

CONSIDERATIONS ON MIXED INITIAL-BOUNDARY VALUE PROBLEMS FOR MICROPOLAR POROUS BODIES

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ABSTRACT. This paper is dedicated to some results in the thermodynamic theory of porous elastic bodies. Unlike other studies, here is included the voidage time derivative among the independent constitutive variables. In order to analyse the spatial behavior of solutions, we use some estimates of Saint-Venant type in the case of bounded bodies, while for the unbounded bodies, the spatial behavior is described by means of some estimates of Phragmén-Lindelöf type.

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1. Introduction

For a more faithful characterization of the behavior of some kinds of materials it is necessary to introduce into continuum theory of the appropriate terms reflecting the microstructure of the materials. Materials which operate at elevated temperature will invariable be subjected to heat flow at some time during normal use. The heat flow will involve a temperature distribution which will inevitably give rise to thermal stresses. The role of the pertinent material properties and other variables can affect the magnitude of thermal stress must be well understood and all possible mode of failure must be considered.

The results of our present study can be useful in other fields of applications which deal with porous materials as geological materials, solid packed granular and many others. Goodman and Cowin made the first investigations on materials with voids, they are the initiators of the granular theory, in the paper [7].

Similar studies appear in the paper [5] where the authors Cowin and Nunziato introduce, as in fact they did Goodman and Cowin, an additional degree of freedom in order to develop the mechanical behavior of porous solids in which the matrix material

is elastic and the interstices are voids material. interstices are voids of material. This theory has found immediate applications to geological materials like rocks and soil and to manufactured porous materials, like ceramics and pressed powders. We want to emphasize that the basic feature of this theory is the introduction of a concept of material for which the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field (see also Iesan and Quidanilla [8]). The theory of Cowin and Nunziato (see also Nunziato and Cowin [15]) is dedicated to non conductor of heat materials. In the context of behavior of solutions, the author of paper [14] consider the Cahn-Hilliard equation and obtain the exponential decay of solutions under suitable assumptions on the data. Also, Quintanilla [17] investigates the spatial behavior of the solutions for a theory for the heat conduction with a delay term. Chirita and Ciarletta, in the paper [3], have used for first time in this context the method of time-weighted surface power function. Ciarletta and Scarpetta in [4] give a variational characterization of Gurtin type for the incremental problem of thermoelasticity for porous dielectric materials, by means of non-standard techniques. In [16] Passarella and his co-workers derive a uniqueness theorem with no positive definiteness assumption on the elastic constitutive coefficients and, under non homogeneous initial conditions, a reciprocal relation and a variational principle. The main result of the paper [6] is the calculation of mechanical properties and of the mechanical behavior of single wall carbon nanotubes. The authors of paper [2] introduce a novel particle approach for elasticity, namely the modified finite particle method. In our paper [11] a minimum principle for dipolar materials with stretch is presented in the study. A kind of weak solutions is presented in our study [10] in the context of thermoelasticity of dipolar materials with voids, and the paper [12] is dedicated to a partition of energy in thermoelasticity of microstretch bodies. In [18] the authors consider the reflection and transmission of waves from imperfect boundary between two heat conducting micropolar thermoelastic materials. Some results on existence and uniqueness of solutions for thermoelastic micropolar materials are made in [9]. A model for a microstretch thermoelastic material with two temperatures can be found in the paper [13].

In our present study we wish to extend the Cowin and Nunziato theory to cover the micropolar thermoelastic materials with voids. The basic premise underlying our paper is the introduction into the set of constitutive variables of the time derivative of the voidage to characterize the inelastic effects.

2. Basic equations

Let us assume that a micropolar thermoelastic material with voids occupies at time $t = 0$ a properly regular region B of Euclidian three-dimensional space R^3 . The

boundary of B , denoted by ∂B , is a sufficiently smooth surface to admit the application of divergence theorem. The closure of B will be denoted by \bar{B} . Throughout this paper we refer the motion of the continuum to a fixed system of rectangular Cartesian axes Ox_i , ($i = 1, 2, 3$) and adopt Cartesian tensor notation. The Greek indices will always assume the values 1, 2, whereas the italic indices will range over the value 1, 2, 3. A comma followed by a subscript denotes partial derivatives with respect to the spatial respective coordinates whereas a superposed dot stands for the material time derivative. Convention of mute indice summation (Einstein convention) on repeated indices is also used. The spatial argument and the time argument of a function will be omitted when there is no likelihood of confusion.

The bulk density ϱ_0 , the matrix density γ and the matrix volume fraction ν , in the reference configuration, are related by

$$\varrho_0 = \gamma_0 \nu_0,$$

where γ_0 and ν_0 are spatially constants.

The motion of the micropolar thermoelastic body with voids is described by means of the following independent variables

- $u_i(x, t)$, $\varphi_i(x, t)$ - the displacement and microrotation fields from reference configuration;
- θ - the change in temperature from T_0 , the absolute temperature of the reference configuration, i.e. $\theta(x, t) = T(x, t) - T_0$;
- σ - the change in volume fraction measured from the reference configuration volum fraction ν_0 , i.e. $\sigma(x, t) = \nu(x, t) - \nu_0$.

If the initial body is stress free and with zero intrinsic equilibrated body force and zero flux rate, we can write the free energy function as follows

$$\begin{aligned} \Psi = & \frac{1}{2} A_{ijmn} \varepsilon_{ij} \varepsilon_{mn} + B_{ijmn} \varepsilon_{ij} \gamma_{mn} + \frac{1}{2} C_{ijmn} \gamma_{ij} \gamma_{mn} \\ & + B_{ij} \sigma \varepsilon_{ij} + C_{ij} \sigma \gamma_{ij} + D_{ijk} \phi_k \varepsilon_{ij} + E_{ijk} \phi_k \gamma_{ij} \\ (1) \quad & - \alpha_{ij} \theta \varepsilon_{ij} - \beta_{ij} \theta \gamma_{ij} - m \theta \sigma + d_i \sigma \phi_i + \gamma_i \theta \phi_i \\ & - \frac{1}{2} a \theta^2 + \frac{1}{2} \xi \sigma^2 + \frac{1}{2} A_{ij} \phi_i \phi_j - \frac{1}{2} \omega \dot{\sigma}^2. \end{aligned}$$

With a suggestion given in [5], the expression $-\omega \dot{\sigma}$ is the dissipation which takes into account of the inelastic behavior of the voids. Here ω is a positive constant.

With the aid of the free energy function, using an usual procedure, we can derive the following constitutive equations (see [10], [16])

$$\begin{aligned} t_{ij} &= C_{ijmn} \varepsilon_{mn} + B_{ijmn} \gamma_{mn} + B_{ij} \sigma + D_{ijk} \phi_k - \beta_{ij} \theta, \\ m_{ij} &= B_{mnij} \varepsilon_{mn} + C_{ijmn} \gamma_{mn} + C_{ij} \sigma + E_{ijk} \phi_k - \alpha_{ij} \theta, \end{aligned}$$

$$\begin{aligned}
(2) \quad h_i &= D_{mni}\varepsilon_{mn} + E_{mni}\gamma_{mn} + d_i\sigma + A_{ij}\phi_j - \gamma_i\theta, \\
g &= -B_{ij}\varepsilon_{ij} - C_{ij}\gamma_{ij} - \xi\sigma - d_i\phi_i + m\theta, \\
\rho\eta &= \alpha_{ij}\varepsilon_{ij} + \beta_{ij}\gamma_{ij} + m\sigma + \gamma_i\phi_i + a\theta, \\
q_i &= k_{ij}\theta_{,j},
\end{aligned}$$

where ε_{ij} , γ_{ij} and ϕ_i are the kinematic characteristics of the strain which are computed by means of the following geometric relations

$$\begin{aligned}
(3) \quad \varepsilon_{ij} &= u_{j,i} + \varepsilon_{jik}\varphi_k, \quad \gamma_{ij} = \varphi_{j,i}, \\
\phi_i &= \sigma_{,i}, \quad \theta = T - T_0, \quad \sigma = \nu - \nu_0.
\end{aligned}$$

By using a procedure similar to that used by Nunziato and Cowin in [15], we obtain the following fundamental equations (see also, [9])

- the equations of motion

$$(4) \quad t_{ij,j} + \rho F_i = \rho \ddot{u}_i, \quad m_{ij,j} + \varepsilon_{ijk}t_{jk} + \rho M_i = I_{ij}\ddot{\varphi}_j;$$

- the balance of the equilibrated forces

$$(5) \quad h_{i,i} + g + \rho L = \rho \kappa \ddot{\sigma};$$

- the energy equation

$$(6) \quad \rho T_0 \dot{\eta} = q_{i,i} + \rho S.$$

In the above equations we used the following notations: ρ -the constant mass density; η -the specific entropy; T_0 -the constant absolute temperature of the body in its reference state; I_{ij} -coefficients of microinertia; κ -the equilibrated inertia; u_i -the components of displacement vector; φ_i -the components of microrotation vector; φ -the volume distribution function which in the reference state is φ_0 ; σ -the change in volume fraction measured from the reference state; θ -the temperature variation measured from the reference temperature T_0 ; ε_{ij} , γ_{ij} , ϕ_i -kinematic characteristics of the strain; t_{ij} -the components of the stress tensor; m_{ij} -the components of the couple stress tensor; h_i -the components of the equilibrated stress vector; q_i -the components of the heat flux vector; F_i -the components of the body forces; M_i -the components of the body couple; S -the heat supply per unit time; g -the intrinsic equilibrated force; L -the extrinsic equilibrated body force; A_{ijmn} , B_{ijmn} , \dots , k_{ij} -the characteristic functions of the material, and they are prescribed functions of the spatial variable and obey the symmetry relations

$$(7) \quad A_{ijmn} = A_{mnij}, \quad C_{ijmn} = C_{mnij}, \quad A_{ij} = A_{ji}, \quad k_{ij} = k_{ji}.$$

A consequence of the entropy inequality is the following useful relation

$$(8) \quad k_{ij}\theta_{,i}\theta_{,j} \geq 0.$$

The equations (4) and (6) are analogous to equations of motion and, respectively, to the balance equation, as in the classical theory.

