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STABILITY SOLUTIONS FOR A SYSTEM OF NONLINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH FUNCTIONAL DELAY

MOUATAZ BILLAH MESMOULI, ABDELOUAHEB ARDJOUNI, AND AHCENE DJOUDI

Applied Mathematics Lab., Faculty of Sciences, Department of Mathematics,
Univ Annaba, P.O. Box 12, Annaba 23000, Algeria
Department of Mathematics and Informatics, Univ Souk Ahras, P.O. Box 1553,
Souk Ahras, 41000, Algeria
Applied Mathematics Lab., Faculty of Sciences, Department of Mathematics, Univ
Annaba, P.O. Box 12, Annaba 23000, Algeria

ABSTRACT. In this paper, we use the fixed point theorem to obtain stability results of the zero solution of a nonlinear neutral system of differential equations with functional delay. Application to the second-order model is given with an example.

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1. INTRODUCTION

The problems of stability of time-delay systems of neutral type have received considerable attention in the last two decades, see [12, 13, 20, 21, 24, 25]. Practical examples of such systems include distributed networks containing lossless transmission lines [1], and population ecology [18], vibration of masses attached to an elastic bar [23].

Lyapunov functions have been the main tool used to obtain boundedness, stability and the existence of periodic solutions of differential equations, differential equations with functional delays and functional differential equations (see [2, 5, 26]). As an example, in the study of differential equations with functional delays by using Lyapunov functionals, many difficulties arise if the delay is unbounded (see [14, 22]). Even more difficult it is to obtain necessary and sufficient conditions. Many authors have examined particular problems which have offered great difficulties for that theory and have presented solutions by means of various fixed point theorems for the last ten years. Burton [3, 4, 6], Burton and Furumochi [7, 8, 9] have shown that many of these problems can be solved using fixed point theory.

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In the current paper, motivated by [4, 15, 16, 17] we study the stability results of the zero solution of the nonlinear neutral system of differential equations

$$\frac{d}{dt}x(t) = A(t)x(t) + \frac{d}{dt}Q(t, x(t-\tau(t))) + G(t, x(t), x(t-\tau(t))), \tag{1.1}$$

with an assumed initial function

$$x(t) = \psi(t), \quad t \in [m_0, t_0],$$

where $\psi \in C([m_0, t_0], \mathbb{R}^n)$, $m_0 = \inf\{t - \tau(t) : t \geq t_0\}$ and $A(\cdot)$ is nonsingular $n \times n$ matrix with continuous real-valued functions as its elements, $\tau(t)$ being scalar, continuous, and $\tau(t) \geq \tau^* > 0$. The functions $Q : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ are continuous in their respective arguments.

In the analysis we use the fundamental matrix solution of

$$x'(t) = A(t)x(t), \tag{1.2}$$

to invert the system (1.1) into an integral system which we derive a fixed point mapping. After then, we define prudently a suitable complete space, depending on the initial condition, so that the mapping is a contraction. Using Banach's contraction mapping principle, we obtain a solution for this mapping, and hence a solution for (1.1), which is asymptotically stable.

The organization of this paper is as follows. In Section 2, we present some definitions, remarks and the inversion of (1.1). In Section 3, we present our main results. Application to the second-order model is given with an example in Section 4.

2. PRELIMINARIES

Let $C(\mathbb{R}, \mathbb{R}^n)$ is the space of all n-vector continuous functions endowed with the supremum norm

$$\|x(\cdot)\| = \sup_{t\in[0,\infty)} |x(t)|,$$

where $|\cdot|$ denotes the infinity norm for $x \in \mathbb{R}^n$. Also, if A is an $n \times n$ real matrix, then we define the norm of A by

$$|A| = \sup_{t\in[0,\infty)} \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}(t)|.$$

Let $\psi \in C([m_0, t_0], \mathbb{R}^n)$ be a given continuous bounded initial function. We denote such a solution by $x(t) = x(t, t_0, \psi)$. From the existence theory we can conclude that for each $\psi \in C([m_0, t_0], \mathbb{R}^n)$, there exists a unique solution $x(t) = x(t, t_0, \psi)$ of (1.1) defined on $[t_0, \infty)$. We define $\|\psi\| = \sup\{|\psi(t)| : m_0 \leq t \leq t_0\}$.

