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SINGULAR SEMI-POSITONE ϕ-LAPLACIAN BVP ON INFINITE INTERVAL IN BANACH SPACES

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ABSTRACT. In this work, we have obtained new existence results of unbounded positive solutions for a second-order ϕ-Laplacian equation subject to nonlinear integral boundary conditions of Riemann-Stieltjes type and posed on the positive half-line. The index fixed point theory on cones of Banach spaces for countably strict set-contractions has been employed. The nonlinearity depends on the solution and its derivative, may change sign, and has time and space singularities in its arguments. It further takes values in a general Banach space and is assumed to have quite general growth conditions. We have illustrated our theoretical results with two examples of application in a finite and in an infinite dimensional space, respectively.

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1. Introduction

Let E be a real Banach space with zero element θ and $\mathcal{P}$ a cone in E which induces a partial ordering $\leq$ in E defined by $x \leq y$ if and only if $y - x \in \mathcal{P}$. If $x \leq y$ and $x \neq y$, we write $x < y$. Let $\mathcal{P}_+ = \mathcal{P}\backslash\{\theta\}$; so $x \in \mathcal{P}_+$ if and only if $\theta < x$. $\mathcal{P}$ is called a normal cone if there exists a constant $N > 0$ such that $\theta \leq x \leq y$ implies that $\|x\| \leq N\|y\|$ for all $x, y \in E$; clearly $N \geq 1$. More details on cone theory in Banach spaces may be found in [6, 12]. In this paper, we are interested in the questions of existence and nonexistence of positive solutions to the following second-order ϕ-Laplacian boundary value problem with integral conditions of Riemann-Stieltjes type and posed on the positive half-line:

$$(1.1)\qquad \begin{cases} -(\phi(y' - y_\infty))'(t) = m(t)f(t, y(t), y'(t)), \quad t > 0 \\ y(0) - \int_0^{+\infty} \mu(t)y'(t)\, d\xi(t) = y_0, \quad \lim\limits_{t\to+\infty} y'(t) = y_\infty, \end{cases}$$

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where $y_0 \in \mathcal{P}$ and $y_\infty \in \mathcal{P}_+$. The case $E = \mathbb{R}$ was studied in [8, 9]. The term $-(\phi(y' - y_\infty))'$ is consistent with the limit condition at positive infinity and will be removed following a shift of the solution. ξ is a nondecreasing function of bounded variation with real values and satisfies $\int_0^{+\infty} \mu(t)\, d\xi(t) > 0$; $\int_0^{+\infty} \mu(t) y'(t)\, d\xi(t)$ denotes the Riemann-Stieltjes abstract integral of y' with respect to ξ and μ is a weighted real function. The coefficient $m \in C(I, \mathbb{R}^+) \cap L^1(I, \mathbb{R}^+)$ may be singular at $t = 0$. The nonlinearity $f := f(t, y, z) \in C(\mathbb{R}^+ \times \mathcal{P}_+ \times E \setminus \{\theta\}, E)$ satisfies a general growth condition and is allowed to have space singularities at $y = \theta$ and/or at $z = \theta$. By space singularity at the origin, we mean that $\|f(t, y, z)\| \to +\infty$, as either $y \to \theta$ or $z \to \theta$. Here and hereafter $I = (0, +\infty)$ and $\mathbb{R}^+ = [0, +\infty)$. The nonlinear derivation operator is represented by an increasing homeomorphism $\phi\colon E \to E$ satisfying $\phi(\theta) = \theta$ and such that ϕ is expansive, i.e.,

$$\|x - y\|_E \le \|\phi(x) - \phi(y)\|_E, \quad \forall\, x, y \in E. \tag{1.2}$$

Example 1.1. (a) In case $E = \mathbb{R}$ equipped with standard norm, if we consider a k-contraction $(0 < k < 1)$ ψ with $\psi(0) = 0$, then $\phi = \frac{I_d - \psi}{1-k}$ satisfies the previous conditions, where I_d is the identity operator. Indeed, it is easily checked that

$$[\phi(x) - \phi(y)](x - y) \ge (x - y)^2$$

and thus ϕ is expansive.

(b) On the Banach space $E = C([0, 1])$ equipped with the sup-norm, define the function $\phi(f) = \lambda f$ with some $\lambda \ge 1$. Then ϕ is an increasing homomorphism which satisfies $\phi(0) = 0$ and

$$\|\phi(f) - \phi(g)| = \lambda \|f - g\| \ge \|f - g\|,$$

for all $f, g \in E$.

The theory of ordinary differential equations in Banach spaces is a wide and important branch of nonlinear functional analysis which has developed during the last couple of years starting from the pioneer works of Guo *et al.* [13], Guo and Lakshmikantham [12], Lakshmikantham and Leela [17], Deimling [5], etc. More recently, this theory has gained particular interest because of extensive applications in many problems arising in applied mathematics (see, e.g., [10, 14, 16, 17, 18, 19, 21, 23, 25, 26, 27, 15] and references therein). In 2009, the authors studied the boundary value problem

$$\begin{cases} -(\phi(y'))'(t) = m(t) f(t, y(t), y'(t)), & t \in I \\ y(0) = \alpha y'(\eta), \quad \lim\limits_{t \to +\infty} y'(t) = 0, \end{cases} \tag{1.3}$$

where $\alpha \ge 0$ and $\eta \in (0, \infty)$ are given real numbers. They obtained the existence of multiple positive solutions using index fixed point theory (see [7]). In 2012, by using the method of upper and lower solutions and the topological degree theory of strict-set-contractions, Y. Zhao *et al.* [28] studied the existence of multiple solutions for the following three-point second-order boundary value problem on the unbounded domain $[0, +\infty)$ in a Banach space E:

$$\begin{cases} y''(t) + q(t)f(t, y(t), y'(t)) = \theta, \ t > 0, \\ y(0) - \alpha y'(0) - \beta y(\eta) = y_0, \quad \lim_{t \to +\infty} y'(t) = y_\infty, \end{cases} \tag{1.4}$$

where $\alpha, \beta \geq 0, \ \eta > 0, \ y_0, y_\infty \in E, \ q \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $f \in C(\mathbb{R}^+ \times E \times E, E)$. Finally the case where $E = \mathbb{R}$ and $\int_0^{+\infty} y'(t)\, d\xi(t) = \sum_{i=1}^n k_i y'(\xi_i)$ with $k_i \geq 0$ and $0 < \xi_1 < \xi_2 < \cdots < \xi_n < \infty$ is considered in [8] where existence of positive solutions is obtained in a weighted Banach functional space.

The main feature of this work is to prove the existence and the nonexistence of positive solutions for the more general semi-positone problem (1.1), where the ϕ-Laplacian derivation operator extends the second-order differential operator y'' and the nonlinear term f takes values in an abstract Banach space and may change sign. Clearly, the Riemann-Stieltjes integral boundary conditions includes multi-point boundary conditions as special cases. Under general growth conditions on the nonlinearity, existence of solutions is obtained in a positive cone. Throughout this work, by a positive solution we mean a function $y \in C^1(\mathbb{R}^+, E)$ with $y(t) > \theta$ on I, $\phi(y) \in C^1(I, E)$ and y satisfies (1.1).

The organization of this paper is as follows. In Section 2, we present some preliminaries and establish several auxiliary lemmas. The main results are formulated and proved in Section 3. More precisely, two results of existence and nonexistence are proved respectively under suitable conditions on the nonlinearity f. Two examples in a finite and in an infinite dimensional spaces respectively are worked out in Section 4 to illustrate the main existence theorem.

2. **Preliminaries**

2.1. **General framework.** An operator $A : D \subset E \longrightarrow E$ in said to be nondecreasing if $x_1 \leq x_2$ $(x_1, \, x_2 \in D)$ implies $Ax_1 \leq Ax_2$.

The following lemma is a consequence of the Hahn-Banach theorem.

Lemma 2.1 ([16])**.** *Let E be a Banach space and $x_0 \neq \theta$ a vector of E. Then there exists a functional $\psi \in E'$ such that $\psi(x_0) = \|x_0\|_E$ and $\|\psi\| = 1$, where E' is the topological dual of E.*

Definition 2.2 (Kuratowski measure of noncompactness [2, 3])**.** Let E be a real Banach space and Ω_E be the class of all bounded subsets of E. The Kuratowski measure of noncompactness $\alpha : \Omega_E \longrightarrow [0, +\infty)$ is defined by

$$\alpha(V) = \inf \left\{ \delta > 0 \mid \; V = \bigcup_{i=1}^{n} V_i \;\text{ and }\; \mathrm{diam}\,(\mathrm{V_i}) \leq \delta, \; \forall\, \mathrm{i} = 1, \dots, \mathrm{n} \right\},$$

where $\mathrm{diam}\,(\mathrm{V_i}) = \sup\{\|\mathrm{x} - \mathrm{y}\|_{\mathrm{E}}, \; \mathrm{x}, \mathrm{y} \in \mathrm{V_i}\}$ is the diameter of V_i.

