

PERIODIC SOLUTIONS TO SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH NON-INSTANTANEOUS IMPULSES

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ABSTRACT. In this paper, we consider a non-instantaneous impulsive system represented by the second order nonlinear differential equations in a Banach space. We use the strongly continuous cosine family of linear operators along with Schauder and Banach fixed point theorems to study the existence and uniqueness of the periodic solutions of the non-instantaneous impulsive system. Moreover, we construct a Poincaré operator, which is a composition of the maps and we apply the techniques of a priori estimate for this operator. Finally, we give an example to illustrate the application of these obtained abstract results.

Keywords. non-instantaneous impulsive system, mild periodic solutions, Poincaré operator, Banach and Schauder fixed point theorems

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1. INTRODUCTION

Periodic motion plays a very significant role not only in natural science but also in social science such as climate, food supplement, insecticide population, sustainable development. Somolinos (1978) has considered the equation

$$x'' + (a/r)x' + (b/r) \sin x(t - r) = 0$$

and has obtained interesting results on the existence of periodic solutions. The study of this problem goes back to the early 1800s and has attracted much attention. It involves the motion of a sunflower plant. The tip of the plant is observed to move from side to side in a periodic fashion.

The dynamics of many evolving processes are subject to abrupt changes, such as shocks, harvesting and natural disaster. In the literature of impulsive differential

equations there are mainly two types of impulses, one is instantaneous and other is non-instantaneous. In the instantaneous impulses, the duration of abrupt changes is negligible in comparison with the duration of an entire evolution. Sometimes time abrupt changes may stay for time intervals such impulses are called non-instantaneous impulses. The importance of the study of non-instantaneous impulsive differential equations lies in its diverse fields of applications such as in the theory of stage by stage rocket combustion, maintaining hemodynamical equilibrium etc. A very well known application of non-instantaneous impulses is the introduction of insulin in the bloodstream which is abrupt change and the consequent absorption which is a gradual process as it remains active for a finite interval of time. The theory of impulsive differential equations has found enormous applications in realistic mathematical modeling of a wide range of practical situations. It has emerged as an important area of research such as modeling of impulsive problems in physics, population dynamics, ecology, biological systems, biotechnology and so forth.

Recently, Hernández and O'Regan [9] studied mild and classical solutions for the impulsive differential equation with non-instantaneous impulses which is of the form

$$\begin{aligned}
 (1.1) \quad & x'(t) = Ax(t) + f(t, x(t)), \quad t \in (s_i, t_{i+1}] \quad i = 0, 1, \dots, m, \\
 & x(t) = g_i(t, x(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \dots, m, \\
 & x(0) = x_0 \in X
 \end{aligned}$$

for a Banach space X with a norm $\|\cdot\|$. Wang and Fečkan [16] have generalized the conditions $x(t) = g_i(t, x(t))$ in (1.1) as follows

$$x(t) = g_i(t, x(t_i^-)), \quad t \in (t_i, s_i], \quad i = 1, 2, \dots, m.$$

Of course then $x(t_i^+) = g_i(t_i, x(t_i^-))$, $i = 1, 2, \dots, m$, in general. The symbols $x(t_i^+) := \lim_{\epsilon \rightarrow 0^+} x(t_i + \epsilon)$ and $x(t_i^-) := \lim_{\epsilon \rightarrow 0^-} x(t_i + \epsilon)$ represent the right and left limits of $x(t)$ at $t = t_i$, respectively. Motivated by above remark, Wang and Fečkan [16] have shown existence, uniqueness and stability of solutions of such general class of impulsive differential equations.

In this paper, we continue in this direction like in [11]. But now we study the existence of periodic mild solutions of the second order nonlinear differential equation with non-instantaneous impulses in a Banach space X of the form

$$\begin{aligned}
 (1.2) \quad & x''(t) = Ax(t) + f(t, x(t)), \quad t \in (s_i, t_{i+1}], \quad i \in \mathbb{N}_0, \\
 & x(t) = J_i^1(t, x(t_i^-)), \quad t \in (t_i, s_i], \quad i \in \mathbb{N}, \\
 & x'(t) = J_i^2(t, x(t_i^-)), \quad t \in (t_i, s_i], \quad i \in \mathbb{N},
 \end{aligned}$$

where $x(t)$ is a state function, $0 = s_0 < t_1 < s_1 < t_2, \dots, t_m < s_m < t_{m+1} < \dots$ with $\lim_{i \rightarrow \infty} t_i = \infty$ and $t_{i+m} = t_i + T$, $i \in \mathbb{N}$, $s_{i+m} = s_i + T$, $i \in \mathbb{N}_0$ for some $m \in \mathbb{N}$ denoting the number of impulsive points between 0 and $T > 0$, and we set