The new balance of equilibrated force (5) can be motivated by a variational argument as in the paper Cowin and Nunziato [5]. It is necessary to assume that the functions coefficients ϱ , κ and a and the above constitutive coefficients are continuous differentiable functions on closure \bar{B} of B . Also, we assume that ϱ , κ and a are strictly positive functions on \bar{B} , that is

$$(9) \quad \varrho(x) \geq \varrho_0 > 0, \quad \kappa(x) \geq \kappa_0 > 0, \quad a(x) \geq a_0 > 0, \quad \varrho_0, \kappa_0, a_0 = \text{constants.}$$

Suppose that the conductivity tensor k_{ij} is symmetric, positive definite and satisfies the following double inequality

$$(10) \quad k_m \theta_{,i} \theta_{,j} \leq k_{ij} \theta_{,i} \theta_{,j} \leq k_M \theta_{,i} \theta_{,j}.$$

Here k_m and k_M are the minimum, respectively, maximum value of the conductivity tensor.

With the help of inequality (10) by taking into account the constitutive equation (2)₆ and the Schwarz's inequality, we obtain:

$$(11) \quad q_i q_i = (k_{ij} \theta_{,j}) q_i \leq (k_{ij} \theta_{,i} \theta_{,j})^{1/2} (k_{mn} q_m q_n)^{1/2} \leq (k_{ij} \theta_{,i} \theta_{,j})^{1/2} (k_M q_n q_n)^{1/2}$$

from where we deduce

$$(12) \quad q_i q_i \leq k_M k_{ij} \theta_{,i} \theta_{,j}.$$

We assume that the free energy function Ψ defined in (1) is a positive definite quadratic form. As such, we deduce that it satisfies the inequalities

$$(13) \quad \mu_m (\varepsilon_{ij} \varepsilon_{ij} + \gamma_{ij} \gamma_{ij} + \phi_i \phi_i + \sigma^2) \leq 2\Psi \leq \mu_M (\varepsilon_{ij} \varepsilon_{ij} + \gamma_{ij} \gamma_{ij} + \phi_i \phi_i + \sigma^2)$$

where μ_m and μ_M are positive constants.

Let us denote by \mathcal{S}_7 the seven-dimensional space of all displacement fields \mathbf{U} , where \mathbf{U} is of the form

$$(14) \quad \mathbf{U} = \{u_i, \varphi_i, \sigma\}$$

On the the space \mathcal{S}_7 we define the following inner product

$$(15) \quad \mathbf{U} \cdot \mathbf{V} = u_i v_i + \varphi_i \psi_i + \sigma \chi, \quad \mathbf{U} = \{u_i, \varphi_i, \sigma\}, \quad \mathbf{V} = \{v_i, \psi_i, \chi\}$$

and, as usual, this inner product induces the following norm

$$(16) \quad |\mathbf{V}| = (\mathbf{V} \cdot \mathbf{V})^{1/2} = (v_i v_i + \psi_i \psi_i + \chi^2)^{1/2}$$

for a vector field $\mathbf{V} = \{v_i, \psi_i, \chi\} \in \mathcal{S}_7$. In order to characterize the state of strain we will use the fields

$$(17) \quad \mathbf{E}(\mathbf{U}) = \{\varepsilon_{ij}(\mathbf{U}), \gamma_{ij}(\mathbf{U}), \phi_i(\mathbf{U}), \sigma\}$$

Taking into account (3), the tensors of the strain from (17) are

$$(18) \quad \varepsilon_{ij}(\mathbf{U}) = u_{j,i} + \varepsilon_{jik}\varphi_k, \quad \gamma_{ij}(\mathbf{U}) = \varphi_{j,i}, \quad \phi_i(\mathbf{U}) = \sigma_{,i}$$

Now, we introduce an other vector space, namely the vector space of the strains, which will be denoted by \mathcal{E} and which consists of elements of the form (17). The norm of this vector space has the form

$$(19) \quad |\mathbf{E}| = (\mathbf{E} \cdot \mathbf{E})^{1/2} = (\varepsilon_{ij}\varepsilon_{ij} + \gamma_{ij}\gamma_{ij} + \phi_i\phi_i + \sigma^2)^{1/2}$$

With a suggestion given by \mathbf{E} from (17) and the constitutive equations (2) we introduce the notations

$$(20) \quad T_{ij}(\mathbf{E}) = C_{ijmn}\varepsilon_{mn} + B_{ijmn}\gamma_{mn} + B_{ij}\sigma + D_{ijk}\phi_k,$$

$$(21) \quad M_{ij}(\mathbf{E}) = B_{mni}\varepsilon_{mn} + C_{ijmn}\gamma_{mn} + C_{ij}\sigma + E_{ijk}\phi_k,$$

$$(22) \quad H_i(\mathbf{E}) = D_{mni}\varepsilon_{mn} + E_{mni}\gamma_{mn} + d_i\sigma + A_{ij}\phi_j,$$

$$(23) \quad G(\mathbf{E}) = -B_{ij}\varepsilon_{ij} - C_{ij}\gamma_{ij} - \xi\sigma - d_i\phi_i,$$

and attach to these notations the quantity $\mathbf{S}(\mathbf{E})$, defined by

$$\mathbf{S}(\mathbf{E}) = \{T_{ij}(\mathbf{E}), M_{ij}(\mathbf{E}), H_i(\mathbf{E}), G(\mathbf{E})\}$$

for every $\mathbf{E} \in \mathcal{E}$.

If we take into account (17) and (19), for every $\mathbf{S}(\mathbf{E}) \in \mathcal{E}$ we define the norm by

$$(24) \quad |\mathbf{S}(\mathbf{E})| = \left\{ T_{ij}(\mathbf{E})T_{ij}(\mathbf{E}) + M_{ij}(\mathbf{E})M_{ij}(\mathbf{E}) + H_i(\mathbf{E})H_i(\mathbf{E}) + G(\mathbf{E})G(\mathbf{E}) \right\}^{1/2}$$

Considering \mathbf{E} , we introduce the bilinear form $\mathcal{F}(\mathbf{E}^{(1)}, \mathbf{E}^{(2)})$ by

$$(25) \quad \begin{aligned} \mathcal{F}(\mathbf{E}^{(1)}, \mathbf{E}^{(2)}) = & \frac{1}{2} \left[A_{ijmn}\varepsilon_{ij}^{(1)}\varepsilon_{mn}^{(2)} + B_{ijmn} \left(\varepsilon_{ij}^{(1)}\gamma_{mn}^{(2)} + \varepsilon_{ij}^{(2)}\gamma_{mn}^{(1)} \right) \right. \\ & + C_{ijmn}\gamma_{ij}^{(1)}\gamma_{mn}^{(2)} + B_{ij} \left(\sigma^{(1)}\varepsilon_{ij}^{(2)} + \sigma^{(2)}\varepsilon_{ij}^{(1)} \right) \\ & + C_{ij} \left(\sigma^{(1)}\gamma_{ij}^{(2)} + \sigma^{(2)}\gamma_{ij}^{(1)} \right) \\ & + D_{ijk} \left(\phi_k^{(1)}\varepsilon_{ij}^{(2)} + \phi_k^{(2)}\varepsilon_{ij}^{(1)} \right) + E_{ijk} \left(\phi_k^{(1)}\gamma_{ij}^{(2)} + \phi_k^{(2)}\gamma_{ij}^{(1)} \right) \\ & \left. + d_i \left(\sigma^{(1)}\phi_i^{(2)} + \sigma^{(2)}\phi_i^{(1)} \right) + \xi\sigma^{(1)}\sigma^{(2)} + A_{ij}\phi_i^{(1)}\phi_j^{(2)} \right]. \end{aligned}$$

for every $\mathbf{E}^{(\alpha)} \in \mathcal{E}$, where $\mathbf{E}^{(\alpha)} = \left\{ \varepsilon_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)}, \phi_i^{(\alpha)}, \sigma^{(\alpha)} \right\}$, $\alpha = 1, 2$.

It is easy to deduce the symmetry relation

$$(26) \quad \mathcal{F}(\mathbf{E}^{(1)}, \mathbf{E}^{(2)}) = \mathcal{F}(\mathbf{E}^{(2)}, \mathbf{E}^{(1)}), \quad \forall \mathbf{E}^{(1)}, \mathbf{E}^{(2)} \in \mathcal{E}$$

taking into account the symmetry relations (7).