Let $J = [0, \lambda]$ be a compact interval. The Kuratowski measures of noncompactness of a bounded set in the spaces E, $C(J, E)$, and X are denoted by $\alpha_E(.)$, $\alpha_C(.)$, and $\alpha_X(.)$ respectively.

Definition 2.3 (Countably strict-set contraction operator [22])**.** Let $A\colon\; D \subset E \to E$ be a continuous and bounded operator. If there exists a constant $k \geq 0$ such that $\alpha(A(S)) \leq k\alpha(S)$, for every countably bounded subset $S \subset D$, then A is called a countably k-set contraction operator. If $k < 1$, A is called a countably strict-set contraction operator.

Definition 2.4. A set of functions $H \subset C(J, E)$ is said to be almost equi-continuous on $\mathbb{R}^+$ if it is equi-continuous on each compact sub-interval of $\mathbb{R}^+$.

Lemma 2.5 ([3, 6])**.** *If $H \subset C(J, E)$ is bounded and equi-continuous, then $\alpha(H(\cdot))$ is continuous on J and $\alpha_C(H) = \sup_{t\in J} \alpha_E(H(t))$, where $H(t) = \{u(t) \mid u \in H\}$, $t \in J$.*

Lemma 2.6 ([20])**.** *Let E be a Banach space and $H \subset C(\mathbb{R}^+, E)$. If H is countable and there exists some $\rho \in L^1(\mathbb{R}^+, \mathbb{R}^+)$ such that $\|u\| \leq \rho(t)$, for all $t \in \mathbb{R}^+$ and $u \in H$, then $\alpha_E(\{u(t) \mid u \in H\})$ is integrable on $[0, +\infty)$, and*

$$\alpha_E \left(\left\{ \int_0^{+\infty} u(t)\, dt, \; u \in H \right\} \right) \leq 2 \int_0^{+\infty} \alpha_E \left(\{u(t) \mid u \in H\} \right) dt.$$

Lemma 2.7 (Corollary 3.1.19, [11])**.** *Let C be a nonempty bounded closed convex set in a Banach space X and let $A : C \longrightarrow C$ be a countably strict-set contraction. Then A has a fixed point in C.*

2.2. **Integral formulation.**

Definition 2.8. A function $\xi : J \to \mathbb{R}$ is said to be of bounded variation if there exists a constant M such that $\sum_{k=1}^{n} |\xi(t_k) - \xi(t_{k-1})| \leq M$, for every partition $\{t_0, t_1, \dots, t_n\}$ of J.

Definition 2.9. Let y and ξ be real-valued functions, $\{t_0, t_1, \dots, t_n\}$ a partition of $[0, \lambda]$, and $c_k \in [t_{k-1}, t_k]$, for $k = 1, 2, \dots, n$. The Riemann-Stieltjes integral of y, with respect to ξ, is defined as

$$\int_0^{\lambda} y(t)\, d\xi(t) = \lim_{\delta \to 0} \sum_{k=1}^{n} y(c_k) \left[\xi(t_k) - \xi(t_{k-1}) \right], \; \text{ where } \delta = \max_{1\leq k\leq n} |t_k - t_{k-1}|.$$

When $\xi(t) = t$, we recover the classical Riemann integral. For more details on Riemann-Stieltjes integration, we recommend [1, 24].

Proposition 2.10. *Let $y \in C(J, \mathbb{R})$ and ξ be a function of bounded variation on J. We have*

1. *If $y(t) \geq 0$ and ξ is nondecreasing function on J, then $\int_0^\lambda y(t)\, d\xi(t) \geq 0$.*
2. *If ξ is nondecreasing function on J, then $|\int_0^\lambda y(t)\, d\xi(t)| \leq \int_0^\lambda |y(t)|\, d\xi(t)$.*

The Riemann-Stieltjes integral can be generalized to the case when either the integrand y or the integrator ξ takes values in a Banach space E. Given a continuous function $y(\cdot) : I \longrightarrow E$ and $\xi(\cdot) : I \longrightarrow \mathbb{R}$, if $\lim_{\varepsilon\to 0^+} \int_\varepsilon^1 y(t)\, d\xi(t)$ and $\lim_{\lambda\to+\infty} \int_1^\lambda y(t)\, d\xi(t)$ exist, then we say that the abstract generalized integral $\int_0^{+\infty} y(t)\, d\xi(t)$ converges; otherwise this abstract integral diverges.

The following lemma allows us to consider only homogenous boundary conditions. The proof is immediate.

Lemma 2.11. *$y \in C^1(\mathbb{R}^+, \mathcal{P})$ is a solution of problem (1.1) if and only if $x = y - \bar{y} \in C^1(\mathbb{R}^+, \mathcal{P})$ is a solution of the problem*

$$\begin{cases} -(\phi(x'))'(t) = m(t)g(t, x(t), x'(t)), \quad t \in I \\ x(0) - \int_0^{+\infty} \mu(t)x'(t)\, d\xi(t) = \theta, \quad \lim\limits_{t\to+\infty} x'(t) = \theta, \end{cases} \tag{2.1}$$

where

$$\begin{cases} g(t,u,v) &= f(t, u + \bar{y}(t), v + y_\infty) \\ \text{and } \ \bar{y}(t) &= y_\infty(t + \int_0^{+\infty} \mu(t)d\xi(t)) + y_0. \end{cases}$$

In order to transform problem (2.1) into a fixed point problem, the following auxiliary lemma is needed. The proof is also omitted.

Lemma 2.12. *Let $v \in C(\mathbb{R}^+, \mathcal{P})$ be such that $\int_0^{+\infty} v(t)\, dt$ exists. Then $x \in C^1([0, +\infty), \mathcal{P})$ is a solution of*

$$\begin{cases} -(\phi(x'))'(t) = v(t), \quad t \in I \\ x(0) - \int_0^{+\infty} \mu(t)x'(t)\, d\xi(t) = \theta, \quad \lim\limits_{t\to+\infty} x'(t) = \theta, \end{cases} \tag{2.2}$$

if and only if

$$x(t) = \int_0^{+\infty} \mu(s)\phi^{-1}\left(\int_s^{+\infty} v(\tau)\, d\tau\right) d\xi(s) + \int_0^t \phi^{-1}\left(\int_s^{+\infty} v(\tau)\, d\tau\right) ds,\ t \in \mathbb{R}^+. \tag{2.3}$$

Now, define a function γ on I by

$$\gamma(t) = \int_0^{+\infty} \mu(s)\phi^{-1}\left(\int_s^{+\infty} m(\tau)y_\infty\, d\tau\right) d\xi(s) + \int_0^t \phi^{-1}\left(\int_s^{+\infty} m(\tau)y_\infty\, d\tau\right) ds. \tag{2.4}$$

Remark 2.13. The properties of ϕ and m imply that γ, γ' are well defined and belong to $\mathcal{P}$. Moreover γ is the unique solution of problem (2.2) for $v \equiv my_\infty$.

The space

$$X = \left\{x \in C^1([0,+\infty), E) \mid \lim_{t\to+\infty} x'(t) = \theta \text{ and } x(0) = \int_0^{+\infty} \mu(t)x'(t)\, d\xi(t)\right\}$$

equipped with the norm

$$\|x\|_X = \sup_{t\geq 0} \|x'(t)\|_E$$

is a Banach space.