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We consider in (1.2) that $x \in C((t_i, t_{i+1}], X)$, $i \in \mathbb{N}$ and there exist $x(t_i^-)$ and $x(t_i^+)$, $i \in \mathbb{N}$ with $x(t_i^-) = x(t_i)$. The functions $J_i^1(t, x(t_i^-))$ and $J_i^2(t, x(t_i^-))$ represent noninstantaneous impulses during the intervals $(t_i, s_i]$, $i \in \mathbb{N}$, so impulses at t_i^- have some duration, namely on intervals $(t_i, s_i]$. A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t \in \mathbb{R}}$ on X . J_i^1, J_i^2 and f are suitable functions and they will be specified later. We construct a Poincaré operator to (1.2) and study its fixed points and dynamics.

Second order differential equations play a very crucial role in the modeling of physical phenomena, for example, modeling the position of the mass attached to spring over time and modeling the motion of a simple pendulum etc. A useful tool for the study of second-order abstract differential equations in the infinite dimensional space is the theory of strongly continuous cosine families of operators. Existence and uniqueness of the solution of second-order nonlinear systems and controllability of these systems in Banach spaces have been studied thoroughly by many authors [1, 2, 4, 12, 13]. Related problems are studied also in [3, 6, 7, 8, 10].

The plan of the paper is as follows. In Section 2, we give some important notations, definitions and assumptions which are required for the establishment of main results of the paper. In Section 3, we study the periodic solutions for the problem (1.2). In the last Section 4, an example is given to show the application of these abstract results.

2. PRELIMINARIES AND ASSUMPTIONS

First, we briefly recall some definitions from the theory of cosine family [5, pp. 32–33].

Definition 2.1. A one parameter family $(C(t))_{t \in \mathbb{R}}$ of bounded linear operators mapping the Banach space X into itself is called a strongly continuous cosine family if and only if

- (i) $C(s+t) + C(s-t) = 2C(s)C(t)$ for all $s, t \in \mathbb{R}$,
- (ii) $C(0)$ is the identity operator,
- (iii) $C(t)x$ is continuous in t on $\mathbb{R}$ for each fixed point $x \in X$.

The sine function $(S(t))_{t \in \mathbb{R}}$ associated to the strongly continuous cosine family $(C(t))_{t \in \mathbb{R}}$ is defined by

$$S(t)x = \int_0^t C(s)x \, ds, \quad x \in X, \quad t \in \mathbb{R}.$$

The domain $D(A)$ of the operator A is defined by

$$D(A) = \{x \in X : C(t)x \text{ is twice continuously differentiable in } t\},$$

which is a Banach space endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|$ for all $x \in D(A)$. We define a set

$$E = \{x \in X : C(t)x \text{ is once continuously differentiable in } t\},$$

which is a Banach space endowed with a suitable norm $\|\cdot\|_E$ (see [5, p. 46]). We also note that if $x : I \rightarrow X$, $I = [0, \infty)$ is a locally integrable function then

$$y(t) = \int_0^t S(t-s)x(s)ds$$

defines an E valued continuous function. We refer to Fattorini and Travis, Webb [5, 14, 15] for more details on the cosine family theory.

In order to prove the existence of the periodic solution for the problem (1.2), we need the following assumptions:

- (A1) A be the infinitesimal generator of a strongly continuous cosine family $(C(t))_{t \in \mathbb{R}}$ of bounded linear operators.
- (A2) $f : I_0 \times X \rightarrow X$, $I_0 = \bigcup_{i=0}^{\infty} [s_i, t_{i+1}]$ is a continuous function and there exists a positive constant K_f such that

$$\|f(t, x_1) - f(t, x_2)\| \leq K_f \|x_1 - x_2\|$$

for every $x_1, x_2 \in X$, $t \in I_0$.

