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PRICING AN INSURANCE PRODUCT THAT INTEGRATES REVERSE MORTGAGE WITH LONG-TERM CARE INSURANCE

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ABSTRACT. We derive in this article the pricing formula for an insurance product that integrates reverse mortgage with long term care. This pricing is based on the principle of balance between the mean gain and mean payment. We compute the expected gain and the expected payment respectively under the continuous and discrete framework. Here, we assume that the dynamics of the housing price is driven by the Black-Scholes model and the interest rate is driven by the Ornstein-Uhlenbeck process. With these assumptions, we present closed-form formulas for the growing perpetuity annuity, the state annuity, and the constant annuity. Furthermore, we discuss the monotonicity property of the annuities, lump sum, and annuity payment factors with respect to the parameters of housing price, interest rate model, and the age of the insured. We present the numerical results for the lump sum, the annuity, and the annuity payment factors, and analyze the sensitivity with respect to the above parameters. We also show that the mean return of housing price has the dominating influence on the lump sum and annuity.

Keywords: Reverse mortgage, Long-term care insurance, Actuarial pricing, Markov model, Vasicek model

1. INTRODUCTION

Reverse Mortgage is an inviting financial lending product offered to any senior citizen who owns a house. In the case of standard home mortgage loan, the borrower repays the loan periodically to the lender (*say*, bank). As the name suggests, the 'reverse mortgage' derives this nomenclature due to the periodic cash flow from the lender (*i.e.*, the financial institution issuing the reverse mortgage) to the borrower (namely, the senior homeowner). The homeowner pledges his property as collateral to

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the particular lending institution in exchange for cash, rather than selling the property for cash. Some special financial institutions, such as the state and local housing agencies, credit unions, insurance companies, and the banks, may offer the reverse mortgage. Any amount that a homeowner can receive depends primarily on the value of the property, the homeowner's age, and the current interest rate. Borrowers can choose one of several types of payment options, such as a lump sum, fixed monthly payment (until death), line of credit, or a combination of payment options. Reverse mortgage has some advantage, including: the reverse mortgage recipient can continue to live in the house; the utilization of the obtained cash is flexible and diverse; the homeowner can defer handing over the house until the homeowner dies; there are no tax requirements for the reverse mortgage borrower. Also, the cash obtained by releasing the borrower's home equity can be used for daily living costs, house maintenance, and long-term care insurance, and other relevant expenses.

Reverse mortgage is normally categorized by law into two categories, *viz.*, collateral reverse mortgage and ownership conversion reserve mortgage. The collateral reverse mortgage can be redeemed and ownership conversion reverse mortgage not. In the case of the collateral reverse mortgage, the borrower is able to redeem the reverse mortgage by repaying the loan principal and accumulated interests through property sale at any time from the mortgage's effective date to due date. Of course, when the reserve mortgage contract is due, the borrower can choose a financial institution to auction off the property to repay loan and due interest. Home Equity Conversion Mortgage System is a typical collateral reverse mortgage in USA. In the case of the ownership conversion reverse mortgage, the borrower enters into a contract with a lending institution to obtain an annuity until his death, and at death the property ownership is transferred to the lender. *Rente Viager* is a typical ownership conversion reverse mortgage in France.

It is for some time now that the funds from the reverse mortgage are being used to pay for long-term care costs. The relevant research mainly discuss qualitatively its feasibility, effectiveness, opportunity, and challenges. Firman (1983) suggests that one should analyze the potential of utilizing home equity conversion mortgage system as a means of supplementing cash income and long-term care insurance. Jacobs and Weissert (1987) estimates that when home equity conversion plan is available, two-thirds of all elderly homeowners could afford an adequate long term care insurance policy. Benejam (1987), Benjamin (1992) and Gibbs (1992) further discuss the utility of home equity conversions to fund long term care for the elderly. Rasmussen *et al.* (1997) presents a more detailed view of reverse mortgage as a financial tool for tapping housing equity for various purposes and at various stages in the life cycle; one of main uses is to fund long-term care. Ahlstrom *et al.* (2004) explains the basic features of reverse mortgage and the long term care insurance. They outline the opportunities and challenges of linking reverse mortgage with the long term care insurance. Stucki (2005) discusses the consumer's attitude toward using reverse mortgages to fund the long term care insurance, product design barriers, and the role of government. Stucki and Group (2006) supports reverse mortgage as a financial instrument to fund the long-term care and shows that it would be beneficial for improving the senior citizen's life after retirement. In addition to the quantitative results, Murtaugh *et al.* (2001) designs a product combining reverse mortgage with long term care insurance. The idea there is to uitilize reverse mortgage to finance the long-term care insurance. Using the data from the 1986 National Mortality Followback Survey in USA, they investigate the premium, annuity, and risk according to the sex, age, and disability. Their research shows that this particular combination of products will give a 21% increases in the potential customers, and hence it is beneficial for the insurer to develop the bundled reverse mortgage and the long-term care insurance.