Let $\mathcal{P}$ be the translate of cone defined by

$$\mathcal{K} = \{x \in X \mid \ x(t) \geq \gamma(t) \text{ and } x'(t) \geq \gamma'(t), \ \ \forall\, t \in \mathbb{R}^+\}.$$

Remark 2.14. If $x \in X$, then it is immediate that

$$\sup_{t\in[0,+\infty)} \frac{\|x(t)\|_E}{t + \int_0^{+\infty} \mu(t)d\xi(t)} \leq \|x\|_X.$$

Indeed, if $x \in X$, we have $x(0) = \int_0^{+\infty} \mu(t)x'(t)\, d\xi(t)$. Then for all positive t

$$\begin{aligned} x(t) &= \int_0^t x'(s)\, ds + x(0) \\ &= \int_0^t x'(s)\, ds + \int_0^{+\infty} \mu(t)x'(t)\, d\xi(t). \end{aligned}$$

Hence $\|x(t)\|_E \leq \|x'(t)\|_E \left(t + \int_0^{+\infty} \mu(t)d\xi(t)\right)$, $\forall\, t \geq 0$ and thus

$$\frac{\|x(t)\|_E}{t + \int_0^{+\infty} \mu(t)d\xi(t)} \leq \|x'(t)\|_E \leq \|x\|_X, \ \forall\, t \geq 0.$$

3. **Main Results**

To abbreviate our presentation, we first set the following assumptions:

($\mathcal{H}_1$)**:** $g \in C\left(\mathbb{R}^+ \times \mathcal{P}_+ \times E \setminus \{\theta\}, E\right)$ and when u, v are bounded, the function $g(t, \left(t + \int_0^{+\infty} \mu(t)d\xi(t)\right) u, v)$ is bounded on $\mathbb{R}^+$. The coefficient $m \in C\left(I, \mathbb{R}^+\right)$ may be singular at $t = 0$, does not vanish identically on any subinterval of I, and satisfies

$$A = \int_0^{+\infty} m(t)dt < \infty.$$

($\mathcal{H}_2$)**:** For all $r > 0$ and every subinterval $[\lambda, \bar{\lambda}] \subset I$, the nonlinearity g is uniformly continuous on $[\lambda, \bar{\lambda}] \times B_E(\theta, r) \times B_E(\theta, r)$, where θ is the zero element of E and $B_E(\theta, r) = \{x \in E : \|x\| \leq r\}$.

$(\mathcal{H}_3)$: There exist nonnegative functions $l_1, l_2 \in L^1(I)$ such that

$$\alpha_E(g(t, D_1, D_2)) \leq l_1(t)\alpha_E(D_1) + l_2(t)\alpha_E(D_2), \quad t \in I,$$

for all countable bounded subsets $D_1 \subset \mathcal{P}$ and $D_2 \subset E$ with

$$\int_0^{+\infty} m(t) \left(\left(t + \int_0^{+\infty} \mu(s) d\xi(s) \right) l_1(t) + l_2(t) \right) dt < \frac{1}{2}.$$

Now, let Ω be a bounded subset of X. Then there exists $K > 0$ such that $\Omega \subset \overline{B_X(0, K)}$. According to Assumption $(\mathcal{H}_1)$, let

$$M_K = \sup \left\{ \|g(t, (t + \textstyle\int_0^{+\infty} \mu(t) d\xi(t))u, v)\|_E, \text{ for } t \geq 0,\ u, v \in \overline{B_E(0, K)} \right\}.$$

Let $x \in \Omega \cap \mathcal{P}$. Then for every $t \geq 0$, we have $\|x'(t)\| \leq K$ and $\frac{\|x(t)\|_E}{t + \int_0^{+\infty} \mu(t)\, d\xi(t)} \leq K$. Thus

$$\begin{aligned}
&\int_0^{+\infty} m(s) \left\| g(s, x(s), x'(s)) \right\|_E ds \\
&= \int_0^{+\infty} m(s) \| g \left(s, \frac{(s + \int_0^{+\infty} \mu(s)\, d\xi(s))\, x(s)}{s + \int_0^{+\infty} \mu(s)\, d\xi(s)}, x'(s) \right) \|_E ds \qquad (3.1)\\
&\leq M_K \int_0^{+\infty} m(s) ds < \infty.
\end{aligned}$$

From Lemma 2.12, we know that the boundary value problem (2.1) is equivalent to

$$x(t) = C + \int_0^t \phi^{-1} \left(\int_s^{+\infty} m(\tau) g\left(\tau, x(\tau), x'(\tau)\right) d\tau \right) ds,$$

where

$$C = \int_0^{+\infty} \mu(s) \phi^{-1} \left(\int_s^{+\infty} m(\tau) g(\tau, x(\tau), x'(\tau)) d\tau \right) d\xi(s).$$

Consider the integral operator

$$F\colon\ \overline{\Omega} \cap \mathcal{P} \longrightarrow C^1(\mathbb{R}^+, E) \qquad (3.2)$$

defined by

$$Fx(t) = C + \int_0^t \phi^{-1} \left(\int_s^{+\infty} m(\tau) g\left(\tau, x(\tau), x'(\tau)\right) d\tau \right) ds.$$

Remark 3.1. In view of Lemmas 2.11, 2.12, if x is a fixed point of F in X and satisfies $\int_0^{+\infty} m(s) \|g(s, x(s), x'(s))\, ds\|_E < \infty$, then it is solution of problem (2.1) which implies that the function $y = x + \bar{y} \in C^1(\mathbb{R}^+, E)$ is an unbounded solution of problem (1.1).

In the next five lemmas, we study the properties of the fixed point operator F.

Lemma 3.2. *Under Assumptions $(\mathcal{H}_1)$ with*

$$g(t, u, v) \geq y_\infty, \quad \text{for } t > 0,\ u \geq \gamma(t), \text{ and } v \geq \gamma'(t), \qquad (\mathcal{CS})$$

the operator F maps the set $\overline{\Omega} \cap \mathcal{K}$ into $\mathcal{K}$.

Proof. We first show that $F:\overline{\Omega}\cap\mathcal{K}\to X$ is well defined. Let $x\in\overline{\Omega}\cap\mathcal{K}$, then

$$\begin{aligned} Fx(0)=C&=\int_0^{+\infty}\mu(s)\phi^{-1}\left(\int_s^{+\infty}m(\tau)g(\tau,x(\tau),x'(\tau))d\tau\right)d\xi(s)\\ &=\int_0^{+\infty}\mu(s)(Fx)'(s)d\xi(s)\end{aligned}$$

and

$$\lim_{t\to+\infty}(Fx)'(t)=\lim_{t\to+\infty}\phi^{-1}\left(\int_t^{+\infty}m(\tau)g(\tau,x(\tau),x'(\tau))d\tau\right)=\phi^{-1}(\theta)=\theta.$$

Now, we claim that $Fx(t)\geq\gamma(t)$ on $\mathbb{R}^+$. Otherwise, there exists some positive real number t_0 such that $(Fx)(t_0)\ngeq\gamma(t_0)$, i.e., $(Fx)(t_0)-\gamma(t_0)\notin\mathcal{P}$. From Assumption $(\mathcal{CS})$ and the property of the function ϕ, we have

$$\begin{aligned} Fx(t_0)-\gamma(t_0)&=\int_0^{+\infty}\mu(s)\phi^{-1}\left(\int_s^{+\infty}m(\tau)g(\tau,x(\tau),x'(\tau))d\tau\right)d\xi(s)\\ &\quad+\int_0^{t_0}\phi^{-1}\left(\int_s^{+\infty}m(\tau)g\left(\tau,x(\tau),x'(\tau)\right)d\tau\right)ds\\ &\quad-\int_0^{+\infty}\mu(s)\phi^{-1}\left(\int_s^{+\infty}m(\tau)y_\infty\,d\tau\right)d\xi(s)\\ &\quad-\int_0^{t_0}\phi^{-1}\left(\int_s^{+\infty}m(\tau)y_\infty\,d\tau\right)ds\\ &\geq\theta.\end{aligned}$$

So $(Fx)(t_0)-\gamma(t_0)\in\mathcal{P}$, which is a contradiction. Finally, we claim that $(Fx)'(t)\geq\gamma'(t)$, for all $t\in\mathbb{R}^+$. On the contrary, there exists some real $t_1\in I$ such that $(Fx)'(t_1)\ngeq\gamma'(t_1)$, then $(Fx)'(t_1)-\gamma'(t_1)\notin\mathcal{P}$. From Assumption $(\mathcal{CS})$ and the property of the function ϕ, we get

$$\begin{aligned}(Fx)'(t_1)-\gamma'(t_1)&=\phi^{-1}\left(\int_{t_1}^{+\infty}m(\tau)g\left(\tau,x(\tau),x'(\tau)\right)d\tau\right)\\ &\quad-\phi^{-1}\left(\int_{t_1}^{+\infty}m(\tau)y_\infty\,d\tau\right)\\ &\geq\theta,\end{aligned}$$

i.e., $(Fx)'(t_1)-\gamma'(t_1)\in\mathcal{P}$, leading to a contradiction. □

Lemma 3.3. *Assume that conditions $(\mathcal{K}_1)$ and $(\mathcal{CS})$ hold. Then for every bounded set $\Omega\subset X$, the operator $F\colon\overline{\Omega}\cap\mathcal{K}\to\mathcal{K}$ is bounded and continuous.*