We recall now some definition for fundamental matrix, see also [10].

Definition 2.1. An $n \times n$ matrix function $t \to \Phi(t)$, defined on an open interval J, is called a *matrix solution* of the homogeneous linear system (1.2) if each of its columns is a (vector) solution.

Definition 2.2. A set of n solutions of the homogeneous linear differential equation (1.2), all defined on the same open interval J, is called a *fundamental set* of solutions on J if the solutions are linearly independent functions on J.

Definition 2.3. A matrix solution is called a fundamental matrix solution if its columns form a fundamental set of solutions. In addition, a fundamental matrix solution $t \to \Phi(t)$ is called the *principal fundamental matrix solution* at $t_0 \in J$ if $\Phi(t_0) = I$, where I denotes the $n \times n$ identity matrix.

Definition 2.4. The state transition matrix for the homogeneous linear system (1.2) on the open interval J is the family of fundamental matrix solutions $t \to \Phi(t, r)$ parametrized by $r \in J$ such that $\Phi(r, r) = I$.

Proposition 2.1 ([10, Proposition 2.14])**.** *If $t \to \Phi(t)$ is a fundamental matrix solution for the system* (1.2) *on J, then $\Phi(t, r) := \Phi(t)\Phi^{-1}(r)$ is the* state transition matrix*. Also, the state transition matrix satisfies the Chapman-Kolmogorov identities*

$$\Phi(r, r) = I, \quad \Phi(t, s)\Phi(s, r) = \Phi(t, r),$$

and the identities

$$\Phi(t, s)^{-1} = \Phi(s, t), \quad \frac{\partial \Phi(t, s)}{\partial s} = -\Phi(t, s) A(s).$$

Throughout this paper, $\Phi(t)$ will denote a fundamental matrix solution of the homogeneous (unperturbed) linear problem (1.2). First, we have to transform (1.1) into an equivalent equation that possesses the same basic structure and properties to define a fixed point mapping.

Lemma 2.5. *$x(\cdot)$ is a solution of the equation* (1.1)*if and only if*

$$\begin{aligned} x(t) &= Q(t, x(t-\tau(t))) + \Phi(t, t_0)[x(t_0) - Q(t_0, x(t_0 - \tau(t_0)))] \\ &\quad + \int_{t_0}^{t} \Phi(t, s)[A(s) Q(s, x(s-\tau(s))) + G(s, x(s), x(s-\tau(s)))]\, ds. \end{aligned} \tag{2.1}$$

Proof. Let x be a solution of (1.1) and $\Phi(t)$ is a fundamental system of solutions of (1.2). Rewrite the equation (1.1) as

$$\frac{d}{dt}[x(t) - Q(t, x(t-\tau(t)))] = A(t) x(t) + G(t, x(t), x(t-\tau(t))).$$

Define a new function z by $z(t) = \Phi^{-1}(t)[x(t) - Q(t, x(t-\tau(t)))]$. We have

$$\begin{aligned} \frac{d}{dt} z(t) &= \frac{d}{dt}\Phi^{-1}(t)[x(t) - Q(t, x(t-\tau(t)))] \\ &\quad + \Phi^{-1}(t)\frac{d}{dt}[x(t) - Q(t, x(t-\tau(t)))]. \end{aligned}$$

By the Proposition 2.1 , it follows that

$$\frac{d}{dt}\Phi^{-1}(t) = -\Phi^{-1}(t)\,A(t).$$

Then

$$\begin{aligned}&\frac{d}{dt}\left[x(t) - Q(t, x(t-\tau(t)))\right]\\&\quad = A(t)\left[x(t) - Q(t, x(t-\tau(t)))\right] + \Phi(t)\frac{d}{dt}z(t).\end{aligned}$$

