(1-q^{1-k-j})}{\prod_{j=0}^{\infty}(1-q^{\nu-k-j})}x(1) \\
&- \frac{(1-q^{\nu-2})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})} \cdot \frac{q^{-k\nu}(q^{k-1}-q^{\nu-1})\prod_{j=0}^{\infty}(1-q^{-k-j})}{(1-q^{-1})\prod_{j=0}^{\infty}(1-q^{\nu-k-j})} \cdot \frac{q^{\nu}-1}{q-1}x(1) \\
=\ & -\frac{q^{-k\nu}(1-q^{\nu-2})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^2}\frac{\prod_{j=0}^{\infty}(1-q^{-1-k-j})}{\prod_{j=0}^{\infty}(1-q^{\nu-k-j})} \cdot \Big[q^{-k}(1-q^{\nu-1})(1-q^{\nu}) \\
&- q^{k-1}(1-q^{1-k})(1-q^{-k}) + (q^{k-1}-q^{\nu-1})(1-q^{-k}) \cdot \frac{q^{\nu}-1}{q-1}\Big]x(1) \\
=\ & -\frac{q^{-k\nu}(1-q^{\nu-2})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^2}\frac{\prod_{j=0}^{\infty}(1-q^{-1-k-j})}{\prod_{j=0}^{\infty}(1-q^{\nu-k-j})} \cdot \Big[q^{-k}(q^{\nu-1}-1)(q^{\nu}-1) \\
&- (q^{k-1}-1)(1-q^{-k}) + (q^{k-1}-q^{\nu-1})(1-q^{-k}) \cdot \frac{q^{\nu}-1}{q-1}\Big]x(1).
\end{aligned}$$

Let $x = q^{\nu}$ and for $q^2 < x < q^3$ and $k \ge 2$, define

$$f(x,q^k) = q^{-k}(q^{-1}x-1)(x-1) - (q^{k-1}-1)(1-q^{-k}) + (q^{k-1}-q^{-1}x)(1-q^{-k}) \cdot \frac{x-1}{q-1}.$$

We have

$$\frac{\partial^2 f(x,q^k)}{\partial^2 x} = \frac{2(q^{-k}-q^{-1})}{q-1} < 0. \tag{4.13}$$

So $f(x, q^k)$ is convex for $q^2 < x < q^3$. When $k = 2$, we have

$$\frac{\partial f(x,q^2)}{\partial x} = (1 + q^{-1} - q^{-2}x - q^{-2})(x-1) - (q-1)(1-q^{-2}). \tag{4.14}$$

$$f(q^2, q^2) = q^{-2}(q-1)(q^2-1) - (q-1)(1-q^{-2}) = 0, \tag{4.15}$$

and

$$\begin{aligned} f(q^3, q^2) &\\ &= q^{-2}(q^2-1)(q^3-1) - (q-1)(1-q^{-2}) + (q-q^2)(1-q^{-2}) \cdot \frac{q^3-1}{q-1} \\ &= -q^3(1-q^{-2})(q-1) \\ &< 0. \end{aligned} \tag{4.16}$$

Let $\frac{\partial f(x,q^2)}{\partial x} = 0$. It is easy to get that $x_0 = \frac{q^2(1+q^{-1})}{2}$. Because of $x_0 < q^2$, we have $f(x, q^2)$ is decreasing on $[q^2, q^3]$. From (4.13), (4.15) and (4.16), we get that

$$f(x, q^2) < 0, \quad \text{for} \quad q^2 < x < q^3. \tag{4.17}$$

When $k \geq 3$, we have that

$$\begin{aligned} f(q^2, q^k) &= q^{-k}(q-1)(q^2-1) - (q^{k-1}-1)(1-q^{-k}) \\ &\quad + (q^{k-1}-q)(1-q^{-k})(q+1) \\ &= q^{-k+3} + q^k - q^2 - q \\ &= (1-q^{-k+2})(q^k-1) \\ &\geq 0. \end{aligned} \tag{4.18}$$

$$\begin{aligned} f(q^3, q^k) &= q^{-k}(q^2-1)(q^3-1) - (q^{k-1}-1)(1-q^{-k}) \\ &\quad + (q^{k-1}-q^2)(1-q^{-k})(q^2+q+1) \\ &= q^{-k+5} + q^{-k+4} + q^{k+1} + q^k - q^4 - q^3 - q^2 - q \\ &= (q^{k-1}-1)[q^2 + q - q^{-k+5} - q^{-k+4}] \\ &\geq 0, \end{aligned} \tag{4.19}$$

for $k \geq 3$. So from (4.13), (4.18) and (4.19), we have

$$f(x, q^2) > 0, \quad \text{for} \quad k \geq 3, \quad q^2 < x < q^3. \tag{4.20}$$