- (A3) There exist nonnegative constants L_f and M_f such that

$$\|f(t, x)\| \leq L_f \|x\| + M_f, \quad x \in X, \quad t \in I_0.$$

- (A4) $f(t, x)$ is T -periodic in t , i.e., $f(t+T, x) = f(t, x)$, $t \in I_0$.
- (A5) $J_i^l \in C(I_i \times X, X)$, $I_i = [t_i, s_i]$ and there are positive constants K_{J_i} , $i \in \mathbb{N}$, such that

$$\max \{ \|J_i^1(t, x_1) - J_i^1(t, x_2)\|, \|J_i^2(t, x_1) - J_i^2(t, x_2)\| \} \leq K_{J_i} \|x_1 - x_2\|$$

for all $t \in I_i$ and $x_1, x_2 \in X$.

- (A6) There exist nonnegative constants L_{J_i} and M_{J_i} , $i \in \mathbb{N}$ such that

$$\max \{ \|J_i^1(t, x)\|, \|J_i^2(t, x)\| \} \leq L_{J_i} \|x\| + M_{J_i}, \quad t \in I_i, \quad x \in X.$$

- (A7) The following periodicity conditions hold: $J_{i+m}^l(t+T, x) = J_i^l(t, x)$, $t \in I_i$, $x \in X$, where $i \in \mathbb{N}$ and $l = 1, 2$. Note $I_i + T = I_{i+m}$. So $K_{J_{i+m}} = K_{J_i}$, $L_{J_{i+m}} = L_{J_i}$ and $M_{J_{i+m}} = M_{J_i}$, $i \in \mathbb{N}$.

Let us set $PC(I, X) = \left\{ x : I \rightarrow X : x \in C([0, t_1], X), x \in C((t_k, t_{k+1}], X), k \in \mathbb{N} \text{ and there exist } x(t_k^-) \text{ and } x(t_k^+), k \in \mathbb{N} \text{ with } x(t_k^-) = x(t_k^+) \right\}$.

In the following definition, we introduce the concept of the mild solution and mild dynamics for the problem (1.2).

Definition 2.2. A function $x \in PC(I, X)$ is called a mild solution of the impulsive problem

$$(2.1) \quad \begin{aligned} x''(t) &= Ax(t) + f(t, x(t)), \quad t \in (s_i, t_{i+1}], \quad i \in \mathbb{N}_0, \\ x(t) &= J_i^1(t, x(t_i^-)), \quad t \in (t_i, s_i], \quad i \in \mathbb{N}, \\ x'(t) &= J_i^2(t, x(t_i^-)), \quad t \in (t_i, s_i], \quad i \in \mathbb{N}, \\ x(0) &= x_0, \quad x'(0) = y_0, \end{aligned}$$

if it satisfies the following relations:

- the non-instantaneous impulse conditions

$$x(t) = J_i^1(t, x(t_i^-)), \quad x'(t) = J_i^2(t, x(t_i^-)), \quad t \in (t_i, s_i], \quad i \in \mathbb{N}$$

- and $x(t)$ is the solution of the following integral equations

$$\begin{aligned} x(t) &= C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s, x(s))ds, \quad t \in [0, t_1], \\ x(t) &= C(t-s_i)(J_i^1(s_i, x(t_i^-))) + S(t-s_i)(J_i^2(s_i, x(t_i^-))) \\ &\quad + \int_{s_i}^t S(t-s)f(s, x(s))ds, \quad t \in [s_i, t_{i+1}], \quad i \in \mathbb{N}. \end{aligned}$$

By the mild dynamics of (2.1) we consider the iteration

$$(2.2) \quad (x_0, y_0) \cup \{(J_i^1(s_i, x(t_i^-)), J_i^2(s_i, x(t_i^-)))\}_{i \in \mathbb{N}}.$$

Definition 2.3. A function $x \in PC(I, X)$ is said to be a T -periodic PC -mild solution of the problem (1.2) if it is a PC -mild solution of the problem (2.1) for some initial conditions x_0, y_0 and the corresponding iteration (2.2) is m -periodic, i.e., it holds

$$\begin{aligned} (x_0, y_0) &= (J_m^1(s_m, x(t_m^-)), J_m^2(s_m, x(t_m^-))) \\ &= (J_{i+m}^1(s_{i+m}, x(t_{i+m}^-)), J_{i+m}^2(s_{i+m}, x(t_{i+m}^-))), \quad \forall i \in \mathbb{N}. \end{aligned}$$

We recall that $s_m = T$ in the above definition.