Considering the free energy function Ψ defined by (1), by direct calculations we deduce that

$$(27) \quad \mathcal{F}(\mathbf{E}, \mathbf{E}) = \Psi(\mathbf{E}), \quad \forall \mathbf{E} \in \mathcal{E}$$

Taking into account the inequalities (13) and using the well known Cauchy-Buniakovschi-Schwarz's inequality, it is no difficult to obtain that

$$(28) \quad \mathcal{F}(\mathbf{E}^{(1)}, \mathbf{E}^{(2)}) \leq [\Psi(\mathbf{E}^{(1)})]^{1/2} [\Psi(\mathbf{E}^{(2)})]^{1/2}, \forall \mathbf{E}^{(1)}, \mathbf{E}^{(2)} \in \mathcal{E}$$

Through a combination of relations (20)–(25) we are lead to

$$(29) \quad \begin{aligned} |\mathbf{S}(\mathbf{E})|^2 &= T_{ij}(\mathbf{E})T_{ij}(\mathbf{E}) + M_{ij}(\mathbf{E})M_{ij}(\mathbf{E}) + H_i(\mathbf{E})H_i(\mathbf{E}) + G(\mathbf{E})^2 \\ &= A_{ijmn}T_{ij}\varepsilon_{mn} + B_{ijmn}T_{ij}\gamma_{mn} + B_{ij}T_{ij}\sigma + D_{ijk}T_{ij}\Phi_k \\ &\quad + B_{mnij}M_{ij}\varepsilon_{mn} + C_{ijmn}M_{ij}\gamma_{mn} + C_{ij}M_{ij}\sigma + E_{ijk}M_{ij}\Phi_k \\ &\quad + D_{mni}\varepsilon_{mn}H_i + E_{mni}\gamma_{mn}H_i + d_i\sigma H_i + A_{ij}\Phi_j H_i \\ &\quad - B_{ij}\varepsilon_{ij}G - C_{ij}\gamma_{ij}G - \xi\sigma G - d_i\Phi_i G = 2\mathcal{F}(\mathbf{E}, \bar{\mathbf{S}}(\mathbf{E})) \end{aligned}$$

where we have used the notation

$$\bar{\mathbf{S}}(\mathbf{E}) = \{T_{ij}(\mathbf{E}), M_{ij}(\mathbf{E}), H_i(\mathbf{E}), -G(\mathbf{E})\}$$

With the help of relations (13), (19), (28) and (29) one obtains

$$(30) \quad |\mathbf{S}(\mathbf{E})|^2 \leq 2\mu_M \Psi(\mathbf{E})$$

Considering the norm (24) and the inequality (30) we get

$$(31) \quad T_{ij}(\mathbf{E})T_{ij}(\mathbf{E}) + M_{ij}(\mathbf{E})M_{ij}(\mathbf{E}) + H_i(\mathbf{E})H_i(\mathbf{E}) \leq 2\mu_M \Psi(\mathbf{E}), \forall \mathbf{E} \in \mathcal{E}$$

such that for any arbitrarily positive number ε , we obtain

$$(32) \quad (T_{ij} + M_{ij})(T_{ij} + M_{ij}) \leq (1 + \varepsilon)T_{ij}T_{ij} + \left(1 + \frac{1}{\varepsilon}\right)M_{ij}M_{ij},$$

As a consequence of the inequality (31) we are led to

$$(33) \quad \begin{aligned} t_{ij}t_{ij} + m_{ij}m_{ij} + h_i h_i &= (T_{ij} - \alpha_{ij}\theta)(T_{ij} - \alpha_{ij}\theta) + \\ &\quad + (M_{ij} - \beta_{ij}\theta)(M_{ij} - \beta_{ij}\theta) + (H_i - \gamma_i\theta)(H_i - \gamma_i\theta) \leq \\ &\leq (1 + \varepsilon)T_{ij}T_{ij} + \left(1 + \frac{1}{\varepsilon}\right)\alpha_{ij}\alpha_{ij}\theta^2 + (1 + \varepsilon)M_{ij}M_{ij} \\ &\quad + \left(1 + \frac{1}{\varepsilon}\right)\beta_{ij}\beta_{ij}\theta^2 + (1 + \varepsilon)H_i H_i + \left(1 + \frac{1}{\varepsilon}\right)\gamma_{ij}\gamma_{ij}\theta^2 \\ &\leq (1 + \varepsilon)2\mu_M \Psi(\mathbf{E}) + \left(1 + \frac{1}{\varepsilon}\right)M^2\theta^2, \forall \varepsilon > 0, \end{aligned}$$

where we took into account relations (2), (20)–(23) and the inequality (32). Also, M^2 is defined by

$$(34) \quad M^2 = \max_B \left(\alpha_{ij}\alpha_{ij} + \beta_{ij}\beta_{ij} + \gamma_i\gamma_i \right)$$

To complete the mixed initial-boundary value problem within context of thermoelastic theory of micropolar bodies with voids we give the boundary and initial conditions. The boundary conditions can be prescribed as in classical elasticity and we must give

the additional data for the surface continuous temperature field on the boundary ∂B of the geometry of the body B and for the time interval for which the solution is desired. As such, we use the initial conditions in the form

$$(35) \quad \begin{aligned} u_i(x, 0) &= u_i^0(x), & \dot{u}_i(x, 0) &= u_i^1(x), & x &\in \bar{B}, \\ \varphi_i(x, 0) &= \varphi_i^0(x), & \dot{\varphi}_i(x, 0) &= \varphi_i^1(x), & x &\in \bar{B}, \\ \theta(x, 0) &= \theta^0(x), & \sigma(x, 0) &= \sigma^0(x), & \dot{\sigma}(x, 0) &= \sigma^1(x), & x &\in \bar{B}, \end{aligned}$$

Also, we take the boundary conditions in the form

$$(36) \quad \begin{aligned} u_i &= \bar{u}_i \text{ on } \partial B_1 \times [0, \infty), & t_i &\equiv t_{ij}n_j = \bar{t}_i \text{ on } \partial B_1^c \times [0, \infty), \\ \varphi_i &= \bar{\varphi}_i \text{ on } \partial B_2 \times [0, \infty), & m_i &\equiv m_{ij}n_j = \bar{m}_i \text{ on } \partial B_2^c \times [0, \infty), \\ \sigma &= \bar{\sigma} \text{ on } \partial B_3 \times [0, \infty), & h &\equiv h_i n_i = \bar{h} \text{ on } \partial B_3^c \times [0, \infty), \\ \theta &= \bar{\theta} \text{ on } \partial B_4 \times [0, \infty), & q &\equiv q_i n_i = \bar{q} \text{ on } \partial B_4^c \times [0, \infty). \end{aligned}$$

Above, the surfaces ∂B_1 , ∂B_2 , ∂B_3 and ∂B_4 with respective complements ∂B_1^c , ∂B_2^c , ∂B_3^c and ∂B_4^c are subsets of the surface ∂B . By n_i we denote the components of the unit outward normal to ∂B .

The given functions u_i^0 , u_i^1 , φ_i^0 , φ_i^1 , θ^0 , σ^0 , σ^1 , \bar{u}_i , \bar{t}_i , $\bar{\varphi}_i$, \bar{m}_i , $\bar{\sigma}$, $\bar{\theta}$, \bar{q} and \bar{h} are continuous functions in their domains.

By a solution of the mixed initial-boundary value problem for the thermoelasticity of micropolar bodies with voids, in the cylinder $\Omega_0 = B \times [0, \infty)$ we mean an ordered array $(u_i, \varphi_i, \sigma, \theta)$ which satisfies the equations (4)–(6) for all $(x, t) \in \Omega_0$, the boundary conditions (36) and the initial conditions (35).

If we substitute constitutive relations (2) into equations (4), (5) and (6), we obtain the following system of equations

$$(37) \quad \begin{aligned} \varrho \ddot{u}_i &= (A_{ijmn}\varepsilon_{mn} + B_{ijmn}\gamma_{mn} + B_{ij}\sigma + D_{ijk}\phi_k - \alpha_{ij}\theta)_{,j} + \varrho F_i, \\ I_{ij}\ddot{\varphi}_j &= (B_{mni}\varepsilon_{mn} + C_{ijmn}\gamma_{mn} + C_{ij}\sigma + E_{ijk}\phi_k - \beta_{ij}\theta)_{,j} \\ &\quad + \varepsilon_{ijk}(A_{jkmn}\varepsilon_{mn} + B_{jkmn}\gamma_{mn} + B_{jk}\sigma + D_{jkm}\phi_m - \alpha_{jk}\theta) + \varrho M_i, \\ \varrho \kappa \ddot{\sigma} &= (D_{mni}\varepsilon_{mn} + E_{mni}\gamma_{mn} + d_i\sigma + A_{ij}\phi_j - \gamma_i\theta)_{,i} + \varrho L \\ &\quad - B_{ij}\varepsilon_{ij} - C_{ij}\gamma_{ij} - \xi\sigma - d_i\phi_i + m\theta, \\ a\dot{\theta} &= \frac{1}{\varrho T_0}(k_{ij}\theta_{,j})_{,i} + \frac{1}{T_0}S - \beta_{ij}\dot{\varepsilon}_{ij} - \alpha_{ij}\dot{\gamma}_{ij} - m\dot{\sigma} - a_i\dot{\phi}_i. \end{aligned}$$

In the following we will address the initial boundary value problem consisting of system of equations (37), the initial conditions (35) and the boundary conditions (36) which will be denoted by \mathcal{P} .