Proof. From (1.2) and (3.1), we have

$$\begin{aligned}\|(Fx)'(t)\|_E\leq\|\phi((Fx)'(t))\|_E&\leq\int_t^{+\infty}m(\tau)\left\|g(\tau,x(\tau),x'(\tau))\right\|_E d\tau\\ &\leq A\,M_K,\ \forall\,t\in\mathbb{R}^+,\ \forall\,x\in\overline{\Omega}\cap\mathcal{K},\end{aligned}$$

which shows that F is bounded. To prove that T is continuous, let $\{x_n\}_n,\ \{x\} \subset \overline{\Omega} \cap \mathcal{K}$ and $\|x_n - x\|_X \to 0$, as $n \to \infty$, hence $\{x_n\}_n$ is bounded in $\overline{\Omega} \cap \mathcal{K}$. Thus, there exists $L > 0$ such that $\max\{\|x_n\|_X\ (n \in \mathbb{N}),\ \|x\|_X\} \leq L$. Letting

$$M_L = \sup\left\{\|g\left(t, \left(t + \int_0^{+\infty} d\xi(t)\right) u, v\right)\|_E,\ \text{for}\ \ t \geq 0,\ u, v \in \overline{B_E(0, L)}\right\},$$

we get

$$\int_0^{+\infty} m(s)\|g(s, x_n(s), x_n'(s)\|_E\, ds \leq AM_L$$

and

$$\int_0^{+\infty} m(s)\|g(s, x(s), x'(s))\|_E\, ds \leq AM_L.$$

The continuity of g and ϕ^{-1} imply that for every $t \in \mathbb{R}^+$

$$\begin{aligned}&\|(Fx_n)'(t) - (Fx)'(t))\|_E\\ &\quad = \|\phi^{-1}\left(\int_t^{+\infty} m(\tau)g(\tau, x_n(\tau), x_n'(\tau))d\tau\right) - \phi^{-1}\left(\int_t^{+\infty} m(\tau)g(\tau, x(\tau), x'(\tau))d\tau\right)\|_E\end{aligned}$$

which tends to 0 as $n \to +\infty$. As a consequence

$$\|Fx_n - Fx\|_X \longrightarrow 0,\ \text{ as }\ n \to +\infty,$$

which means that the operator F is continuous. □

Lemma 3.4. *Assume that $(\mathcal{H}_1)$ holds and let $\Omega \subset X$ be a bounded set. Then*

- (a)**:** *The functions belonging to the set $\mathcal{A} = \{z \mid z(t) = \phi((Fx)'(t)),\ x \in \overline{\Omega} \cap \mathcal{K}\}$ are almost equi-continuous on $\mathbb{R}^+$.*
- (b)**:** *For any $\varepsilon > 0$, there exists a constant $T > 0$ such that*

$$\|\phi((Fx)'(t_1)) - \phi((Fx)'(t_2))\|_E < \varepsilon,\ \textit{for all}\ t_1, t_2 \geq T\ \ \textit{and}\ \ x \in \overline{\Omega} \cap \mathcal{K}.$$

Proof. We first prove (a). There exists $K > 0$ such that, for every $x \in \overline{\Omega} \cap \mathcal{K}$, $\|x\|_X \leq K$. For every $T \in I$ and $t_1, t_2 \in [0, T]$ $(t_2 < t_1)$, we have

$$\begin{aligned}&\|\phi((Fx)'(t_2)) - \phi((Fx)'(t_1))\|_E\\ &\quad = \|\int_{t_2}^{+\infty} m(s)g(s, x(s), x'(s))ds - \int_{t_1}^{+\infty} m(s)g(s, x(s), x'(s))ds\|_E\\ &\quad = \int_{t_2}^{t_1} m(s)\|g(s, x(s), x'(s))\|_E\, ds\\ &\quad \leq \max\Bigg\{\|g\left(t, \left(t + \int_0^{\infty} \mu(t)d\xi(t)\right) u, v\right)\|_E,\\ &\qquad t \in [t_2, t_1],\quad u, v \in \overline{B_E(0, K)}\Bigg\} \int_{t_2}^{t_1} m(s)ds.\end{aligned}$$

The right-hand term tends to 0, as $|t_1 - t_2| \to 0$. This proves that the set $\mathcal{A}$ is almost equi-continuous in $\mathbb{R}^+$.

Now, we prove (b). $(\mathcal{H}_1)$ yields $\lim_{t\to+\infty}\int_t^{+\infty} m(s)\,ds = 0$; then for all $\varepsilon > 0$, there exists $T > 0$ such that $\int_T^{+\infty} m(s)\,ds \le \frac{\varepsilon}{2M_K}$ and so

$$\begin{aligned}&\|\phi((Fx)'(t_1)) - \phi((Fx)'(t_2))\|_E\\ &= \|\int_{t_1}^{+\infty} m(\tau)g(\tau, x(\tau), x'(\tau))\,d\tau - \int_{t_2}^{+\infty} m(\tau)g(\tau, x(\tau), x'(\tau))\,d\tau\|_E\\ &\le 2M_K\int_T^{+\infty} m(s)\,ds \le 2M_K\frac{\varepsilon}{2M_K} = \varepsilon,\end{aligned}$$

uniformly with respect to $x \in \overline{\Omega}\cap\mathcal{K}$, as $t_1, t_2 \ge T$. This completes the proof of Lemma 3.4. □

In order to prove that F is a countably strict-set contraction, we first consider

Lemma 3.5. *Let Assumptions $(\mathcal{H}_1)$ and $(\mathcal{CS})$ be satisfied, Ω be a bounded subset of X, and V be a subset of $\overline{\Omega}\cap\mathcal{K}$. Then*

$$\alpha_X(FV) \le \sup_{t\in\mathbb{R}^+}\alpha_E\left(\phi((FV)'(t))\right) \tag{3.3}$$

and

$$\max\left(\sup_{t\in\mathbb{R}^+}\alpha_E\left(\frac{(FV)(t)}{t+\int_0^{+\infty}\mu(t)d\xi(t)}\right),\ \sup_{t\in\mathbb{R}^+}\alpha_E\left((FV)'(t)\right)\right) \le \alpha_X(FV). \tag{3.4}$$

Proof. By Lemma 3.3, we know that FV is bounded subset of X; then

$$\alpha_X\left(FV\right) < \infty.$$

Step 1. We first prove (3.3). By Lemma 3.4, for all $\varepsilon > 0$, there exists $T > 0$ such that for $t_1,\ t_2 \ge T$,

$$\|\phi((Fx)'(t_1)) - \phi((Fx)'(t_2))\| \le \varepsilon,\quad \text{uniformly for } x \in V. \tag{3.5}$$

Denote by $\phi((FV)')|_{[0,T]}$ the restriction of $\phi((FV)')$ on $[0,T]$. Since $\phi((FV)'(t))$ is equi-continuous on $[0,T]$, by Lemma 2.5, we have

$$\alpha_C(\phi((FV)')|_{[0,T]}) = \sup_{t\in[0,T]}\alpha_E\left(\phi((FV)'(t))\right) \le \sup_{t\in\mathbb{R}^+}\alpha_E\left(\phi((FV)'(t))\right),$$

where

$$\phi((FV)')|_{[0,T]} = \{\phi(x'(t)) \mid t\in[0,T],\quad x \in FV\}.$$

By the definition of the MNC α_C, there exists a partition $\{V_i\}_{i=1}^n$ such that $V = \bigcup_{i=1}^n V_i$ satisfies

$$\phi((FV)')|_{[0,T]} = \bigcup_{i=1}^n \phi((FV_i)')|_{[0,T]}$$

and

$$\operatorname{diam}_C(\phi((FV_i)')|_{[0,T]}) \le \sup_{t\in\mathbb{R}^+}\alpha_E\left(\phi((FV)'(t))\right) + \varepsilon,\quad i = 1,\dots,n, \tag{3.6}$$

where $\text{diam}_C(.)$ denotes the diameter of the bounded subsets of $C([0,T],E)$ and

$$\phi((FS)')_{[0,T]} = \{\phi((Fx)')(t) : x \in S,\ t \in [0,T]\}.$$

Furthermore, for i fixed, for all $Fx_1,\ Fx_2 \in FV_i$ and $t \geq T$, (3.5), (3.6), and (1.2) guarantee that

$$\begin{aligned} (3.7) \qquad & \|(Fx_1)'(t) - (Fx_2)'(t)\|_E \\ & \leq \|(Fx_1)'(t) - (Fx_1)'(T)\|_E + \|(Fx_1)'(T) - (Fx_2)'(T)\|_E \\ & \quad + \|(Fx_2)'(T) - (Fx_2)'(t)\|_E \\ & \leq \|\phi((Fx_1)'(t)) - \phi((Fx_1)'(T))\|_E + \|\phi((Fx_1)'(T)) - \phi((Fx_2)'(T))\|_E \\ & \quad + \|\phi((Fx_2)'(T)) - \phi((Fx_2)'(t))\|_E \\ & < \varepsilon + \sup_{t \in \mathbb{R}^+} \alpha_E\left(\phi((FV)'(t))\right) + \varepsilon + \varepsilon. \end{aligned}$$