3. EXISTENCE AND UNIQUENESS OF PERIODIC SOLUTIONS

By [11, Theorem 3.1], we have

Theorem 3.1. *If the assumptions (A1), (A2), (A3), (A5) and (A6) are satisfied, then the second order problem (2.1) has a unique mild solution.*

Remark 3.2. If $x \in PC(I, X)$ is a T -periodic PC -mild solution of the problem (1.2) then $x(t+T) = x(t)$, $\forall t \geq 0$ by the uniqueness result of Theorem 3.1. On the other hand, if $x_0 \in E$ then $x(t+T) = x(t)$, $\forall t \geq 0$ implies the m -periodicity of (2.2), since

$$x(0) = x_0, \quad x'(0^+) = y_0, \quad x(s_i) = J_i^1(s_i, x(t_i^-)), \quad x'(s_i^-) = J_i^2(s_i, x(t_i^-)), \quad i \in \mathbb{N}.$$

Moreover, if $x_0 \in E$ and $J_i^1(s_i, E) \subset E$ for any $i \in \mathbb{N}$, then $x \in C^1((t_i, t_{i+1}), X)$, $i \in \mathbb{N}$ and there exist $x'(t_i^-)$ and $x'(t_i^+)$, $i \in \mathbb{N}$, so assuming $x'(t_i^-) = x'(t_i)$, we get $x'(t+T) = x'(t)$, $\forall t \geq 0$.

Now we define a Poincaré operator $P : X \times X \rightarrow X \times X$, $P = (\tilde{P}_1, \tilde{P}_2)$ by

$$\begin{aligned}
 \tilde{P}_1(x_0, y_0) &= J_m^1(T, x(t_m^-, x_0)) \\
 (3.1) \quad &= J_m^1\left(T, C(t_m - s_{m-1})(J_{m-1}^1(s_{m-1}, x(t_{m-1}^-, x_0)))\right. \\
 &\quad \left.+ S(t_m - s_{m-1})(J_{m-1}^2(s_{m-1}, x(t_{m-1}^-, x_0)))\right. \\
 &\quad \left.+ \int_{s_{m-1}}^{t_m} S(t_m - s)f(s, x(s))ds\right)
 \end{aligned}$$

and

$$\begin{aligned}
 \tilde{P}_2(x_0, y_0) &= J_m^2(T, x(t_m^-, x_0)) \\
 (3.2) \quad &= J_m^2\left(T, C(t_m - s_{m-1})(J_{m-1}^1(s_{m-1}, x(t_{m-1}^-, x_0)))\right. \\
 &\quad \left.+ S(t_m - s_{m-1})(J_{m-1}^2(s_{m-1}, x(t_{m-1}^-, x_0)))\right. \\
 &\quad \left.+ \int_{s_{m-1}}^{t_m} S(t_m - s)f(s, x(s))ds\right).
 \end{aligned}$$

It is easily to show that fixed points of P defined in (3.1) and (3.2) give rise to a periodic solution to the problem (1.2), i.e., the following result holds.

Lemma 3.3. *The problem (1.2) has a T -periodic PC-mild solution if and only if P has a fixed point.*

We see that P is a composition of the maps:

$$(3.3) \quad P = \mathbb{J}_m \circ P_{m-1} \circ \mathbb{J}_{m-1} \circ P_{m-2} \circ \mathbb{J}_{m-2} \circ \cdots \circ \mathbb{J}_1 \circ P_0,$$

where

$$\begin{aligned}
 \mathbb{J}_i : X &\rightarrow X \times X, \quad \mathbb{J}_i(u) = (J_i^1(s_i, u), J_i^2(s_i, u)), \quad i = 1, 2, \dots, m; \\
 (3.4) \quad P_i(z) &= x(t_{i+1}, z), \quad i = 0, 1, 2, \dots, m-1, \\
 x(t, z) &= C(t - s_i)u + S(t - s_i)v + \int_{s_i}^t S(t - s)f(s, x(s, z))ds, \quad t \in [s_i, t_{i+1}],
 \end{aligned}$$

where we set $z = (u, v) \in X \times X$ and consider a norm $\|z\| = \max\{\|u\|, \|v\|\}$.