3. Preliminary results

To demonstrate the main results with regard to the spatial behavior of the solutions of the problem \mathcal{P} , we need some integral identities that are proved in next three theorems.

Theorem 1. *Let us consider an arbitrary solution $(u_i, \varphi_i, \sigma, \theta)$ of the problem \mathcal{P} . Then takes place the following conservation law of total energy*

$$\begin{aligned}
 & \int_B e^{-\lambda t} \left\{ \frac{1}{2} [\varrho \dot{u}_i(t) \dot{u}_i(t) + I_{ij} \dot{\varphi}_i(t) \dot{\varphi}_j(t) + \varrho \kappa \dot{\sigma}^2(t)] + \Psi(\mathbf{E}(t)) + \frac{1}{2} a \theta^2(t) \right\} dV \\
 & + \int_0^t \int_B e^{-\lambda s} \frac{\lambda}{2} [\varrho \dot{u}_i(s) \dot{u}_i(s) + I_{ij} \dot{\varphi}_i(s) \dot{\varphi}_j(s) + \varrho \kappa \dot{\sigma}^2(s)] dV ds \\
 (38) \quad & + \int_0^t \int_B e^{-\lambda s} \left[\lambda \Psi(\mathbf{E}(s)) + \frac{\lambda}{2} a \theta^2(s) + \frac{1}{T_0} k_{ij} \theta_{,i}(s) \theta_{,j}(s) \right] dV ds \\
 & = \int_B \left\{ \frac{1}{2} [\varrho \dot{u}_i(0) \dot{u}_i(0) + I_{ij} \dot{\varphi}_i(0) \dot{\varphi}_j(0) + \varrho \kappa \dot{\sigma}^2(0)] + \Psi(\mathbf{E}(0)) + \frac{1}{2} a \theta^2(0) \right\} dV \\
 & + \int_0^t \int_B e^{-\lambda s} \varrho \left[\dot{u}_i(s) F_i(s) + \dot{\varphi}_i(s) M_i(s) + \dot{\sigma}(s) L(s) + \frac{1}{T_0} \theta(s) S(s) \right] dV ds \\
 & + \int_0^t \int_{\partial B} e^{-\lambda s} \left[t_i(s) \dot{u}_i(s) + m_i(s) \dot{\varphi}_i(s) + h(s) \dot{\sigma}(s) + \frac{1}{T_0} q(s) \theta(s) \right] dA ds,
 \end{aligned}$$

Here λ is a given positive parameter and quantities t_i, m_i, h and q are defined in (35).

Proof. Taking into account the system of equations (37), the constitutive equations (2), the geometric relations (3) and the symmetry relations (7), one obtains

$$\begin{aligned}
 & \frac{d}{ds} \left\{ \frac{1}{2} [\varrho \dot{u}_i(s) \dot{u}_i(s) + I_{ij} \dot{\varphi}_i(s) \dot{\varphi}_j(s) + \varrho \kappa \dot{\sigma}^2(s)] + \Psi(\mathbf{E}(s)) + \frac{1}{2} a \theta^2(s) \right\} \\
 (39) \quad & + \frac{1}{T_0} k_{ij} \theta_{,i}(s) \theta_{,j}(s) \\
 & = \varrho \left[\dot{u}_i(s) F_i(s) + \dot{\varphi}_i(s) M_i(s) + \dot{\sigma}(s) L(s) + \frac{1}{T_0} \theta(s) S(s) \right] \\
 & + \left[t_{ij}(s) \dot{u}_i(s) + m_{ik}(s) \dot{\varphi}_i(s) + h_j(s) \dot{\sigma}(s) + \frac{1}{T_0} q_j(s) \theta(s) \right]_{,j}
 \end{aligned}$$

In this equality we by $e^{-\lambda s}$ and then integrate the obtained identity over the cylinder $B \times [0, t]$. But, by hypothesis, the surface ∂B was assumed be sufficient smooth such that we can apply the divergence theorem. With the help of this theorem we are led to the desired result (38) and Theorem 1 is concluded. \square

Theorem 2. *Let $(u_i, \varphi_i, \sigma, \theta)$ be a solution of the mixed initial-boundary value problem consists of the equations (11), the boundary conditions (10) and the initial conditions (9). Then we have the following identity:*

$$2 \int_B [\varrho u_i(t) \dot{u}_i(t) + I_{ij} \varphi_i(t) \dot{\varphi}_j(t) + \varrho \kappa \sigma(t) \dot{\sigma}(t)] dV$$

$$\begin{aligned}
& + 2 \int_B \left[\frac{1}{T_0} k_{ij} \left(\int_0^t \theta_{,i}(s) ds \right) \left(\int_0^t \theta_{,j}(s) ds \right) \right] dV \\
& = 2 \int_0^t \int_B [\varrho \dot{u}_i(s) \dot{u}_i(s) + I_{ij} \dot{\varphi}_i(s) \dot{\varphi}_j(s) + \varrho \kappa \dot{\sigma}^2(s) - 2\Psi(\mathbf{E}(s)) - a\theta^2(s)] dV ds \\
(40) \quad & + 2 \int_0^t \int_B \varrho \eta(0) \theta(s) dV ds \\
& + 2 \int_B [\varrho u_i(0) \dot{u}_i(0) + I_{ij} \varphi_i(0) \dot{\varphi}_j(0) + \varrho \kappa \sigma(0) \dot{\sigma}(0)] dV \\
& + 2 \int_0^t \int_B \varrho \left[F_i(s) u_i(s) + M_i(s) \varphi_i(s) + L(s) \sigma(s) + \frac{1}{T_0} \theta(s) \int_0^s S(z) dz \right] dV ds \\
& + 2 \int_0^t \int_B \varrho \eta(0) \theta(s) dV ds + 2 \int_B [\varrho u_i(0) \dot{u}_i(0) + I_{ij} \varphi_i(0) \dot{\varphi}_j(0) + \varrho \kappa \sigma(0) \dot{\sigma}(0)] dV \\
& + 2 \int_0^t \int_{\partial B} \left[t_i(s) u_i(s) + m_i(s) \varphi_i(s) + h(s) \sigma(s) + \frac{1}{T_0} \theta(s) \int_0^s q(z) dz \right] dAd s
\end{aligned}$$

Proof. By direct calculations, taking into account the equations of motion (4)₁ and the geometric relations (3), one obtains

$$(41) \quad \frac{d}{ds} [\varrho u_i(s) \dot{u}_i(s)] = \varrho \dot{u}_i(s) \dot{u}_i(s) + [t_{ji}(s) u_i(s)]_{,j} - t_{ji}(s) u_{i,j}(s) + \varrho u_i(s) F_i(s)$$

In a similar way, using the motion equations (4)₂ and the geometric relations (3) we are led to

$$(42) \quad \begin{aligned} \frac{d}{ds} [I_{ij} \varphi_i(s) \dot{\varphi}_i(s)] &= I_{ij} \dot{\varphi}_i(s) \dot{\varphi}_i(s) + [m_{ji}(s) \varphi_i(s)]_{,j} \\ &\quad - m_{ji}(s) \varphi_{i,j}(s) + \varepsilon_{ijk} t_{jk}(s) \varphi_i(s) + \varrho \varphi_i(s) M_i(s) \end{aligned}$$

Now, we add the relations (41) and (42) therefore we deduce that

$$(43) \quad \begin{aligned} \frac{d}{ds} [\varrho u_i(s) \dot{u}_i(s) + I_{ij} \varphi_i(s) \dot{\varphi}_j(s)] &= \varrho \dot{u}_i(s) \dot{u}_i(s) + I_{ij} \dot{\varphi}_i(s) \dot{\varphi}_j(s) \\ &\quad + [t_{ji}(s) u_i(s) + m_{ji}(s) \varphi_i(s)]_{,j} - t_{ij}(s) \varepsilon_{ij}(s) - m_{ij}(s) \gamma_{ij}(s) \end{aligned}$$

If we take into account the constitutive equation (2)₁, we can write

$$(44) \quad \begin{aligned} t_{ij}(s) \varepsilon_{ij}(s) &= A_{ijmn} \varepsilon_{ij}(s) \varepsilon_{mn}(s) + B_{ijmn} \varepsilon_{ij}(s) \gamma_{mn}(s) + 2B_{ij} \sigma(s) \varepsilon_{ij}(s) \\ &\quad + 2D_{ijk} \phi_k(s) \varepsilon_{ij}(s) - [B_{ij} \sigma(s) \varepsilon_{ij}(s) + D_{ijk} \phi_k(s) \varepsilon_{ij}(s) + \alpha_{ij} \theta(s) \varepsilon_{ij}(s)] \end{aligned}$$

In a similar way, using the constitutive equation (2)₂ we can write:

$$(45) \quad \begin{aligned} m_{ij}(s) \gamma_{ij}(s) &= B_{mnij} \varepsilon_{ij}(s) \gamma_{mn}(s) + C_{ijmn} \gamma_{ij}(s) \gamma_{mn}(s) + 2C_{ij} \sigma(s) \gamma_{ij}(s) \\ &\quad + 2E_{ijk} \phi_k(s) \gamma_{ij}(s) - [C_{ij} \sigma(s) \gamma_{ij}(s) + E_{ijk} \phi_k(s) \gamma_{ij}(s) + \beta_{ij} \theta(s) \gamma_{ij}(s)] \end{aligned}$$