Therefore (3.6) and (3.7) guarantee that

$$\text{diam}_X(FV_i) \leq \sup_{t \in \mathbb{R}^+} \alpha_E\left(\phi((FV)'(t))\right) + 3\varepsilon.$$

Noting that $FV = \bigcup_{i=1}^n FV_i$ implies

$$\alpha_X(FV) \leq \sup_{t \in \mathbb{R}^+} \alpha_E\left(\phi((FV)'(t))\right) + 3\varepsilon$$

and ε being arbitrary, we deduce that

$$\alpha_X(FV) \leq \sup_{t \in \mathbb{R}^+} \alpha_E\left(\phi((FV)'(t))\right).$$

Step 2. We prove (3.4). By the definition of the MNC α_X, for every $\varepsilon > 0$, there exists a partition $U_i \subset V$, $i = 1, \ldots, m$ such that

$$FV = \bigcup_{i=1}^m FU_i \quad \text{and} \quad \text{diam}_X(FU_i) \leq \alpha_X(FV) + \varepsilon.$$

Then for fixed i, for all $t \in \mathbb{R}^+$ and all $x_1,\ x_2 \in U_i$, we have

$$\|(Fx_1)'(t) - (Fx_2)'(t)\|_E \leq \|Fx_1 - Fx_2\|_X < \alpha_X(FV) + \varepsilon.$$

In accordance with $(FV)'(t) = \bigcup_{i=1}^m (FU)_i'(t)$, we have $\alpha_E\left((FV)'(t)\right) \leq \alpha_X(FV) + \varepsilon$, where

$$(FS)'(t) = \{(Fx)'(t) \mid x \in S\}.$$

The parameter ε being arbitrary, we deduce that

$$\sup_{t \in \mathbb{R}^+} \alpha_E\left((FV)'(t)\right) \leq \alpha_X(FV).$$

Since for all $x \in \overline{\Omega} \cap \mathcal{K}$, $\sup_{t \in \mathbb{R}^+} \frac{\|x(t)\|_E}{t + \int_0^{+\infty} \mu(t)\, d\xi(t)} \leq \|x\|_X$, for the same reason, we have

$$\sup_{t \in \mathbb{R}^+} \alpha_E\left(\frac{(FV)(t)}{t + \int_0^{+\infty} \mu(t) d\xi(t)}\right) \leq \alpha_X(FV),$$

proving (3.4). □

Lemma 3.6. *Suppose that $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ hold together with $(\mathcal{CS})$. Then for every bounded set $\Omega \subset X$, F is a countably strict set-contraction operator on $\overline{\Omega} \cap \mathcal{K}$.*

Proof. Let $V = \{x_n \mid n = 1, 2, \ldots\} \subset \overline{\Omega} \cap \mathcal{K}$; we prove that there exists a constant $0 \leq k < 1$ such that $\alpha_X(FV) \leq k\alpha_X(V)$. By Lemma 3.5, it is enough to verify that

$$\sup_{t\in\mathbb{R}^+} \alpha_E(\phi((FV)'(t))) \leq k\alpha_X(V), \tag{3.8}$$

where $(FV)' = \{(Fx_n)',\ x_n \in V,\ n = 1, 2, \ldots\}$. Using Lemmas 2.6, 3.5 together with Assumption $(\mathcal{H}_3)$, we obtain, for every $t \in \mathbb{R}^+$, the following estimates:

$$\begin{aligned}
\alpha_E(\phi((FV)'(t))) &= \alpha_E\left(\left\{\int_t^{+\infty} m(\tau)g(\tau, x_n(\tau), x_n'(\tau))\,d\tau,\ x_n \in V\right\}\right)\\
&\leq 2\int_t^{+\infty} m(\tau)\alpha_E(g(\tau, V(\tau), V'(\tau)))\,d\tau\\
&\leq 2\int_t^{+\infty} m(\tau)\left[l_1(t)\alpha_E(V(\tau)) + l_2(t)\alpha_E(V'(\tau))\right]\,d\tau\\
&\leq 2\sup_{\tau\in\mathbb{R}^+}\alpha_E\left(\frac{V(\tau)}{\tau + \int_0^{+\infty} d\xi(\tau)}\right)\int_t^{+\infty} m(\tau)\left(\tau + \int_0^{+\infty}\mu(\tau)d\xi(\tau)\right)l_1(\tau)d\tau\\
&\quad + 2\sup_{\tau\in\mathbb{R}^+}\alpha_E(V'(\tau))\int_t^{+\infty} m(\tau)l_2(\tau)\,d\tau\\
&\leq 2\,\alpha_X(V)\int_0^{+\infty} m(\tau)\left[\left(\tau + \int_0^{+\infty}\mu(\tau)d\xi(\tau)\right)l_1(\tau) + l_2(\tau)\right]d\tau.
\end{aligned}$$

Passing to the supremum over t in $\mathbb{R}^+$ yields

$$\sup_{t\in\mathbb{R}^+}\alpha_E(\phi((FV)'(t))) \leq k\,\alpha_X(V),$$

where, by Assumption $(\mathcal{H}_3)$,

$$k = 2\int_0^{+\infty} m(\tau)\left[\left(\tau + \int_0^{+\infty}\mu(s)d\xi(s)\right)l_1(\tau) + l_2(\tau)\right]d\tau < 1.$$

We then immediately deduce the estimate

$$\alpha_X(TV) \leq k\,\alpha_X(V)$$

which means that F is a countably strict set-contraction operator on $\overline{\Omega} \cap \mathcal{K}$. □

Now, we are ready to prove our main existence result.

Theorem 3.7. *Assume that $\mathcal{P}$ is normal cone with constant of normality N. Suppose that Assumptions $(\mathcal{H}_2)$–$(\mathcal{H}_3)$ and $(\mathcal{CS})$ hold together with*

($\mathcal{H}_4$): *$f \in C(\mathbb{R}^+ \times \mathcal{P}_+ \times E|\{\theta\}, E)$ ($f(t,.,.) \not\equiv \theta$ on $\mathbb{R}^+$) and there exist functions $a, b, c, d \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that for all $(t,u,v) \in \mathbb{R}^+ \times \mathcal{P}_+ \times E|\{\theta\}$*

$$\|f(t,u,v)\|_E \leq \left(a\left(\frac{\|u\|_E}{t + \int_0^{+\infty} \mu(t)\, d\xi(t)}\right) + b\left(\frac{\|u\|_E}{t + \int_0^{+\infty} \mu(t)\, d\xi(t)}\right)\right) \left(c\left(\|v\|_E\right) + d\left(\|v\|_E\right)\right),$$

where a, c are nonincreasing functions such that $\frac{b}{a}, \frac{d}{c}$ are nondecreasing functions and

$$\Pi = \int_0^{+\infty} m(t) a\left(\frac{\|\gamma(t)\|}{N(t + \int_0^{+\infty} \mu(t) d\xi(t))}\right) c\left(\frac{1}{N}\|\gamma'(t)\|_E\right) dt < +\infty,$$

γ being defined by (2.4).

($\mathcal{H}_5$): *There exits $R > 0$ such that*

$$\left(1 + \frac{b}{a}\left(R + \sup_{t \geq 0} \frac{\|\bar{y}(t)\|_E}{t + \int_0^{+\infty} \mu(t) d\xi(t)}\right)\right)\left(1 + \frac{d}{c}\left(R + \|y_\infty\|_E\right)\right)\Pi < R. \tag{3.9}$$

Then, problem (1.1) has at least one unbounded positive solution $y \in \mathcal{K}$ ($y \not\equiv \theta$) satisfying $y(t) \geq \gamma(t) + \bar{y}(t) > \theta$, $\forall\, t \in I$ and

$$\sup_{t \in \mathbb{R}^+} \frac{\|y(t) - \bar{y}(t)\|_E}{t + \int_0^{+\infty} \mu(t) d\xi(t)} \leq \sup_{t \in \mathbb{R}^+} \|y'(t) - y_\infty\|_E < R.$$