Next, from [5, Theorem 1.1], there is $K \geq 1$ and $\omega > 0$ such that $\|C(t)\| \leq Ke^{\omega t}$ for any $t \geq 0$. Then we derive $\|S(t)\| \leq \frac{K}{\omega}e^{\omega t}$. By assumption (A3), we have

$$\begin{aligned}
 (3.5) \quad \|x(t, z)\| &\leq Ke^{\omega(t-s_i)}\|z\| + \frac{K}{\omega}e^{\omega(t-s_i)}\|z\| \\
 &\quad + \frac{KL_f}{\omega} \int_{s_i}^t e^{\omega(t-s)}(1 + \|x(s, z)\|)ds.
 \end{aligned}$$

Setting $y(t, z) = e^{-\omega(t-s_i)}\|x(t, z)\|$, (3.5) is reduced to

$$\begin{aligned} y(t, z) &\leq K \left(1 + \frac{1}{\omega}\right) \|z\| + \frac{KL_f}{\omega} \int_{s_i}^t [e^{-\omega(s-s_i)} + y(s, z)] ds \\ &\leq K \left(1 + \frac{1}{\omega}\right) \|z\| + \frac{KL_f}{\omega^2} + \frac{KL_f}{\omega} \int_{s_i}^t y(s, z) ds. \end{aligned}$$

Applying Gronwall inequality, we get

$$y(t, u) \leq K \left[\left(1 + \frac{1}{\omega}\right) \|z\| + \frac{L_f}{\omega^2} \right] e^{\frac{KL_f(t-s_i)}{\omega}}.$$

Hence

$$(3.6) \quad \|P_i(z)\| = \|x(t_{i+1}, z)\| \leq K \left[\left(1 + \frac{1}{\omega}\right) \|z\| + \frac{L_f}{\omega^2} \right] e^{\left(\frac{KL_f}{\omega} + \omega\right)(t_{i+1}-s_i)}.$$

By assumption (A6) and inequality (3.6), we have

$$\|(\mathbb{J}_i \circ P_{i-1})(z)\| \leq a + L_{J_i} K \left(1 + \frac{1}{\omega}\right) e^{\left(\frac{KL_f}{\omega} + \omega\right)(t_i-s_{i-1})} \|z\|, \quad i = 1, 2, \dots, m,$$

where

$$a = \max_{i=1,2,\dots,m} \left[M_{J_i} + \frac{L_{J_i} K L_f}{\omega^2} e^{\left(\frac{KL_f}{\omega} + \omega\right)(t_i-s_{i-1})} \right]$$

Setting

$$\begin{aligned} b &= K \left(1 + \frac{1}{\omega}\right) \max_{i=1,2,\dots,m} L_{J_i} e^{\left(\frac{KL_f}{\omega} + \omega\right)(t_i-s_{i-1})}, \\ A &= a \sum_{i=0}^{m-1} b^i, \\ \theta &= K^m \left(1 + \frac{1}{\omega}\right)^m \prod_{i=1}^m L_{J_i} e^{\left(\frac{KL_f}{\omega} + \omega\right) \sum_{i=1}^m (t_i-s_{i-1})}, \end{aligned}$$

by an elementary computation, we have

$$\begin{aligned} \|P(z)\| &= \|(\mathbb{J}_m \circ P_{m-1} \circ \mathbb{J}_{m-1} \circ P_{m-2} \circ \mathbb{J}_{m-2} \circ \dots \circ \mathbb{J}_1 \circ P_0)(z)\| \\ &\leq a(1 + b + \dots + b^{m-1}) \\ &\quad + K^m \left(1 + \frac{1}{\omega}\right)^m \prod_{i=1}^m L_{J_i} e^{\left(\frac{KL_f}{\omega} + \omega\right) \sum_{i=1}^m (t_i-s_{i-1})} \|z\| \\ (3.7) \quad &\leq A + \theta \|z\|. \end{aligned}$$

By (3.7), we have the following result.

Lemma 3.4. *All PC-mild solutions of the problem (2.1) corresponding to initial conditions in a bounded subset of X are uniformly bounded.*

Next, we verify that P defined by (3.3) is a continuous and compact operator under our assumptions. If we can show P_i , $i = 0, 1, \dots, m-1$ are continuous and

compact, then we can use the continuity of $\mathbb{J}_i$, $i = 1, 2, \dots, m$ to derive that P is continuous and compact.