Note that in (4.12), $\prod_{j=0}^{\infty}(1 - q^{\nu-k-j}) < 0$, for $k = 2$ and $\prod_{j=0}^{\infty}(1 - q^{\nu-k-j}) > 0$ for $k \geq 3$. So from (4.12), (4.17) and (4.20), we get that for $k \geq 2$

$$\begin{aligned} &-\frac{(1-q^{\nu-2})(1-q^{\nu-1})(1-q^{\nu})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})^2}(q^k - q^{-1})_{q^{-1}}^{(-\nu-1)}x(1) \\ &+\frac{q(1-q^{\nu-2})(q^k-q)_{q^{-1}}^{(-\nu+1)}}{\Gamma_{q^{-1}}(3-\nu)(q-1)^2}x(1) \end{aligned} \tag{4.21}$$

$$
\begin{aligned}
&-\frac{(1-q^{\nu-2})}{\Gamma_{q^{-1}}(3-\nu)(1-q^{-1})}\Big[\frac{(1-q^{\nu-1})(1-q^{\nu})q}{(1-q^{-1})}(q^k-1)_{q^{-1}}^{(-\nu-1)}\\
&+\frac{(1-q^{\nu-1})}{(1-q^{-1})}(q^k-q)_{q^{-1}}^{(-\nu)}+\frac{(q^k-q)_{q^{-1}}^{(-\nu+1)}}{(q-1)}\Big]x(q)\\
&\geq 0.
\end{aligned}
$$

Taking $k=2$ in (4.6) and using (4.21) for $k=2$ and $(q^2-q)_{q^{-1}}^{(-\nu+1)}<0$, we get that

$$\nabla_q^2 x(q^2) \geq 0.$$

Suppose that $\nabla_q^2 x(q^i) \geq 0$ for $i=2,3,\ldots,k-1$. From (4.6), (4.21) and using $(q^k-q^{-1+k})_{q^{-1}}^{(-\nu+1)}<0$ for $k\geq 2$, we have $\nabla_q^2 x(q^k)\geq 0$ and the proof is complete

□

REFERENCES

[1] R. P. Agarwal, Certain fractional q-integral and q-derivative, Proc. Camb. Phil. Soc. **66** (1969) 365–370.

[2] W. A. Al-Salam, Some fractional q-integrals and q-derivatives, Proc. Edin. Math. Soc. **15** (1966) 135–140.

[3] G. Anastassiou, Foundations of nabla fractional calculus on time scales and inequalities, Comput. Math. Appl. **59** (2010) 3750–3762.

[4] F. M. Atici, P. W. Eloe, Initial value problems in discrete fractional calculus, Proc. Amer. Math. Soc. **137** (2009) 981–989.

[5] F. M. Atici, P. W. Eloe, Discrete fractional calculus with the nabla operator, Electron. J. Qual. Theory Differ. Equ. **3** (2009) 1–12.

[6] F. M. Atici, P. W. Eloe, Linear systems of fractional nabla difference equations, Rocky Mountain J. Math., **41** (2011) 353–370.

[7] F. M. Atici, P. W. Eloe, Fractional q-calculus on a time scale, Journal of Nonlinear Mathematical Physics, **14** (2007) 341–352.

[8] M. Bohner and A. Peterson, Dynamic Equations on time Scales: An Introduction with Applications Birkhäuser, Boston (2001).

[9] M. Bohner and A. Peterson, Advances in Dynamic Equations on Time Scales, Editors, Birkhäuser, Boston (2003).

[10] J. Čermák and L. Nechvátal, On (q,h)-analogue of fractional calculus, Journal of Nonlinear Mathematical Physics, **17** (2010) 51–68.

[11] J. Čermák, T. Kisela and L. Nechvátal, Discrete Mittag-Leffler functions in linear fractional difference equations, Abstract and Applied Analysis, Vol. 2011, Article ID 565067, 21 pages. doi:10.1155/2011/565067.

[12] J. D. Diaz and T. J. Osler, Differences of fractional order, Math. Comp. **28** (1974) 185–202.

[13] R. Dahal and C. Goodrich, A monotonicity result for discrete fractional difference operators, Arch. Math. **102**(2014), 293–299.

[14] R. A. C. Ferreira, A discrete fractional Gronwall inequality, Proc. Amer. Math. Soc. **140** (2012) 1605–1612.

[15] R. A. C. Ferreira and D. F. M. Torres, Fractional h-difference equations arising from the calculus of variations, Appl. Anal. Discrete Math. **5** (2011) 110–121.

[16] C. Goodrich, A convexity result for fractional differences., Applied Mathematics Letter, **35**(2014), 58–62.

[17] C. Goodrich and A. Peterson, Discrete Fractional Calculus, Springer, 2015.

[18] B. Jia, L. Erbe and A. Peterson, Two monotonicity results for nabla and delta fractional difference, Arch Math., **104**(2015) 589–597.

[19] B. Jia, L. Erbe and A. Peterson, Convexity for nabla and delta fractional differences, J. Difference Equ. Appl., **21** (2015) 360–373.

[20] B. Jia, L. Erbe and A. Peterson, Comparison theorems and asymptotic behavior of solutions of discrete fractional equations, Electron J.Qual.Theory Differ. Equ. 2015, No.89, 1–18.

[21] W. Kelley and A. Peterson, Difference Equations: An Introduction With Applications, Second Edition, Harcourt/Academic Press 2001.

[22] A. Nagai, On a certain fractional q-difference and its Eigen function, Journal of Nonlinear Mathematical Physics, **10** (2003) 133–142.

[23] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.