If we add member by member the relations (44) and (45), we are led to the following identity

$$t_{ij}(s) \varepsilon_{ij}(s) + m_{ij}(s) \gamma_{ij}(s) = A_{ijmn} \varepsilon_{ij}(s) \varepsilon_{mn}(s)$$

$$\begin{aligned}
 (46) \quad & + 2B_{mnij}\varepsilon_{ij}(s)\gamma_{mn}(s) + C_{ijmn}\gamma_{ij}(s)\gamma_{mn}(s) + 2B_{ij}\sigma(s)\varepsilon_{ij}(s) \\
 & + 2D_{ijk}\phi_k(s)\varepsilon_{ij}(s) + 2C_{ij}\sigma(s)\gamma_{ij}(s) + 2E_{ijk}\phi_k(s)\gamma_{ij}(s) \\
 & - [B_{ij}\sigma(s)\varepsilon_{ij}(s) + D_{ijk}\phi_k(s)\varepsilon_{ij}(s) + \alpha_{ij}\theta(s)\varepsilon_{ij}(s)] \\
 & - [C_{ij}\sigma(s)\gamma_{ij}(s) + E_{ijk}\phi_k(s)\gamma_{ij}(s) + \beta_{ij}\theta(s)\gamma_{ij}(s)]
 \end{aligned}$$

If we use formulas (2)₃–(2)₅ and (3) we can obtain equivalent expressions for the last two parentheses in (46)

$$\begin{aligned}
 (47) \quad & [B_{ij}\varepsilon_{ij}(s) + C_{ij}\gamma_{ij}(s)]\sigma(s) + [D_{ijk}\varepsilon_{ij}(s) + E_{ijk}\gamma_{ij}(s)]\phi_k(s) \\
 & + [\alpha_{ij}\varepsilon_{ij}(s) + \beta_{ij}\gamma_{ij}(s)]\theta(s) = g(s)\sigma(s) - \xi\sigma^2(s) - 2d_i\phi_i(s)\sigma(s) \\
 & + [h_i(s)\sigma(s)]_{,i} - h_{i,i}(s)\sigma(s) - A_{ij}\phi_i(s)\phi_j(s) - a\theta^2(s) + \varrho\eta(s)\theta(s)
 \end{aligned}$$

Integrating the energy equation (6) over interval $[0, s]$, we deduce

$$(48) \quad \varrho\eta(s) = \frac{1}{T_0} \int_0^s q_{i,i}(z)dz + \frac{\varrho}{T_0} \int_0^s S(z)dz + \varrho\eta(0)$$

such that, in view of balance of the equilibrated forces (5) and above relation (48), we are led to

$$\begin{aligned}
 (49) \quad & [g(s) + h_{i,i}(s)]\sigma(s) - \varrho\eta(s)\theta(s) = [\varrho\kappa\ddot{\sigma}(s) - \varrho L(s)]\sigma(s) - \varrho\eta(0)\theta(s) \\
 & - \frac{\varrho}{T_0} \int_0^s S(z)dz - \left[\frac{1}{T_0}\theta(s) \int_0^s q_i(z)dz \right]_{,i} + \frac{1}{T_0}\theta_{,i}(s) \int_0^s q_{i,i}(z)dz
 \end{aligned}$$

With the help of the constitutive equation (2)₆, the equality (49) can be restated in the form

$$\begin{aligned}
 (50) \quad & [g(s) + h_{i,i}(s)]\sigma(s) - \varrho\eta(s)\theta(s) = -\varrho\kappa\dot{\sigma}^2(s) - \varrho\eta(0)\theta(s) \\
 & + \frac{d}{ds} \left[\varrho\kappa\sigma(s)\dot{\sigma}(s) + \frac{1}{2T_0}k_{ij} \left(\int_0^s \theta_{,i}(z)dz \right) \left(\int_0^s \theta_{,j}(z)dz \right) \right] \\
 & - \varrho \left[L(s)\sigma(s) + \frac{1}{T_0}\theta(s) \int_0^s S(z)dz \right] - \left[\frac{1}{T_0}\theta(s) \int_0^s q_i(z)dz \right]_{,i}
 \end{aligned}$$

By substituting the relations (46), (47) and (50) into equality (43), then it received the form

$$\begin{aligned}
 (51) \quad & \frac{d}{ds} \left[2\varrho u_i(s)\dot{u}_i(s) + 2I_{ij}\varphi_i(s)\dot{\varphi}_j(s) + 2\varrho\kappa\sigma(s)\dot{\sigma}(s) + \frac{1}{T_0}k_{ij} \left(\int_0^s \theta_{,i}(z)dz \right) \left(\int_0^s \theta_{,j}(z)dz \right) \right] \\
 & = 2\varrho\dot{u}_i(s)\dot{u}_i(s) + 2I_{ij}\dot{\varphi}_i(s)\dot{\varphi}_j(s) + 2\varrho\kappa\dot{\sigma}^2(s) - 2 [2\Psi(\mathbf{E}(s)) + a\theta^2(s)] \\
 & + 2\varrho \left[F_i(s)u_i(s) + M_i(s)\varphi_i(s) + L(s)\sigma(s) + \frac{1}{T_0}\theta(s) \int_0^s S(z)dz \right] \\
 & + 2 \left[t_{ji}(s)u_i(s) + m_{ji}(s)\varphi_i(s) + h_j(s)\sigma(s) + \frac{1}{T_0}\theta(s) \int_0^s q_j(z)dz \right]_{,j} \\
 & + 2\varrho\eta(0)\theta(s).
 \end{aligned}$$

Now, we integrate the equality (51) onto the cylinder $B \times [0, t]$ and apply the divergence. In this way we get to the desired identity (40) and the proof of Theorem 2 is complete. \square

Theorem 3. *For a solution $(u_i, \varphi_i, \sigma, \theta)$ of the mixed initial-boundary value problem consisting of the equations (11), the boundary conditions (10) and the initial conditions (9), we have the following identity:*

$$\begin{aligned}
& 2 \int_B \left[\varrho u_i(t) \dot{u}_i(t) + I_{ij} \varphi_i(t) \dot{\varphi}_j(t) + \varrho \kappa \sigma(t) \dot{\sigma}(t) + \frac{1}{T_0} k_{ij} \left(\int_0^t \theta_{,i}(s) ds \right) \left(\int_0^t \theta_{,j}(s) ds \right) \right] dV \\
&= \int_B \left\{ \varrho [u_i(0) \dot{u}_i(2t) + \dot{u}_i(0) u_i(2t)] + I_{ij} [\varphi_i(0) \dot{\varphi}_j(2t) + \dot{\varphi}_j(0) \varphi_i(2t)] \right\} dV \\
&+ \int_B \varrho \kappa [\sigma(0) \dot{\sigma}(2t) + \dot{\sigma}(0) \sigma(2t)] dV + \int_0^t \int_B \varrho \eta(0) [\theta(t-s) - \theta(t+s)] dV ds \\
&+ \int_0^t \int_B \varrho [u_i(t+s) F_i(t-s) - u_i(t-s) F_i(t+s)] dV ds \\
&+ \int_0^t \int_B I_{ij} [\varphi_i(t+s) M_i(t-s) - \varphi_i(t-s) M_i(t+s)] dV ds \\
(52) \quad &+ \int_0^t \int_B [\sigma(t+s) L(t-s) - \sigma(t-s) L(t+s)] dV ds \\
&+ \int_0^t \int_B \frac{1}{T_0} \left[\theta(t-s) \int_0^{t+s} S(z) dz - \theta(t+s) \int_0^{t-s} S(z) dz \right] dV ds \\
&+ \int_0^t \int_{\partial B} [u_i(t+s) t_i(t-s) - u_i(t-s) t_i(t+s)] dA ds \\
&+ \int_0^t \int_{\partial B} [\varphi_i(t+s) m_i(t-s) - \varphi_i(t-s) m_i(t+s)] dA ds \\
&+ \int_0^t \int_{\partial B} [\sigma(t+s) h(t-s) - \sigma(t-s) h(t+s)] dA ds \\
&+ \int_0^t \int_{\partial B} \frac{1}{T_0} \left[\theta(t-s) \int_0^{t+s} q(z) dz - \theta(t+s) \int_0^{t-s} q(z) dz \right] dA ds
\end{aligned}$$

Proof. It is easy to deduce the obvious identity

$$\begin{aligned}
& -\frac{d}{ds} \left\{ \varrho [u_i(t+s) \dot{u}_i(t-s) + \dot{u}_i(t+s) u_i(t-s)] \right\} \\
(53) \quad &= \varrho [u_i(t+s) \ddot{u}_i(t-s) - u_i(t-s) \ddot{u}_i(t+s)], \quad s \in [0, t], \quad t \in [0, \infty)
\end{aligned}$$