Lemma 3.5. *If $(C(t))_{t \geq 0}$ is a compact cosine family, then P is a continuous and compact operator.*

Proof. Suppose $x(\cdot, z_1)$ and $z(\cdot, z_2)$ are the PC -mild solutions defined in (3.4) corresponding to the initial conditions $z_i = (u_i, v_i) \in X \times X$, $i = 1, 2$, respectively. By assumption (A2), we have

$$\begin{aligned} \|x(t, z_1) - x(t, z_2)\| &\leq K e^{\omega(t-s_i)} \|z_1 - z_2\| + \frac{K}{\omega} e^{\omega(t-s_i)} \|z_1 - z_2\| \\ &\quad + \frac{K K_f}{\omega} \int_{s_i}^t e^{\omega(t-s)} \|x(s, z_1) - x(s, z_2)\| ds \\ &\leq K e^{\omega T} \left(1 + \frac{1}{\omega}\right) \|z_1 - z_2\| \\ &\quad + \frac{K K_f e^{\omega T}}{\omega} \int_{s_i}^t \|x(s, z_1) - x(s, z_2)\| ds \\ &\quad t \in [s_i, t_{i+1}], \quad i = 0, 1, \dots, m. \end{aligned}$$

By applying Gronwall inequality, we have

$$(3.8) \quad \|x(t_{i+1}, z_1) - x(t_{i+1}, z_2)\| \leq K e^{\omega T} \left(1 + \frac{1}{\omega}\right) e^{\frac{T K K_f e^{\omega T}}{\omega}} \|z_1 - z_2\|, \\ i = 0, 1, \dots, m-1.$$

Hence, P_i , $i = 0, 1, \dots, m-1$ are continuous operators on $X \times X$.

Now, we shall show that P_i , $i = 0, 1, \dots, m-1$ are compact. Let Γ be a bounded subset of $X \times X$. Then $\Gamma \subset B_r \times B_r$ for a closed ball B_r in X centered at 0 with a radius $r > 0$. Recalling (3.4), we have

$$(3.9) \quad P_i(z) = C(t_{i+1} - s_i)u + S(t_{i+1} - s_i)v + \int_{s_i}^{t_{i+1}} S(t_{i+1} - s)f(s, x(s, z))ds.$$

It is easy to see that $R : [s_i, t_{i+1}] \rightarrow L(X)$ given as $R(s) = S(t_{i+1} - s)$ is continuous and $R(s)$ is compact for any $s \in [s_i, t_{i+1}]$. Then the sets $C(t_{i+1} - s_i)(B_r)$ and $S(t_{i+1} - s_i)(B_r)$ are precompact. Next, for any $n \in \mathbb{N}$, we consider continuous and linear operators $\Lambda, \Lambda_k : \Upsilon_i = C([s_i, t_{i+1}], X) \rightarrow X$, $k = 0, 1, \dots, n-1$ given by

$$\Lambda x = \int_{s_i}^{t_{i+1}} R(s)x(s)ds, \quad \Lambda_k x = \sum_{k=0}^{n-1} R(r_k) \int_{r_k}^{r_{k+1}} x(s)ds$$

for $r_k = \frac{(n-k)s_i}{n} + \frac{kt_{i+1}}{n}$, $k = 0, 1, \dots, n-1$. It is well-known that Υ_i is a Banach space with a norm

$$\|x\|_\infty = \max_{s \in [s_i, t_{i+1}]} \|x(s)\|.$$

By

$$\|\Lambda x - \Lambda_k x\| \leq \|x\|_\infty \sum_{k=0}^{n-1} \int_{r_k}^{r_{k+1}} \|R(s) - R(r_k)\| ds,$$

we get

$$(3.10) \quad \|\Lambda - \Lambda_k\| \leq \sum_{k=0}^{n-1} \int_{r_k}^{r_{k+1}} \|R(s) - R(r_k)\| ds.$$

Since $[s_i, t_{i+1}]$ is compact, then $R : [s_i, t_{i+1}] \rightarrow L(X)$ is uniformly continuous and so (3.10) implies $\Lambda_k \rightarrow \Lambda$ in $L(\Upsilon_i, X)$. On the other hand, since each $x \rightarrow \int_{r_k}^{r_{k+1}} x(s)$ is linear and continuous, so bounded, as a mapping $\Upsilon_i \rightarrow X$ and $R(r_k)$ are compact, we see that each Λ_k is compact. But then Λ is also compact. Next, the Nemytskii operator $F : \Upsilon_i \rightarrow \Upsilon_i$ defined as $(Fx)(s) = f(s, x(s))$ is continuous and bounded, since by (A3), $\|F(x)\|_\infty \leq K_f \|x\|_\infty + M_f$. Hence F maps bounded subsets to bounded ones. Then the mapping $G : \Upsilon_i \rightarrow X$ defined as

$$G(x) = \int_{s_i}^{t_{i+1}} S(t_{i+1} - s) f(s, x(s)) ds$$

is compact and continuous, since $G = \Lambda \circ F$. According to (3.10), P_i is a sum of continuous and compact maps, so it is also continuous and compact. We just presented an alternative way of proving the continuity of P_i .