Proof. We need to prove the existence of a fixed point in $\mathcal{K}$ of operator F defined by (3.2). Consider the open set

$$\Omega_R := \{x \in X \mid \|x\|_X < R\},$$

where R is as defined by ($\mathcal{H}_5$). From Lemmas 3.3 and 3.6, $F : \overline{\Omega}_R \cap \mathcal{K} \to \mathcal{K}$, is a countably strict set-contraction operator. Next we show that $F(\overline{\Omega}_R \cap \mathcal{K}) \subset \Omega_R$. Let $x \in \overline{\Omega}_R \cap \mathcal{K}$; then $\|x\|_X \leq R$, $\|x(t)\|_E \geq \frac{1}{N}\|\gamma(t)\|_E$, and $\|x'(t)\|_E \geq \frac{1}{N}\|\gamma'(t)\|_E$. By Assumption of Theorem 3.7 and the inequality (1.2), the following estimates hold

$$\begin{aligned}
&\|(Fx)'(t)\|_E \leq \|\phi\left((Fx)'(t)\right)\|_E = \left\|\int_t^{+\infty} m(\tau) g(\tau, x(\tau), x'(\tau)) d\tau\right\|_E \\
&\leq \int_0^{+\infty} m(\tau)\left\|f(\tau, x(\tau) + \bar{y}(\tau), x'(\tau) + y_\infty)\right\|_E d\tau \\
&\leq \int_0^{+\infty} m(\tau)\left(a\left(\frac{\|x(\tau) + \bar{y}(\tau)\|_E}{\tau + \int_0^{+\infty} \mu(s) d\xi(s)}\right) + b\left(\frac{\|x(\tau) + \bar{y}(\tau)\|_E}{\tau + \int_0^{+\infty} \mu(s) d\xi(s)}\right)\right) \\
&\quad \times \left(c\left(\|x'(\tau) + y_\infty\|_E\right) + d\left(\|x'(\tau) + y_\infty\|_E\right)\right) d\tau \\
&= \int_0^{+\infty} m(\tau)\left(1 + \frac{b\left(\frac{\|x(\tau)+\bar{y}(\tau)\|_E}{\tau + \int_0^{+\infty} \mu(s) d\xi(s)}\right)}{a\left(\frac{\|x(\tau)+\bar{y}(\tau)\|_E}{\tau + \int_0^{+\infty} \mu(s) d\xi(s)}\right)}\right)\left(1 + \frac{d\left(\|x'(\tau) + y_\infty\|_E\right)}{c\left(\|x'(\tau) + y_\infty\|_E\right)}\right)
\end{aligned}$$

$$\times\, a\left(\frac{\|x(\tau)+\bar{y}(\tau)\|_E}{\tau+\int_0^{+\infty}\mu(s)d\xi(s)}\right) c\left(\|x'(\tau)+y_\infty\|_E\right) d\tau$$
$$\leq \int_0^{+\infty} m(\tau)\left(1+\frac{b}{a}\left(\|x\|_X+\sup_{t\geq 0}\frac{\|\bar{y}(t)\|_E}{t+\int_0^{+\infty}\mu(s)d\xi(s)}\right)\right)\left(1+\frac{d}{c}\left(\|x\|_X+\|y_\infty\|_E\right)\right)$$
$$\times\, a\left(\frac{\frac{1}{N}\|\gamma(\tau)\|_E}{\tau+\int_0^{+\infty}\mu(s)d\xi(s)}\right) c\left(\frac{1}{N}\|\gamma'(\tau)\|_E\right) d\tau$$
$$=\left(1+\frac{b}{a}\left(R+\sup_{t\geq 0}\frac{\|\bar{y}(t)\|_E}{t+\int_0^{+\infty}\mu(s)d\xi(s)}\right)\right)\left(1+\frac{d}{c}\left(R+\|y_\infty\|_E\right)\right)\Pi$$
$$< R.$$

Passing to the supremum over t yields

$$\|Fx\|_X < R, \quad \forall\, x\in\overline{\Omega}_R\cap\mathcal{K}.$$

Finally, Lemma 2.7 guarantees that the operator F has at least one fixed point $x\in\Omega_R\cap\mathcal{K}$; i.e., $0\leq\|x\|_X<R$, $x(t)\geq\gamma(t)$ and $x'(t)\geq\gamma'(t), \forall\, t\in\mathbb{R}^+$. Moreover

$$\int_0^{+\infty} m(s)\|g(s,x(s),x'(s))\|_E\, ds \leq AM_R,$$

where

$$M_R=\sup\left\{\|g\left(t,(t+\int_0^{+\infty}\mu(t)d\xi(t)\right)\, u,v)\|_E,\ \text{for } t\geq 0,\ u,v\in\overline{B_E(0,R)}\right\}.$$

Appealing to Remark 3.1, we then conclude that $y=x+\bar{y}$ is an unbounded positive solution of problem (1.1) and satisfies

$$y(t)\geq\gamma(t)+\bar{y}(t)>\theta, \forall\, t\in I \ \text{ and } \ y'(t)\geq\gamma'(t)+y_\infty>\theta, \forall\, t\in\mathbb{R}^+.$$

Furthermore

$$\sup_{t\in I}\frac{\|y(t)-\bar{y}(t)\|_E}{t+\int_0^{+\infty}\mu(s)\, d\xi(s)}\leq\sup_{t\in\mathbb{R}^+}\|y'(t)-y_\infty\|_E<R.$$

□

To end this section, we prove a nonexistence result for (1.1).

Theorem 3.8. *Assume that* $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ *and* $(\mathcal{CS})$ *hold and there exists a real number* $\mu>0$ *such that for every* $t\geq\mu$, $y(\cdot)\in\mathcal{P}_+$, *and* $z(\cdot)\in E$

$$\Psi\left(f(t,y(t),z(t))\right)>\frac{\|\phi(z(\mu)-y_\infty)\|_E}{\int_\mu^{+\infty}m(t)\, dt},\ \textit{for all } \Psi\in E' \textit{ with } \|\Psi\|=1.$$

Then problem (1.1) *has no positive solution.*

Proof. Assume on the contrary that problem (1.1) has a positive solution $y\in X$. So $x=y-\bar{y}$ is a solution of (2.1), i.e., F has a fixed point $x\in\mathcal{K}$. Let μ be a positive number; then $\phi\left(x'(\mu)\right)\neq\theta$ for otherwise $f(\mu,x(\mu)+\bar{y}(\mu),x'(\mu)+y_\infty)=\theta$, leading to a contradiction. By Lemma 2.1, there is some $\Psi \in E'$ such that $\Psi\left(\phi\left((Fx)'(\mu)\right)\right) = \|\phi\left((Fx)'(\mu)\right)\|$. Then for any $t \geq \mu$, we have

$$\begin{aligned}
\|\phi\left(x'(\mu)\right)\|_E &= \left\|\int_\mu^{+\infty} m(\tau)g(\tau,x(\tau),x'(\tau))\,d\tau\right\|_E \\
&= \Psi\left(\int_\mu^{+\infty} m(\tau)g(\tau,x(\tau),x'(\tau))\,d\tau\right) \\
&\geq \int_\mu^{+\infty} m(\tau)\Psi\left(f(\tau,x(\tau)+\bar{y}(\tau),x'(\tau)+y_\infty)\right)\,d\tau \\
&> \|\phi\left(x'(\mu)\right)\|_E,
\end{aligned}$$

which is a contradiction, completing the proof. □

Remark 3.9. If the nonlinearity $f(t,y,y') \equiv f(t,y)$ in problem (1.1) does not depend on the first derivative, then a similar method can be used to study problem (1.1) in the Banach space

$$Y = \left\{ y \in C(\mathbb{R}^+, E) : \lim_{t\to+\infty} \frac{\|y(t)\|_E}{t + \int_0^{+\infty} \mu(t)\,d\xi(t)} = 0 \right\}$$

equipped with the norm $\|y\|_Y = \sup_{t\in[0,+\infty)} \frac{\|x(t)\|_E}{t+\int_0^{+\infty}\mu(t)\,d\xi(t)}$. We obtain the same result as in Theorem 3.7 under Assumptions $(\mathcal{H}_2)$–$(\mathcal{H}_4)$ and $(\mathcal{CS})$ with $c(v)+d(v)=1$, $l_2 \equiv 0$. We omit the details of the proof.

4. Applications

To illustrate our main results, two boundary value problems are investigated.