Taking into account the equations of motion $(4)_1$, the right side term of identity (53) received the form

$$\begin{aligned}
& \varrho [u_i(t+s) \ddot{u}_i(t-s) - u_i(t-s) \ddot{u}_i(t+s)] \\
(54) \quad &= \varrho [u_i(t+s) F_i(t-s) - u_i(t-s) F_i(t+s)] \\
&+ [u_i(t+s) t_{ji}(t-s) - u_i(t-s) t_{ji}(t+s)],_j
\end{aligned}$$

$$+ [u_{i,j}(t-s)t_{ji}(t+s) - u_{i,j}(t+s)t_{ji}(t-s)]$$

With the help of relation (54), the identity (53) can be restated in the form

$$(55) \quad \begin{aligned} & -\frac{d}{ds} \left\{ \varrho [u_i(t+s)\dot{u}_i(t-s) + \dot{u}_i(t+s)u_i(t-s)] \right\} \\ & = \varrho [u_i(t+s)F_i(t-s) - u_i(t-s)F_i(t+s)] \\ & \quad + [u_i(t+s)t_{ji}(t-s) - u_i(t-s)t_{ji}(t+s)],_j \\ & \quad + [u_{i,j}(t-s)t_{ji}(t+s) - u_{i,j}(t+s)t_{ji}(t-s)] \end{aligned}$$

It is easy to see that

$$(56) \quad \begin{aligned} & -\frac{d}{ds} \left\{ I_{ij} [\varphi_i(t+s)\dot{\varphi}_j(t-s) + \dot{\varphi}_i(t+s)\varphi_i(t-s)] \right\} \\ & = I_{ij} [\varphi_i(t+s)\ddot{\varphi}_i(t-s) - \varphi_i(t-s)\ddot{\varphi}_i(t+s)], \quad s \in [0, t], \quad t \in [0, \infty) \end{aligned}$$

If we take into account the equations of motion (4)₂, the right side term from equality (56) can be rewritten in the form

$$(57) \quad \begin{aligned} & I_{ij} [\varphi_i(t+s)\ddot{\varphi}_i(t-s) - \varphi_i(t-s)\ddot{\varphi}_i(t+s)] \\ & = \varrho [\varphi_i(t+s)M_i(t-s) - \varphi_i(t-s)M_i(t+s)] \\ & \quad + [\varphi_i(t+s)m_{ji}(t-s) - \varphi_i(t-s)m_{ji}(t+s)],_j \\ & \quad + [\varphi_{i,j}(t-s)m_{ji}(t+s) - \varphi_{i,j}(t+s)m_{ji}(t-s)] \\ & \quad + \varepsilon_{ijk} [\varphi_i(t+s)t_{jk}(t-s) - \varphi_i(t-s)t_{jk}(t+s)] \end{aligned}$$

introducing the relation (57) into the identity (56), we obtain

$$(58) \quad \begin{aligned} & -\frac{d}{ds} \left\{ I_{ij} [\varphi_i(t+s)\dot{\varphi}_j(t-s) + \dot{\varphi}_i(t+s)\varphi_i(t-s)] \right\} \\ & = \varrho [\varphi_i(t+s)M_i(t-s) - \varphi_i(t-s)M_i(t+s)] \\ & \quad + [\varphi_i(t+s)m_{ji}(t-s) - \varphi_i(t-s)m_{ji}(t+s)],_j \\ & \quad + [\varphi_{i,j}(t-s)m_{ji}(t+s) - \varphi_{i,j}(t+s)m_{ji}(t-s)] \\ & \quad + \varepsilon_{ijk} [\varphi_i(t+s)t_{jk}(t-s) - \varphi_i(t-s)t_{jk}(t+s)] \end{aligned}$$

If we add the relations (55) and (58), term by term, and use the geometric relations (3), we are led to

$$(59) \quad \begin{aligned} & -\frac{d}{ds} \left\{ \varrho [u_i(t+s)\dot{u}_i(t-s) + \dot{u}_i(t+s)u_i(t-s)] \right\} \\ & \quad - \frac{d}{ds} \left\{ I_{ij} [\varphi_i(t+s)\dot{\varphi}_j(t-s) + \dot{\varphi}_i(t+s)\varphi_i(t-s)] \right\} \\ & = \varrho [u_i(t+s)F_i(t-s) - u_i(t-s)F_i(t+s)] \\ & \quad + \varrho [\varphi_i(t+s)M_i(t-s) - \varphi_i(t-s)M_i(t+s)] \\ & \quad + [u_i(t+s)t_{ji}(t-s) - u_i(t-s)t_{ji}(t+s)],_j \\ & \quad + [\varphi_i(t+s)m_{ji}(t-s) - \varphi_i(t-s)m_{ji}(t+s)],_j \end{aligned}$$

$$\begin{aligned}
& + [t_{ij}(t+s)\varepsilon_{ij}(t-s) - t_{ij}(t-s)\varepsilon_{ij}(t+s)] \\
& + [m_{ij}(t+s)\gamma_{ij}(t-s) - m_{ij}(t-s)\gamma_{ij}(t+s)]
\end{aligned}$$

Now, we try to find another form for the last two parenthesis from equality (59). To this aim, first, we use the constitutive equations (2)₁-(2)₅ in order to obtain

$$\begin{aligned}
(60) \quad & [t_{ij}(t+s)\varepsilon_{ij}(t-s) - t_{ij}(t-s)\varepsilon_{ij}(t+s)] \\
& + [m_{ij}(t+s)\gamma_{ij}(t-s) - m_{ij}(t-s)\gamma_{ij}(t+s)] \\
& = [\sigma(t-s)g(t+s) - \sigma(t+s)g(t-s)] \\
& + [h_i(t-s)\phi(t+s) - h_i(t+s)\phi(t-s)] \\
& + \varrho [\theta(t-s)\eta(t+s) - \theta(t+s)\eta(t-s)]
\end{aligned}$$

Considering the geometric equations (3) and the balance of the equilibrated forces (5), we are lead to

$$\begin{aligned}
(61) \quad & h_i(t-s)\phi(t+s) - h_i(t+s)\phi(t-s) \\
& = [h_i(t-s)\sigma(t+s) - h_i(t+s)\sigma(t-s)],_i \\
& + [\sigma(t+s)g(t-s) - \sigma(t-s)g(t+s)] \\
& + \varrho [\sigma(t+s)L(t-s) - \sigma(t-s)L(t+s)] \\
& + \varrho\kappa [\sigma(t-s)\check{\sigma}(t+s) - \sigma(t+s)\check{\sigma}(t-s)]
\end{aligned}$$

Analogous, if we take into account the equation of energy (6) we obtain a similar identity

$$\begin{aligned}
(62) \quad & \varrho [\theta(t-s)\eta(t+s) - \theta(t+s)\eta(t-s)] = \varrho\eta(0) [\theta(t-s) - \theta(t+s)] \\
& + \frac{\varrho}{T_0} \left[\theta(t-s) \int_0^{t+s} S(z)dz - \theta(t+s) \int_0^{t-s} S(z)dz \right] \\
& + \frac{1}{T_0} \left[\theta(t-s) \int_0^{t+s} q_i(z)dz - \theta(t+s) \int_0^{t-s} q_i(z)dz \right],_i \\
& + \frac{1}{T_0} k_{ij} \left[\theta_{,i}(t+s) \int_0^{t-s} \theta_{,j}(z)dz - \theta_{,i}(t-s) \int_0^{t+s} \theta_{,i}(z)dz \right]
\end{aligned}$$

We introduce the results from equalities (62) and (61) into (60) and then the resulting equality is introduced in (59). As a consequence, we obtain

$$\begin{aligned}
& - \frac{d}{ds} \left\{ \varrho [u_i(t+s)\dot{u}_i(t-s) + \dot{u}_i(t+s)u_i(t-s)] \right\} \\
& - \frac{d}{ds} \left\{ I_{ij} [\varphi_i(t+s)\dot{\varphi}_j(t-s) + \dot{\varphi}_i(t+s)\varphi_j(t-s)] \right\} \\
& - \frac{d}{ds} \left\{ \varrho\kappa [\sigma(t-s)\dot{\sigma}(t+s) + \sigma(t+s)\dot{\sigma}(t-s)] \right\} \\
& - \frac{d}{ds} \left[\frac{1}{T_0} k_{ij} \left(\int_0^{t+s} \theta_{,i}(z)dz \right) \left(\int_0^{t-s} \theta_{,j}(z)dz \right) \right]
\end{aligned}$$

$$\begin{aligned}
 &= \varrho [u_i(t+s)F_i(t-s) - u_i(t-s)F_i(t+s)] \\
 (63) \quad &+ \varrho [\varphi_i(t+s)M_i(t-s) - \varphi_i(t-s)M_i(t+s)] \\
 &+ \varrho [\sigma(t+s)L(t-s) - \sigma(t-s)L(t+s)] \\
 &+ \frac{\varrho}{T_0} \left[\theta(t-s) \int_0^{t+s} S(z)dz - \theta(t+s) \int_0^{t-s} S(z)dz \right] \\
 &+ \varrho \eta(0) [\theta(t-s) - \theta(t+s)] \\
 &+ [u_i(t+s)t_{ji}(t-s) - u_i(t-s)t_{ji}(t+s)]_{,j} \\
 &+ [\varphi_i(t+s)m_{ji}(t-s) - \varphi_i(t-s)m_{ji}(t+s)]_{,j} \\
 &+ [h_j(t-s)\sigma(t+s) - h_j(t+s)\sigma(t-s)]_{,j} \\
 &+ \frac{1}{T_0} \left[\theta(t-s) \int_0^{t+s} q_j(z)dz - \theta(t+s) \int_0^{t-s} q_j(z)dz \right]_{,j}
 \end{aligned}$$

To get the desired result, that is, the identity (52) we must just to integrate the equality (63) over cylinder $B \times [0, t]$ and use the divergence theorem. Thus the proof of Theorem 3 is complete. \square

4. Behaviour of solutions

We start this section with some auxiliary results. These will be used to prove the main results of our study, that is, the spatial behavior of solutions of the problem \mathcal{P} , defined at the end of Section 2.