Thus,

$$\begin{aligned}&A(t)\,x(t) + G(t, x(t), x(t-\tau(t)))\\&\quad = A(t)\left[x(t) - Q(t, x(t-\tau(t)))\right] + \Phi(t)\frac{d}{dt}z(t),\end{aligned}$$

and

$$\frac{d}{dt}z(t) = \Phi^{-1}(t)\left[A(t)\,Q(t, x(t-\tau(t))) + G(t, x(t), x(t-\tau(t)))\right]. \tag{2.2}$$

Also note that $z(t_0) = \Phi^{-1}(t_0)\left[x(t_0) - Q(t_0, x(t_0 - \tau(t_0)))\right]$.

An integration of the equation (2.2) from t_0 to t yields

$$z(t) - z(t_0) = \int_{t_0}^{t} \Phi^{-1}(s)\left[A(s)\,Q(s, x(s-\tau(s))) + G(s, x(s), x(s-\tau(s)))\right] ds.$$

Or, in other words,

$$\begin{aligned}&\Phi^{-1}(t)\left[x(t) - Q(t, x(t-\tau(t)))\right]\\&\quad = \Phi^{-1}(t_0)\left[x(t_0) - Q(t_0, x(t_0-\tau(t_0)))\right]\\&\qquad + \int_{t_0}^{t} \Phi^{-1}(s)\left[A(s)\,Q(s, x(s-\tau(s))) + G(s, x(s), x(s-\tau(s)))\right] ds. \qquad (2.3)\end{aligned}$$

By the Definition 2.4, (2.3) can be expressed by

$$\begin{aligned}x(t) &= Q(t, x(t-\tau(t))) + \Phi(t, t_0)\left(x(t_0) - Q(t_0, x(t_0-\tau(t_0)))\right)\\&\quad + \int_{t_0}^{t} \Phi(t, s)\left[A(s)\,Q(s, x(s-\tau(s))) + G(s, x(s), x(s-\tau(s)))\right] ds.\end{aligned}$$

The converse implication is easily obtained and the proof is complete. □

If $x : [t_0, \infty) \to \mathbb{R}^n$ is a given solution of (1.1), then discussing the behavior of another solution y of this equation relative to the solution x, i.e. discussing the behavior of the difference $y - x$ is equivalent to studying the behavior of the solution $z = y - x$ of the equation

$$\begin{aligned}z'(t) &= A(t)\left[y(t) - x(t)\right]\\&\quad + \frac{d}{dt}\left[Q(t, z(t-\tau(t)) + x(t-\tau(t))) - Q(t, x(t-\tau(t)))\right]\\&\quad + G(t, z(t) + x(t), z(t-\tau(t)) + x(t-\tau(t))) - G(t, x(t), x(t-\tau(t))),\end{aligned}$$

relative to the trivial solution $z \equiv 0$. Thus we may, without loss in generality, assume that (1.1) has the trivial solution as a reference solution, i.e.

$$Q(t,0) = G(t,0,0) \equiv 0,$$

an assumption we shall henceforth make.

In this paper we assume that, for $t \in \mathbb{R}$, x, y, z, $w \in \mathbb{R}^n$, the functions $Q(t,x)$ and $G(t,x,y)$ are globally Lipschitz continuous in x and in x and y, respectively. That, there are positive constants k_1, k_2, k_3 such that

$$|Q(t,x) - Q(t,y)| \leq k_1 \|x-y\|, \tag{2.4}$$

$$|G(t,x,y) - G(t,z,w)| \leq k_2 \|x-z\| + k_3 \|y-w\|. \tag{2.5}$$

3. MAIN RESULTS

Our aim here is to give a necessary and sufficient condition for asymptotic stability of the zero solution of (1.1). Stability definitions may be found in [4], for example. By the Lemma 2.5, let a mapping $\mathcal{H}$ given by $(\mathcal{H}\varphi)(t) = \psi(t)$ for $t \in [m_0, t_0]$ and for $t \geq t_0$