4.1. **Example 1.** Consider the following singular boundary value problem of a finite system of scalar differential equations in the euclidian space $E = \mathbb{R}^n$ normed by $\|y\| = |(y_1,y_2,\dots,y_n)| = \max_{1\leq i\leq n} |y_i|$:

$$\left\{\begin{array}{l} -(\phi(y_i'-1))'(t) = e^{-kt} f_i(t,y_1(t),\dots,y_n(t),y_1'(t),\dots,y_n'(t)), \quad t>0 \\ y_i(0) - \frac{1}{4}y_i'(\frac{1}{3}) - \frac{1}{8}y_i'(\frac{1}{2}) - \frac{13}{8}y_i'(1) = \frac{1}{2}, \quad \lim_{t\to+\infty} y'(t) = 1, \end{array}\right. \tag{4.1}$$

where $k \geq 1$ and, for $i = 1,2,\dots,n$,

$$f_i(t,y_1,\dots,y_n,z_1,\dots,z_n) = \left(2 + \frac{y_i}{t+2}\right)\left(-\frac{1}{3k^2}e^{-kt} + z_i + \frac{1}{\sqrt{\max_{1\leq j\leq n}|z_j|}}\right).$$

Let $\mathcal{P} = \{y = (y_1,y_2,\dots,y_n) \in \mathbb{R}^n : y_i \geq 0,\ i = 1,2,\dots,n\}$. Then $\mathcal{P}$ is a normal cone in $\mathbb{R}^n$ with constant of normality $N = 1$ and system (4.1) can be rewritten in E under the form (1.1). Also

$$\phi(y_1,\dots,y_n) = 3\left\{\begin{array}{ll} (y_1^2,\dots,y_n^2), & \text{if } (y_1,\dots,y_n) \geq (1,\dots,1) \\ (y_1,\dots,y_n), & \text{otherwise }, \end{array}\right.$$

$$\mu \equiv 1 \text{ in } \mathbb{R}^+, \text{ and } \xi(t) = \frac{1}{4} I_{[\frac{1}{3},\frac{1}{2})}(t) + \frac{3}{8} I_{[\frac{1}{2},1)}(t) + 2I_{[1,+\infty)}(t),$$

where $I_{[\lambda,\bar{\lambda})}(t) = \begin{cases} 1, & \text{if } t \in [\lambda, \bar{\lambda}], \\ 0, & \text{else} \end{cases}$ is the characteristic function of the interval $[\lambda, \bar{\lambda})$. Clearly ϕ is continuous and bijective with

$$\phi^{-1}(y_1, \dots, y_n) = \frac{1}{3} \begin{cases} (\sqrt{y_1}, \dots, \sqrt{y_n}), & \text{if } (y_1, \dots, y_n) \geq (1, \dots, 1) \\ (y_1, \dots, y_n), & \text{otherwise }, \end{cases}$$

We have

$$\int_0^{+\infty} \mu(s)\, d\xi(s) = \left[\xi\left(\frac{1^+}{3}\right) - \xi\left(\frac{1^-}{3}\right)\right] + \left[\xi\left(\frac{1^+}{2}\right) - \xi\left(\frac{1^-}{2}\right)\right] + \left[\xi(1^-) - \xi(1^-)\right] = 2.$$

Clearly f_i is uniformly continuous on $[\lambda, \bar{\lambda}] \times B_E(\theta, r) \times B_E(\theta, r)$, for all $[\lambda, \bar{\lambda}] \subset I$ and $r > 0$, where $\theta = (0, \dots, 0)$. Then for any $i = 1, 2, \dots, n$

$$\begin{aligned} |f_i(t, y_1, \dots, y_n, z_1, \dots, z_n)| &= \left|2 + \frac{y_i}{t+2}\right| \left|-\frac{1}{3k} e^{-kt} + |z_i| + \frac{1}{\sqrt{\max_{1 \leq j \leq n} |z_j|}}\right| \\ &\leq \left(2 + \frac{\max_{1 \leq j \leq n} |y_j|}{t+2}\right) \left(\frac{1}{3k} + \max_{1 \leq j \leq n} |z_j| + \frac{1}{\sqrt{\max_{1 \leq j \leq n} |z_j|}}\right), \end{aligned}$$

which implies that for all $(t, y, z) \in \mathbb{R}^+ \times (\mathbb{R}^+ \setminus \{0\})^n \times (\mathbb{R} \setminus \{0\})^n$

$$\|f(t, y, z)\| \leq \left(a\left(\frac{\|y\|}{t+2}\right) + b\left(\frac{\|y\|}{t+2}\right)\right) \left[c\left(\|z\|\right) + d\left(\|z\|\right)\right],$$

where $a(u) = 2,\ b(u) = u,\ c(u) = \frac{1}{\sqrt{u}}$ and $d(u) = \frac{1}{3k} + u$. In addition

$$A = \int_0^{+\infty} e^{-kt}\, dt = \frac{1}{k}, \quad \gamma(t) = (\gamma_1(t), \dots, \gamma_n(t)),$$

where

$$\begin{aligned} \gamma_i(t) &= \int_0^{+\infty} \phi^{-1}\left(\frac{1}{k} e^{-ks}\right) d\xi(s) + \int_0^t \phi^{-1}\left(\frac{1}{k} e^{-ks}\right) ds \\ &= \frac{1}{3k} \int_0^{+\infty} e^{-ks}\, d\xi(s) + \frac{1}{3k} \int_0^t e^{-ks}\, ds \\ &= -\frac{1}{3k^2} e^{-kt} + \frac{1}{12k} e^{-\frac{1}{3}k} + \frac{1}{24k} e^{-\frac{1}{2}k} + \frac{13}{24k} e^{-k} + \frac{1}{3k^2}, \end{aligned}$$

and so

$$\gamma_i'(t) = \frac{1}{3k} e^{-kt}.$$

Moreover, for $y_0 = (\frac{1}{2}, \dots, \frac{1}{2})$ and $y_\infty = (1, \dots, 1)$, we have

$$\bar{y}(t) = y_\infty \left(t + \int_0^{+\infty} d\xi(t)\right) + y_0 = \left(t + \frac{5}{2}, \dots, t + \frac{5}{2}\right) \text{ and } \sup_{t \geq 0} \frac{\|\bar{y}(t)\|_E}{t + \int_0^{+\infty} d\xi(t)} = \frac{5}{4}.$$

Then, for each $i = 1, 2, \dots, n$

$$g_i(t, y, z) = f_i(t, y + \bar{y}, z + y_\infty)$$

$$= \left(2 + \frac{y_i + t + \frac{5}{2}}{t+2}\right)\left(-\frac{1}{3k}e^{-kt} + z_i + \frac{1}{\sqrt{\max|z_i+1|}} + 1\right)$$
$$\geq 1, \quad \text{for} \ \ t \geq 0, \, y_i \geq 0, \text{ and } \ z_i \geq \gamma_i'(t).$$

In order to check the inequality (3.9) in Assumption ($\mathcal{H}_5$), we take $k = 2000$ and $R = 5$ to get

$$\Pi = 2\int_0^{+\infty} \frac{e^{-ks}}{\sqrt{\frac{1}{3k}e^{-ks}}} ds < \frac{4}{25}.$$

Therefore

$$\left(1 + \frac{b}{a}\left(R + \sup_{t\geq 0}\frac{\|\bar{y}(t)\|_E}{t + \int_0^{+\infty} d\xi(t)}\right)\right)\left(1 + \frac{d}{c}\left(R + \|y_\infty\|_E\right)\right)\Pi$$
$$< \frac{1}{50}(4R+13)\left(1 + (R + \frac{1+3k}{3k})\sqrt{R+1}\right)$$
$$\simeq 4.4734 < R.$$

Finally, for every bounded subsets $D_1, D_2 \subset \mathcal{K}$ and for all $t \in \mathbb{R}^+$, $y \in D_1$, $z \in D_2$, we have

$$\|g(t,y,z)\| \leq \left(\frac{13}{12} + \frac{\|y\|}{2}\right)\left(\frac{1}{3k} + 1 + \max_{1\leq i\leq n}\|z\| + \frac{1}{\sqrt{\|z\|}}\right).$$

Moreover, for all $0 < t_1 < t_2 < +\infty$, $y \in D_1$, and $z \in D_2$, we have

$$\lim_{t_1\to t_2}|g_i(t_1,y,z) - g_i(t_2,y,z)|$$
$$\leq \lim_{t_1\to t_2}\left|\left(2 + \frac{y_i + t_1 + \frac{5}{2}}{t_1+2}\right)\left(-\frac{1}{3k}e^{-kt_1} + z_i + \frac{1}{\sqrt{\max|z_i+1|}} + 1\right)\right.$$
$$\left. - \left(2 + \frac{y_i + t_2 + \frac{5}{2}}{t_2+2}\right)\left(-\frac{1}{3k}e^{-kt_2} + z_i + \frac{1}{\sqrt{\max|z_i+1|}} + 1\right)\right|$$
$$= 0, \ \forall\, i = 1, \ldots, n.$$

Then $\lim_{t_1\to t_2}\|g(t_1,y,z) - g(t_2,y,z)\| = 0$ and

$$\lim_{t\to+\infty}|g_i(t,y,z) - \lim_{s\to+\infty}g_i(s,y,z)|$$
$$\leq \lim_{t\to+\infty}\left|\left(2 + \frac{y_i + t + \frac{5}{2}}{t+2}\right)\left(-\frac{1}{3k}e^{-kt} + z_i + \frac{1}{\sqrt{\max|z_i+1|}} + 1\right)\right.$$
$$\left. - 3\left(1 + z_i + \frac{1}{\sqrt{\max|z_i+1|}}\right)\right| = 0, \ \forall\, i = 1, \ldots, n.$$

Hence $\lim_{t\to+\infty}\|g(t,y,z) - \lim_{s\to+\infty}g(s,y,z)\| = 0$. As a consequence, Corduneanu's compactness criterion ([4], p. 62) ensures that $f(t, D_1, D_2)$ is relatively compact in $\mathbb{R}^n$. So $\alpha(f(t, D_1, D_2)) = 0$, for all $t \in \mathbb{R}^+$ and all bounded subset $D_1, D_2 \subset \mathcal{P}$; hence Assumption ($\mathcal{H}_3$) is satisfied for $l_1 \equiv l_2 \equiv 0$. Therefore all of Assumptions

$(\mathcal{H}_2)$-$(\mathcal{H}_5)$ and $(\mathcal{CS})$ are met. As a consequence, the singular system (4.1) has at least one nontrivial positive solution.