We will assume that the boundary of B , denoted by ∂B , is a sufficiently smooth surface to enable us the application of divergence theorem. Also, we denote the closure of B by \bar{B} .

For fixed $T > 0$ define the space Ω_T of all $\mathbf{x} \in \bar{B}$ such that

1. If $\mathbf{x} \in B$, then

$$\begin{aligned}
 &u_i^0(\mathbf{x}) \neq 0 \text{ or } u_i^1(\mathbf{x}) \neq 0 \text{ or } \varphi_i^0(\mathbf{x}) \neq 0 \text{ or } \varphi_i^1(\mathbf{x}) \neq 0 \text{ or} \\
 (64) \quad &\sigma^0(\mathbf{x}) \neq 0 \text{ or } \sigma^1(\mathbf{x}) \neq 0 \text{ or } \theta^0(\mathbf{x}) \neq 0 \text{ or } \eta^0(\mathbf{x}) \neq 0 \text{ or}
 \end{aligned}$$

$$(65) \quad F_i(\mathbf{x}, t) \neq 0 \text{ or } M_i(\mathbf{x}, t) \neq 0 \text{ or } L(\mathbf{x}, t) \neq 0 \text{ or } S(\mathbf{x}, t) \neq 0, \quad t \in [0, T]$$

2. If $\mathbf{x} \in \partial B$, then

$$\begin{aligned}
 &\bar{u}_i(\mathbf{x}, t) \neq 0 \text{ or } \bar{t}_i(\mathbf{x}, t) \neq 0 \text{ or } \bar{\varphi}_i(\mathbf{x}, t) \neq 0 \text{ or } \bar{m}_i(\mathbf{x}, t) \neq 0 \text{ or} \\
 (66) \quad &\bar{\sigma}(\mathbf{x}, t) \neq 0 \text{ or } \bar{h}(\mathbf{x}, t) \neq 0 \text{ or } \bar{\theta}(\mathbf{x}, t) \neq 0 \text{ or } \bar{\eta}(\mathbf{x}, t) \neq 0, \quad t \in [0, T]
 \end{aligned}$$

If we analyze the above relations, we find that the space Ω_T is the support of the initial and boundary data and the body supplies in the problem \mathcal{P} on the interval $[0, T]$.

Now, we define the set Ω_r by

$$(67) \quad \Omega_r = \left\{ \bar{x} \in \bar{B} : \Omega_r^* \cap \bar{S}(\mathbf{x}, t) \neq \emptyset \right\}$$

where r is a non-negative number, $r \geq 0$. Also, we have denoted by Ω_r^* the smallest regular surface of ∂B that include Ω_r . In the case that Ω_r is an empty set, then Ω_r^* is an arbitrary nonempty regular subsurface of ∂B . Also, in (67) $\bar{S}(\mathbf{x}, t)$ represents, as usual, the notation of the closure of the ball with radius r and center at \mathbf{x} . Other two new notations will be used in what follows. First, we note by B_r the part of B such that $B_r = B \setminus D_r$ and for $r_1 > r_2$ we set $B(r_1, r_2) = B_{r_2} \setminus B_{r_1}$. The second notation is S_r and it stand for the subsurface of ∂B_r contained inside of B and whose outward unit normal vector is forwarded to the exterior of D_r . For a solution $(u_i, \varphi_i, \sigma, \theta)$ of the problem \mathcal{P} we will associate time-weighted surface power function, defined as follows

$$(68) \quad I(r, t) = - \int_0^t \int_{S_r} e^{-\lambda s} \left[t_i(s) \dot{u}_i(s) + m_i(s) \dot{\varphi}_i(s) + h(s) \dot{\sigma}(s) + \frac{1}{T_0} q(s) \theta(s) \right] dA ds$$

The function $I(r, t)$ is defined for any $r \geq 0$ and $t \in [0, T]$. Also, by λ we denoted a positive parameter which is given and the functions $t_i(s)$, $m_i(s)$, $h(s)$ and $q(s)$ are defined in (36).

Also, we will denote by J the integral of function I , that is

$$(69) \quad J(r, t) = \int_0^r I(r, s) ds, \quad r \geq 0, t \in [0, T]$$

The main properties of the time-weighted surface power function I , defined in (68), are formulated and demonstrated in the following theorem.

Theorem 4. *For each $r \geq 0$ and $t \in [0, T]$, the time-weighted surface power function $I(r, t)$, associated with the solution $(u_i, \varphi_i, \sigma, \theta)$ of problem \mathcal{P} , has the following properties*

i). If $0 \leq r_2 \leq r_1$, then

$$(70) \quad \begin{aligned} & I(r_1, t) - I(r_2, t) \\ &= \int_{B(r_1, r_2)} e^{-\lambda t} \left\{ \frac{1}{2} [\varrho \dot{u}_i(t) \dot{u}_i(t) + I_{ij} \dot{\varphi}_i(t) \dot{\varphi}_j(t) + \varrho \kappa \dot{\sigma}^2(t)] + \Psi(\mathbf{E}(t)) + \frac{1}{2} a \theta^2(t) \right\} dV \\ & - \int_0^t \int_{B(r_1, r_2)} e^{-\lambda s} \left\{ \frac{1}{2} [\varrho \dot{u}_i(s) \dot{u}_i(s) + I_{ij} \dot{\varphi}_i(s) \dot{\varphi}_j(s) + \varrho \kappa \dot{\sigma}^2(s)] \right\} dV ds \\ & - \int_0^t \int_{B(r_1, r_2)} e^{-\lambda s} \left\{ \lambda \Psi(\mathbf{E}(s)) + \frac{\lambda}{2} a \theta^2(s) + \frac{1}{T_0} k_{ij} \theta_{,i}(s) \theta_{,j}(s) \right\} dV ds \end{aligned}$$

ii). The function $I(r, t)$ is continuous differentiable with respect to r and its derivative has the expression

$$\begin{aligned} \frac{\partial I}{\partial r}(r, t) &= \int_{S_r} e^{-\lambda t} \left\{ \frac{1}{2} [\varrho \dot{u}_i(t) \dot{u}_i(t) + I_{ij} \dot{\varphi}_i(t) \dot{\varphi}_j(t) + \varrho \kappa \dot{\sigma}^2(t)] + \Psi(\mathbf{E}(t)) + \frac{1}{2} a \theta^2(t) \right\} dA \\ (71) \quad &- \int_0^t \int_{S_r} e^{-\lambda s} \left\{ \frac{1}{2} [\varrho \dot{u}_i(s) \dot{u}_i(s) + I_{ij} \dot{\varphi}_i(s) \dot{\varphi}_j(s) + \varrho \kappa \dot{\sigma}^2(s)] \right\} dA ds \\ &- \int_0^t \int_{S_r} e^{-\lambda s} \left\{ \lambda \Psi(\mathbf{E}(s)) + \frac{\lambda}{2} a \theta^2(s) + \frac{1}{T_0} k_{ij} \theta_{,i}(s) \theta_{,j}(s) \right\} dA ds \end{aligned}$$

iii). The function $I(r, t)$ is non-increasing with respect to r .

iv). The function $I(r, t)$ satisfies a first order differential inequality, for each $r \geq 0$, namely

$$(72) \quad \frac{\lambda}{c} |I(r, t)| + \frac{\partial I}{\partial r}(r, t) \leq 0,$$

where we denoted by c the constant defined by

$$(73) \quad c = \sqrt{\frac{(1 + \varepsilon_0) \mu_M}{\varrho_0}}$$

Also, ε_0 represents the positive root of the following second order algebraic equation

$$(74) \quad x^2 + x \left(1 - \frac{M^2}{a_0 \mu_M} - \frac{\lambda \varrho_0 k_M}{2 a_0 T_0 \mu_M} \right) - \frac{M^2}{a_0 \mu_M} = 0$$

v). For each $r \geq 0$ and $t \in [0, T]$, $I(r, t)$ is a positive function.