$$\begin{aligned}(\mathcal{H}\varphi)(t) &\\ &= Q(t, \varphi(t-\tau(t))) + \Phi(t,t_0)\left[\psi(t_0) - Q(t_0, \psi(t_0 - \tau(t_0)))\right] \\ &\quad + \int_{t_0}^{t} \Phi(t,s)\left[G(s,\varphi(s),\varphi(s-\tau(s))) + A(s)Q(s,\varphi(s-\tau(s)))\right] ds, \end{aligned} \tag{3.1}$$

and define the space $\mathcal{S}_\psi$ by

$$\begin{aligned}\mathcal{S}_\psi &= \{\varphi : \mathbb{R} \to \mathbb{R}^n,\ \varphi(t) = \psi(t) \text{ if } m_0 \leq t \leq t_0,\ \varphi(t) \to 0 \text{ as } t \to \infty, \\ &\qquad \varphi \in C \text{ is bounded}\}. \end{aligned} \tag{3.2}$$

Then, $(\mathcal{S}_\psi, \|\cdot\|)$ is a complete metric space where $\|\cdot\|$ is the supremum norm.

Theorem 3.1. *Assume* (2.4) *and* (2.5) *hold. Further assume that*

$$\Phi(t) \to 0 \text{ as } t \to \infty, \tag{3.3}$$

$$t - \tau(t) \to \infty \text{ as } t \to \infty, \tag{3.4}$$

and there is $\alpha > 0$ *such that*

$$k_1 + \int_{t_0}^{t} |\Phi(t,s)| (k_2 + k_3 + |A| k_1)\, ds \leq \alpha < 1, \quad t \geq t_0, \tag{3.5}$$

hold. Then every solution $x(t,t_0,\psi)$ *of* (1.1) *with small continuous initial function* ψ*, is bounded and asymptotically stable. Moreover, the zero solution is stable at* t_0*.*

Proof. Let the mapping $\mathcal{H}$ defined by (3.1). Since Q, G are continuous, it is easy to show that $\mathcal{H}$ is. Let ψ be a small given continuous initial function with $\|\psi\| < \delta$ $(\delta > 0)$. Since $\varphi \in \mathcal{S}_\psi$ then there exist a positive constant K, such that $\|\varphi\| \leq K$, this and the condition (3.5) imply

$$\begin{aligned}
&\left|\left(\mathcal{H}\varphi\right)(t)\right| \\
&\quad \leq \left|Q\left(t,\varphi\left(t-\tau\left(t\right)\right)\right)\right| + \left|\Phi\left(t,t_0\right)\right|\left[\left|\psi\left(t_0\right)\right| + \left|Q\left(t_0,\psi\left(t_0-\tau\left(t_0\right)\right)\right)\right|\right] \\
&\qquad + \int_{t_0}^{t}\left|\Phi\left(t,s\right)\right|\left[\left|G\left(s,\varphi\left(s\right),\varphi\left(s-\tau\left(s\right)\right)\right)\right| + \left|A\left(s\right)\right|\left|Q\left(s,\varphi\left(s-\tau\left(s\right)\right)\right)\right|\right]ds \\
&\quad \leq k_1K + \left|\Phi\right|\delta\left(1+k_1\right) + K\int_{t_0}^{t}\left|\Phi\left(t,s\right)\right|\left(k_2+k_3+\left|A\right|k_1\right)ds \\
&\quad \leq \left|\Phi\right|\delta\left(1+k_1\right) + \alpha K,
\end{aligned}$$