4.2. **Example 2.** Consider the infinite singular system of scalar second order multi-point boundary value problem ($n \in \{1, 2, \ldots\}$):

$$(4.2) \quad \begin{cases} -(\phi(y_n' - \frac{1}{n}))'(t) = \frac{e^{-t}}{20\sqrt{t}}(3 - \ln(1 + \frac{1}{n}) + \ln(1 + \frac{y_n}{t+10}) + \sin(z_n)), \quad t > 0 \\ y_i(0) - 5\, y_i'(1) - 10\, y_i'(3) = \frac{1}{2n}, \quad \lim\limits_{t \to +\infty} y_n'(t) = \frac{1}{n}, \end{cases}$$

in the Banach space

$$E = l^\infty = \{y = (y_n)_n \mid \|y\| = \sup_n |y_n| < \infty\}$$

furnished with the norm $\|y\| = \sup_n |y_n|$. Noting $y = (y_1, \ldots, y_n, \ldots)$, $z = (z_1, \ldots, z_n, \ldots)$, and $f = (f_1, \ldots, f_n, \ldots)$, we take

$$\phi(y) = \begin{cases} (y_1^2, \ldots, y_n^2, \ldots), & \text{if } (y_1, \ldots, y_n, \ldots) \geq (1, \ldots, 1, \ldots), \\ (y_1, \ldots, y_n, \ldots), & \text{otherwise}. \end{cases}$$

Let

$$\mathcal{P} = \{y = (y_n)_n \in l^\infty \mid y_n \geq 0, \ n = 1, 2, \ldots\}.$$

It is easy to verify that $\mathcal{P}$ is a normal cone in l^∞ with constant of normality $N = 1$. System (4.2) can be regarded as a BVP of the form (1.1) in l^∞, where $m(t) = \frac{e^{-t}}{20\sqrt{t}}$, $t > 0$, $y_1 = (\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2n}, \ldots)$, $y_\infty = (1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots)$, $\mu \equiv 1$ in $\mathbb{R}^+$ and

$$\xi(t) = \begin{cases} 0, & t \in [0, 1), \\ 5, & t \in [1, 3), \\ 10, & t \in [3, +\infty) \end{cases}$$

so that $\int_0^{+\infty} \mu(s)\, d\xi(s) = 10$. Moreover

$$\begin{aligned} |f_n(t, y, z)| &= |(3 - \ln(1 + \frac{1}{n}) + \ln(1 + \frac{y_n}{t + 10}) + \sin(z_n))| \\ &\leq 4 + \ln 2 + \ln(1 + \frac{\|y\|}{t + 10}). \end{aligned}$$

Then, for every $(t, y, z) \in \mathbb{R}^+ \times \mathcal{P} \times l^\infty$

$$\|f(t, y, z)\| \leq \left(a\left(\frac{\|y\|}{t + 2}\right) + b\left(\frac{\|y\|}{t + 2}\right)\right)\left[c\left(\|z\|\right) + d\left(\|z\|\right)\right],$$

where $a(u) = 4 + \ln 2$, $b(u) = \ln(1 + u)$, and $c(u) = d(u) = 1$. Moreover the function f_n is uniformly continuous on $[\lambda, \bar{\lambda}] \times B_E(\theta, r) \times B_E(\theta, r)$, for all $[\lambda, \bar{\lambda}] \subset I$ and $r > 0$, where $\theta = (0, \ldots, 0)$. Also $A = \int_0^{+\infty} \frac{e^{-t}}{20\sqrt{t}}\, dt = \frac{\sqrt{\pi}}{20}$, $\bar{y}(t) = y_\infty(t + \int_0^{+\infty} d\xi(t)) + y_0 = (t + \frac{21}{2}, \ldots, \frac{t+10}{n} + \frac{1}{2n}, \ldots)$, and $\sup\limits_{t \geq 0} \frac{\|\bar{y}(t)\|_E}{t + \int_0^{+\infty} d\xi(t)} = \frac{21}{20}$. Then, for all n

$$\begin{aligned} g_n(t, y, z) &= f_n(t, y + \bar{y}, z + y_\infty) \\ &= \left(3 - \ln(1 + \frac{1}{n}) + \ln\left(1 + \frac{1}{n} + \frac{y_n}{t + 10} + \frac{1}{2n(t + 10)}\right) + \sin\left(z_n + \frac{1}{n}\right)\right) \end{aligned}$$

$$\geq 1, \quad \text{for} \quad t \geq 0,\, y_n \geq 0, \quad \text{and} \quad z_n \geq 0.$$

In order to check the inequality (3.9) in Assumption $(\mathcal{H}_5)$, we choose $R = 3$. Then

$$\Pi = (4 + \ln 2) \int_0^{+\infty} \frac{e^{-t}}{20\sqrt{t}}\, dt = (4 + \ln 2)\frac{\sqrt{\pi}}{20}$$

and

$$\begin{aligned}
&\left(1 + \frac{b}{a}\left(R + \sup_{t\geq 0} \frac{\|\bar{y}(t)\|_E}{t + \int_0^{+\infty} d\xi(t)}\right)\right)\left(1 + \frac{d}{c}\left(R + \|y_\infty\|_E\right)\right)\Pi \\
&= (4 + \ln 2)\frac{\sqrt{\pi}}{10}\left(1 + \frac{\ln(R + \frac{41}{20})}{4 + \ln 2}\right) \\
&= \frac{(4 + \ln 2)\sqrt{\pi}}{10}\left(1 + \frac{\ln \frac{101}{20}}{4 + \ln 2}\right) \\
&< R\cdot
\end{aligned}$$

Now let $D_1 \subset \mathcal{P}, D_2 \subset E$ be bounded sets; thus for any $t \in \mathbb{R}^+$, $y \in D_1$, $z \in D_2$, we have the estimates

$$\begin{aligned}
&|g_n(t, y, z) - g_n(t, \overline{\overline{y}}, \overline{\overline{z}})| \\
&\leq |\ln\left(1 + \frac{1}{n} + \frac{y_n}{t + 10} + \frac{1}{2n(t + 10)}\right) - \ln\left(1 + \frac{1}{n} + \frac{\overline{\overline{y}}_n}{t + 10} + \frac{1}{2n(t + 10)}\right)| \\
&\quad + |\sin\left(z_n + \frac{1}{n}\right) - \sin\left(\overline{\overline{z}}_n + \frac{1}{n}\right)| \\
&\leq \frac{1}{(1 + \frac{1}{n})(t + 10) + \xi_n + \frac{1}{2n}}|y_n - \overline{y}_n| + 2|\cos\left(\frac{z_n + \overline{z}_n + \frac{2}{n}}{2}\right)||\sin\left(\frac{z_n - \overline{z}_n}{2}\right)|,
\end{aligned}$$

where ξ_n lies between y_n and $\overline{y}_n$. Then

$$\|g_n(t, y, z) - g_n(t, \overline{y}, \overline{z})\| \leq \frac{1}{t + 10}\|y_n - \overline{y}_n\| + \|z_n - \overline{z}_n\|, \forall\, t \in \mathbb{R}^+,\ y \in D_1,\ \text{and}\ z \in D_2$$

with

$$\int_0^{+\infty} m(t)\left[(t + 10)\, l_1(t) + l_2(t)\right] dt = \int_0^{+\infty} \frac{e^{-t}}{10\sqrt{t}}\, dt = 0.1772 < \frac{1}{2}\cdot$$

Hence, Assumption $(\mathcal{H}_3)$ is satisfied for $l_1(t) = \frac{1}{t+10}$ and $l_2 \equiv 1$. Therefore all Assumptions of Theorem 3.7 are satisfied. As a consequence, the semi-positone singular system (4.2) has at least one nontrivial positive solution.

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