Proof. We will put $B(r_1, r_2)$ instead of B in Theorem 1, for $r_1 \geq r_2 \geq 0$. Considering the definitions of $B(r_1, r_2)$ and $I(r, t)$, with the help of equality (38) from Theorem 1, we are led to i). Taking into account the assumptions (9) and (10), from the identity (70) we deduce the assertion ii). Also, if we take into account inequalities (13), from the identity (70), we deduce point iii). Now, we wish to prove the assertion iv). To this aim we will apply the Schwarz's inequality and the arithmetic-geometric mean inequality in (68) such that we are led to

$$\begin{aligned} |I(r, t)| &\leq \int_0^t \int_{S_r} e^{-\lambda s} \left\{ \frac{\varepsilon_1}{2 \varrho_0} [t_{ij}(s) t_{ij}(s) + m_{ij}(s) m_{ij}(s) + h_i(s) h_i(s)] \right. \\ &\quad \left. + \frac{1}{2 \varepsilon_1} [\varrho \dot{u}_i(s) \dot{u}_i(s) + I_{ij} \dot{\varphi}_i(s) \dot{\varphi}_i(s) + \varrho \kappa \dot{\sigma}^2] + \frac{\varepsilon_2}{2 T_0 a_0} q_i(s) q_i(s) + \frac{1}{2 T_0 \varepsilon_2} a \theta^2(s) \right\} dA ds \\ (75) \quad &\leq \int_0^t \int_{S_r} e^{-\lambda s} \left\{ \frac{1}{\lambda \varepsilon_1} \cdot \frac{\lambda}{2} [\varrho \dot{u}_i(s) \dot{u}_i(s) + I_{ij} \dot{\varphi}_i(s) \dot{\varphi}_i(s) + \varrho \kappa \dot{\sigma}^2(s)] \right. \\ &\quad \left. + \frac{\varepsilon_1 (1 + \varepsilon) \mu_M}{\lambda \varrho_0} \cdot \lambda \Psi(\mathbf{E}(s)) + \left[\frac{\varepsilon_1 M^2}{\lambda a_0 \varrho_0} \left(\varepsilon + \frac{1}{\varepsilon} \right) + \frac{1}{\lambda T_0 \varepsilon_2} \right] \cdot \frac{\lambda}{2} a \theta^2(s) \right\} \end{aligned}$$

$$\left. + \frac{\varepsilon_2 k_M}{2a_0} \cdot \frac{1}{T_0} k_{ij} \theta_{,i}(s) \theta_{,j}(s) \right\} dAds, \quad r \geq 0, \quad 0 \leq t \leq T, \quad \forall \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$$

In the last integral from (75) we equate the coefficients of energetic terms such that we are lead to

$$(76) \quad \frac{1}{\lambda \varepsilon_1} = \frac{\varepsilon_1 (1 + \varepsilon) \mu_M}{\lambda \varrho_0} = \frac{\varepsilon_1 M^2}{\lambda a_0 \varrho_0} \left(\varepsilon + \frac{1}{\varepsilon} \right) + \frac{1}{\lambda T_0 \varepsilon_2} = \frac{\varepsilon_2 k_M}{2a_0}$$

and from these equalities we can set

$$(77) \quad \varepsilon_1 = \frac{1}{c}, \quad \varepsilon_2 = \frac{2a_0 c}{\lambda k_M}.$$

The constant c has the expression defined in (73).

Now, considering the relations (71) and (75) together, we obtain the relation (72). Finally, we will use the definition (67) of the set Ω_T and the definition (68) of the time-weighted surface power function $I(r, t)$, such that considering the result of point iii), the property v) is obtained. \square

In the following corollary, we state and prove a first-order differential inequality satisfied by the function $J(r, t)$ defined in (69).

Corollary. *The main property of the function $J(r, t)$ is the following*

$$(78) \quad t\gamma(t) \frac{\partial J}{\partial r}(r, t) + \left| J(r, t) \right| \leq 0, \quad r \geq 0, \quad 0 \leq t \leq T,$$

where the function $\gamma(t)$ is defined by

$$(79) \quad \gamma(t) = \sqrt{\frac{(1 + \delta_0(t)) \mu_M}{\varrho_0}}$$

We denoted by $\delta_0(t)$ the positive root of the second order algebraic equation

$$(80) \quad x^2 + x \left(1 - \frac{M^2}{a_0 \mu_M} - \frac{\varrho_0 k_M}{2ta_0 T_0 \mu_M} \right) - \frac{M^2}{a_0 \mu_M} = 0.$$

Proof. Inequality that follows is evident

$$(81) \quad \int_0^t \int_0^s f^2(z) dz ds \leq t \int_0^s f^2(z) dz$$

Using a procedure that is analogous to that used in deduction of point iv) from Theorem 4, with the help of inequality (81) we obtain the inequality (78). \square

Now we are able to formulate and to demonstrate the main result regarding the spatial behavior of solutions of the problem \mathcal{P} in the case of a bounded domain B . This behavior will be evaluated with the help of functions $I(r, t)$ and $J(r, t)$.

Theorem 5. *Let us consider the problem \mathcal{P} which is defined on the bounded domain B and the time-weighted surface power function $I(r, t)$ associated with the solution of problem \mathcal{P} . We suppose that the initial and boundary data and the body supplies have*

the support Ω_T on the interval $[0, T]$. Then the solution of problem \mathcal{P} decays spatially with regard to functions $I(r, t)$ and $J(r, t)$, for each $t \in [0, T]$, that is,

$$(82) \quad I(r, t) \leq I(0, t)e^{-\lambda r/c}, \quad 0 \leq r \leq D$$

$$(83) \quad J(r, t) \leq J(0, t)e^{-r/(t\gamma(t))}, \quad 0 \leq r \leq D.$$

We denoted by D the diameter of the domain $B \setminus \Omega_T^*$.

Proof. According to point v) of Theorem 4, $I(r, t)$ is a positive function. Considering the definition of function $J(r, t)$, we can write the differential inequalities satisfied by the functions $I(r, t)$ and, respectively, $J(r, t)$, in the form

$$(84) \quad \frac{\partial}{\partial r} [e^{\lambda r/c} I(r, t)] \leq 0, \quad 0 \leq r \leq D$$

$$(85) \quad \frac{\partial}{\partial r} [e^{r/(t\gamma(t))} J(r, t)] \leq 0, \quad 0 \leq r \leq D$$

From inequality (84), integrating with respect to r , we get estimate (82). Also, integrating inequality (85) with respect to r , we obtain the estimate (83) and this ends the proof of Theorem 5. □

Finally, we study the spatial behavior of solution of problem \mathcal{P} in the case that the micropolar thermoelastic medium occupies an unbounded domain. To this aim will be useful some estimates of Phragmén-Lindelöf type, [1].

Theorem 6. Consider the problem \mathcal{P} which is defined on the unbounded domain B and the time-weighted surface power function $I(r, t)$ associated with the solution $(u_i, \varphi_i, \sigma, \theta)$. The initial and boundary data and the body supplies have the support Ω_T on the interval $[0, T]$. The the solution of problem \mathcal{P} decays spatially with regard to functions $I(r, t)$ and $J(r, t)$, for each fixed $t \in [0, T]$, according to one of the following situations:

1. If $I(r, t) \geq 0$ for all $r \geq 0$, then

$$(86) \quad I(r, t) \leq I(0, t)e^{-\lambda r/c}, \quad r \geq 0$$

$$(87) \quad J(r, t) \leq J(0, t)e^{-r/(t\gamma(t))}, \quad r \geq 0$$

2. Assume that $\exists r_1 \geq 0$ such that $I(r_1, t) < 0$. Then we have $I(r, t) \leq I(r_1, t) < 0$ and $J(r, t) < 0$, for all $r \geq r_1$. Also, the following estimations are true

$$(88) \quad -I(r, t) \geq -I(r_1, t)e^{\lambda(r-r_1)/c}, \quad r \geq r_1$$

$$(89) \quad -J(r, t) \geq -J(r_1, t)e^{(r-r_1)/c}, \quad 0 \leq r \geq r_1$$

Proof. Because $I(r, t)$ is a non-increasing function with respect to r , we can use point iii) from Theorem 4 in order to deduce that $I(r, t) \geq 0$, for all $r \geq 0$. As a consequence, the differential inequality (72), satisfied by function $I(r, t)$, can be written in the form (84). As such we obtain estimate (86). Analogously, the

differential inequality (78), satisfied by function $J(r, t)$, can be written in the form (85) from where we obtain estimate (87).

Suppose now that we are in the situation 2. of Theorem, that is, there exists $r_1 \geq 0$ such that $I(r, t) \leq 0$, then from point iii) of Theorem 4 we deduce that $I(r, t) < I(r_1, t) \leq 0$ for all $r \geq r_1$. As a consequence, the differential inequality (72) receives the form

$$(90) \quad \frac{\partial}{\partial r} [e^{-\lambda r/c} I(r, t)] \leq 0, \quad r \leq r_1.$$

Therefore, if we integrate with respect to r , we obtain the estimate (88). On the other hand, because $I(r, t) \leq 0$ we deduce $J(r, t) \leq 0$, taking into account the definition (69) of the function $J(r, t)$. Hence, inequality (78) acquires the form

$$\frac{\partial}{\partial r} [e^{-\lambda r/(t\gamma(t))} J(r, t)] \leq 0, \quad r \leq r_1$$

from where, if we integrate with respect to r , we obtain the desired estimate (89). Thus, the proof of Theorem 6 is complete. \square

5. Conclusions

If we look back previous relations, it is not difficult to find that the evaluations of the form (82), (86) and (88) are suitable for appropriate short values of time, while evaluations of the form (83), (87) and (89) are suitable for appropriate large values of time. As such, we coupled the demonstrations of the the previous inequalities, as follows: (82) coupled with (83), (86) coupled with (87) and (88) coupled with (89). This coupling allows a complete description for the spatial behavior of the solutions of problem \mathcal{P} .

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