which implies $\mathcal{H}\varphi$ is bounded, for the right δ. Next we show that $\left(\mathcal{H}\varphi\right)(t) \to 0$ as $t \to \infty$. The first term on the right side of (3.1) tends to zero, by condition (3.4). Also, the second term on the right side tends to zero, because of (3.3) and the fact that $\varphi \in \mathcal{S}_\psi$. Let $\epsilon > 0$ be given, then there exists a $t_1 > t_0$ such that for $t > t_1$, $\left|\varphi\left(t-\tau\left(t\right)\right)\right| < \epsilon$. By the condition (3.3), there exists a $t_2 > t_1$ such that for $t > t_2$ implies that

$$\left|\Phi\left(t,t_2\right)\right| < \frac{\epsilon}{\alpha K}.$$

Thus for $t > t_2$, we have

$$\begin{aligned}
&\int_{t_0}^{t}\left|\Phi\left(t,s\right)\right|\left(k_2\left|\varphi\left(s\right)\right| + k_3\left|\varphi\left(s-\tau\left(s\right)\right)\right| + \left|A\right|k_1\left|\varphi\left(s-\tau\left(s\right)\right)\right|\right)ds \\
&\quad \leq K\int_{t_0}^{t_1}\left|\Phi\left(t,s\right)\right|\left(k_2+k_3+\left|A\right|k_1\right)ds \\
&\qquad + \epsilon\int_{t_1}^{t}\left|\Phi\left(t,s\right)\right|\left(k_2+k_3+\left|A\right|k_1\right)ds \\
&\quad \leq K\left|\Phi\left(t,t_2\right)\right|\int_{t_0}^{t_1}\left|\Phi\left(t_2,s\right)\right|\left(k_2+k_3+\left|A\right|k_1\right)ds + \alpha\epsilon \\
&\quad \leq \alpha K\left|\Phi\left(t,t_2\right)\right| + \alpha\epsilon < \alpha\epsilon + \epsilon.
\end{aligned}$$

Hence, $\left(\mathcal{H}\varphi\right)(t) \to 0$ as $t \to \infty$. It is natural now to prove that $\mathcal{H}$ is contraction under the supremum norm. Let, $\varphi_1, \varphi_2 \in \mathcal{S}_\psi$. Then

$$\begin{aligned}
&\left|\left(\mathcal{H}\varphi_1\right)(t) - \left(\mathcal{H}\varphi_2\right)(t)\right| \\
&\quad \leq \left|Q\left(t,\varphi_1\left(t-\tau\left(t\right)\right)\right) - Q\left(t,\varphi_2\left(t-\tau\left(t\right)\right)\right)\right| \\
&\qquad + \int_{t_0}^{t}\left|\Phi\left(t,s\right)\right|\left(k_2\left\|\varphi_1-\varphi_2\right\| + k_3\left\|\varphi_1-\varphi_2\right\| + k_1\left|A\right|\left\|\varphi_1-\varphi_2\right\|\right)ds \\
&\quad \leq \alpha\left\|\varphi_1-\varphi_2\right\|.
\end{aligned}$$

Hence, the contraction mapping principle implies, $\mathcal{H}$ has a unique fixed point in $\mathcal{S}_\psi$ which solves (1.1), bounded and asymptotically stable. The stability of the zero solution of (1.1) follows simply by replacing K by ϵ. □

4. APPLICATION TO SECOND-ORDER MODE

Consider the second-order nonlinear neutral differential equation

$$\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t)=\frac{d}{dt}V(t,x(t-\tau(t)))+W(t,x(t),x(t-\tau(t))), \tag{4.1}$$

where p and q are continuous real-valued functions. The function $V:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is differentiable and $W:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ continuous in their respective arguments. They are also globally Lipschitz continuous in x and in x and y, respectively. That is, there are positive constants l_1, l_2, l_3 such that