Finally, we can use the continuity of $\mathbb{J}_i$, $i = 1, 2, \dots, m$ to derive that P is continuous and compact. $\square$

Now we present the main theorems of this paper.

Theorem 3.6. *Assume that (A1)–(A7) are satisfied. If $(C(t))_{t \geq 0}$ is compact and $\theta < 1$, then the equation (1.2) has at least a T -periodic PC-mild solution.*

Proof. By Lemma 3.5, the operator P is continuous and compact. Inequality (3.7) implies $P : B_{\theta_0} \times B_{\theta_0} \rightarrow B_{\theta_0} \times B_{\theta_0}$ for $\theta_0 = \frac{A}{1-\theta}$. Clearly $B_{\theta_0} \times B_{\theta_0}$ is bounded, convex and closed. So by Schauder fixed point theorem, there exist $x_0, y_0 \in X$ such that $P(z_0) = z_0$ for $z_0 = (x_0, y_0)$. By applying Lemma 3.3, we can get a T -periodic PC-mild solution $x(\cdot, z_0)$ of Cauchy problem (2.1) corresponding to the initial conditions $x(0) = x_0$ and $x'(0) = y_0$. Thus, $x(\cdot, z_0)$ is a T -periodic PC-mild solution of the problem (1.2). The proof is done. $\square$

When $(C(t))_{t \geq 0}$ is not compact, then we can still have the following results.

Theorem 3.7. *Assume that (A1)–(A7) are satisfied. If $\mathbb{J}_i$ is compact for some $i \in \{1, 2, \dots, m\}$ and $\theta < 1$, then the equation (1.2) has at least a T -periodic PC-mild solution.*

Proof. The result follows directly from the proof of Theorem 3.6, since now P is still compact. $\square$

Theorem 3.8. *Assume that (A1)–(A7) are satisfied. If $(S(t))_{t \geq 0}$ is compact and $\theta < 1$, then the equation (1.2) has at least a T -periodic PC-mild solution provided that one of the following conditions holds:*

- (a) $J_i^1(s_i, \cdot) : X \rightarrow X$ is compact for some $i \in \{1, 2, \dots, m-1\}$,
- (b) $J_m^1(s_m, \cdot) : X \rightarrow Z$ for a finite dimensional subspace $Z \subset X$.

Proof. Assuming (a), by (3.9) and arguments below it, we see that $P_{i+1} \circ \mathbb{J}_i$ is compact. Hence P is also compact. So the result follows again directly from the proof of Theorem 3.6. Assuming (b), we have $P : Z \times X \rightarrow Z \times X$. Next, by (3.9) and arguments below it, we see that $P_0 : Z \times X \rightarrow X$ is compact. Hence $P : (Z \cap B_{\theta_0}) \times X \rightarrow (Z \cap B_{\theta_0}) \times X$ is also compact. So the result follows from Schauder fixed point theorem. The proof is finished. $\square$

By (3.7) we derive

$$\|P^k(z)\| \leq A \frac{1 - \theta^k}{1 - \theta} + \theta^k \|z\| \leq \theta_0 + \theta^k \|z\|$$

for any $k \in \mathbb{N}_0$ and $z \in X \times X$. This implies that the mild dynamics of (1.2) is dissipative and there is its global compact attractor

$$\mathcal{A} = \bigcap_{k=0}^{\infty} P^k(B_{\theta_0} \times B_{\theta_0}) \subset B_{\theta_0} \times B_{\theta_0}$$

for Theorems 3.6, 3.7 and 3.8(a), while we have

$$\mathcal{A} = \bigcap_{k=0}^{\infty} P^k((Z \cap B_{\theta_0}) \times B_{\theta_0}) \subset (Z \cap B_{\theta_0}) \times B_{\theta_0}$$

for Theorem 3.8(b). Finally, we can just apply Banach fixed point theorem in the general case.