$$|V(t,x)-V(t,y)|\le l_1\|x-y\|, \tag{4.2}$$

and

$$|V(t,x,y)-V(t,z,w)|\le l_2\|x-z\|+l_3\|y-w\|. \tag{4.3}$$

To show the stability solutions, we transform (4.1) by letting

$$\begin{cases} x_1=x,\\ x_2=x',\end{cases}$$

into a following system

$$\begin{aligned}\begin{pmatrix} x_1(t)\\ x_2(t)\end{pmatrix}'&=\begin{pmatrix}0 & 1\\ -q(t) & -p(t)\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}+\frac{d}{dt}\begin{pmatrix}0\\ V(t,x_1(t-\tau(t)))\end{pmatrix}\\ &\quad+\begin{pmatrix}0\\ W(t,x_1(t),x_1(t-\tau(t)))\end{pmatrix},\end{aligned} \tag{4.4}$$

where

$$A(\cdot)=\begin{pmatrix}0 & 1\\ -q(\cdot) & -p(\cdot)\end{pmatrix},\ Q(t,x(t-\tau(t)))=\begin{pmatrix}0\\ V(t,x_1(t-\tau(t)))\end{pmatrix},$$

$$G(t,x(t),x(t-g(t)))=\begin{pmatrix}0\\ W(t,x_1(t),x_1(t-\tau(t)))\end{pmatrix}.$$

and ψ be a small continuous initial function, $\psi\in C([m_0,t_0],\mathbb{R}^2)$ with $\|\psi\|<\delta$ $(\delta>0)$.

Example 4.1. Let $q(t)=4$, $p(t)=5$, $\tau(t)=\frac{t}{2}$, $V(t,w)=\lambda_1\sin(t)\,w^2$, $W(t,z,w)=\lambda_2\cos(t)\,z-\lambda_3 w$. Consider the Banach space

$$\begin{aligned}\mathcal{S}_\psi=\big\{&\varphi:\mathbb{R}\to\mathbb{R}^2,\ \varphi(t)=\psi(t)\ \text{if}\ -\infty\le t\le t_0,\ \varphi(t)\to 0\ \text{as}\ t\to\infty,\\ &\varphi\in C\ \text{is bounded}\big\}.\end{aligned} \tag{4.5}$$

Let $\varphi = (\varphi_1, \varphi_2)^t$, $\phi = (\phi_1, \phi_2)^t$. Then, we have

$$\begin{aligned}&\|G(\cdot, \varphi(\cdot), \varphi(\cdot - g(\cdot))) - G(\cdot, \phi(\cdot), \phi(\cdot - g(\cdot)))\| \\ &\qquad \leq \lambda_2 \|\varphi - \phi\| + \lambda_3 \|\varphi - \phi\|.\end{aligned}$$

Hence $k_2 = \lambda_2$, $k_3 = \lambda_3$, in the same way $k_1 = 2\lambda_1 K$, and

$$\Phi(t, t_0) = e^{(t-t_0)A} = Pe^{(t-t_0)D}P^{-1},$$

where

$$D = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}, \; e^{(t-t_0)D} = \begin{pmatrix} e^{-1(t-t_0)} & 0 \\ 0 & e^{-4(t-t_0)} \end{pmatrix}, \; P = \begin{pmatrix} -1 & 1 \\ 1 & -4 \end{pmatrix}.$$

Consequently, $\Phi(t, t_0) \to 0$ as $t \to \infty$, $t - \frac{t}{2} = \frac{t}{2} \to \infty$ as $t \to \infty$ and

$$2\lambda_1 K + \int_{t_0}^{t} \left| Pe^{(t-s)D}P^{-1} \right| (\lambda_2 + \lambda_3 + |A| \, 2\lambda_1 K) \, ds < 1,$$

is satisfied for λ_i, $1 \leq i \leq 3$ small enough. Then (2.4), (2.5) and (3.3)–(3.5) of the Theorem 3.1 are satisfied, which imply the zero solution of (4.1) is asymptotically stable.

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