Theorem 3.9. *Assume that (A1)–(A7) are satisfied. If*

$$\Upsilon = K^m \left(1 + \frac{1}{\omega}\right)^m \prod_{i=1}^m K_{J_i} e^{(\frac{KK_f}{\omega} + \omega) \sum_{i=1}^m (t_i - s_{i-1})} < 1,$$

then the equation (1.2) has a unique T -periodic PC-mild solution, which a global attractor.

Proof. By following above arguments to (3.7) and (3.8), we have

$$\|P(z_1) - P(z_2)\| \leq \Upsilon \|z\|.$$

Since $\Upsilon < 1$, the result is a consequence of Banach fixed point theorem. $\square$

4. APPLICATION

We consider the following partial differential equation

$$\begin{aligned}
 \partial_{tt}x(t, y) &= \partial_{yy}x(t, y) + \frac{|x(t, y)|}{1 + |x(t, y)|} + y \sin t, \quad y \in (0, 1), \quad t \in (0, \pi] \\
 x(t, 0) &= x(t, 1) = 0, \quad t \in [0, 2\pi], \\
 x(0, y) &= x_0(y), \quad y \in (0, 1), \\
 \partial_t x(0, y) &= y_0(y), \quad y \in (0, 1), \\
 x(t, y) &= \frac{|x(\pi^-, y)|}{1 + |x(\pi^-, y)|} \sin t, \quad y \in (0, 1), \quad t \in (\pi, 2\pi], \\
 (4.1) \quad \partial_t x(t, y) &= \frac{|x(\pi^-, y)|}{1 + |x(\pi^-, y)|} \cos t, \quad y \in (0, 1), \quad t \in (\pi, 2\pi].
 \end{aligned}$$

The equation (4.1) can be reformulated as the following abstract equation in $X = L^2(0, 1)$:

$$\begin{aligned}
 x''(t) &= Ax(t) + f(t, x(t)), \quad t \in (0, \pi], \\
 x(t) &= J_1^1(t, x(t_i^-)), \quad t \in (\pi, 2\pi], \\
 x'(t) &= J_1^2(t, x(t_i^-)), \quad t \in (\pi, 2\pi], \\
 (4.2) \quad x(0) &= x_0, \quad x'(0) = y_0,
 \end{aligned}$$

where $0 = t_0 = s_0$, $t_1 = \pi$, $s_1 = T = 2\pi$, $x(t)(y) = x(t, y)$, $y \in (0, 1)$, and

$$\begin{aligned}
 Ax &= x_{yy}, \\
 D(A) &= \{x \in X : x_{yy} \in X, x(0) = x(1) = 0\}, \\
 f(t, x)(y) &= \frac{|x(\cdot, y)|}{1 + |x(\cdot, y)|} + y \sin t, \\
 J_1^1(t, x(t_i^-)) &= \frac{|x(\pi^-, y)|}{1 + |x(\pi^-, y)|} \sin t, \\
 J_1^2(t, x(t_i^-)) &= \frac{|x(\pi^-, y)|}{1 + |x(\pi^-, y)|} \cos t.
 \end{aligned}$$

A has an infinite series representation

$$Ax = \sum_{n=1}^{\infty} -n^2(x, x_n)x_n, \quad x \in D(A),$$

where $x_n(y) = \sqrt{2} \sin n\pi y$, $n \in \mathbb{N}$ is an orthonormal set of eigenfunctions of A . Moreover, the operator A is the infinitesimal generator of a strongly continuous cosine family $C(t)_{t \in \mathbb{R}}$ on X which is given by

$$C(t)x = \sum_{n=1}^{\infty} \cos nt(x, x_n)x_n, \quad x \in X,$$

and the associated sine family $S(t)_{t \in \mathbb{R}}$ on X is given by

$$S(t)x = \sum_{n=1}^{\infty} \frac{1}{n} \sin nt(x, x_n)x_n, \quad x \in X.$$

Clearly, $C(t)_{t \in \mathbb{R}}$ is not compact, while $S(t)_{t \in \mathbb{R}}$ is compact. Next, we have $K_f = K_{J_1} = 1$, $L_f = L_{J_1} = 0$ and $M_f = M_{J_1} = 1$. Hence $\theta = 0$. Since $J_1^1(s_1, u) = 0$, we can apply Theorem 3.8(b) with $Z = \{0\}$ to get a 2π -periodic PC -mild solution of (4.1).

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