The risk of integrating the reverse mortgage with long term care insurance involves particularly the housing price risk, interest rate risk, disability risk, and the longevity risk. In order to rationally price this combined product, one must build an appropriate model that takes into account the above risks. In general, the risk of housing price is modeled in two ways. The first one is to fit the time series model based on the historical data of the housing price, as discussed by Nothaft *et al.* (1995), Chinloy *et al.* (1997), Chen *et al.* (2010b), and Li *et al.* (2010). The second one is to assume directly that the dynamics of housing price is driven by a forward stochastic differential equation, as in Bardhan *et al.* (2006), Wang *et al.* (2008), Mizrach (2008), Huang *et al.* (2011), Chen *et al.* (2010a), Lee *et al.* (2012), and Tsay *et al.* (2014).

The literature on classical interest rate model includes: the Dothan (1978) model, Vasicek (1977) model, Cox, Ingersoll and Ross (1985) model, Exponential Vasicek model, Hull and White (1990) model, Black and Karasinski (1991) model, Mercurio and Moraleda (2000) model, the CIR++ model, and the Extended Exponential Vasicek model (Brigo and Mercurio, 2006). Du Pasquier (1912, 1913) introduce a three-state (active, disabled and dead) Markov model to describe the invalid or sickness process, and derive the full differential equations for the transition probabilities. His work lays the foundations for the application of the multi-state Markov model to the long-term care insurance, disability insurance, and critical illness. Fong *et al.* (2015) estimates the transition intensities of the above three-state Markovian model with the generalized linear model based on a large sample of elderly in the USA.

This article is organized as follows. Section 2 presents the models of risk factors. In Section 3, we first design the product that integrates reverse mortgage with long term care insurance, and then derive the pricing model for the bundled product by the principle of expected balance between gain and payment. We also analyze the monotonicity of the lump sum, annuity, and annuity payment factors for the parameters of housing price and interest rate model. Section 4 provides numerical results to examine how the housing price risk, interest rate risk, and longevity risk impact the annuity payment, the expectation of total annuity present value, and the annuity payment factors. Finally, in Section 5 we draw conclusions about our findings.

2. MODELS OF RISK FACTORS

The main risks involved with reverse mortgage, as pointed out by Szymanoski (1994), include longevity risk, interest rate risk, and property value risk. In addition to these risks, the product integrating reverse mortgage with long-term care still faces other risks such as the disability risk. In order to obtain a suitable model to price the combined product, we must first explore how to describe these risk factors that the combined product enforces. In this section we employ the Black-Scholes model to simulate the dynamics of home price, the Ornstein-Uhlenbeck process to drive the instantaneous interest rate, and a three-state Markov chain to model the disability risk and longevity risk.

2.1. **Home Pricing Model.**

We assume that the home price at time t follows the Black-Scholes model

$$dH(t) = H(t)\left[\mu_h dt + \sigma_h dW_h(t)\right], \quad 0 \le t \le T,\ H(0) = H_0, \tag{1}$$

where $H(t)$ is defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_t\}_{t=0}^{T})$ with the natural filtration $\mathcal{F} = \sigma\{\mathcal{F}_t,\ 0 \le t \le T\}$. Here, $\{W_h(t),\ 0 \le t \le T\}$ is a $\mathcal{P}$-standard Brownian motion, μ_h is the annual average return rate of housing price, and σ_h is the annual volatility of housing price, assuming $\sigma_h > 0$.

Applying Itô's formula to $\ln(H(t))$, we have

$$d[\ln(H(t))] = \left(\mu_h - \frac{1}{2}\sigma_h^2\right) dt + \sigma_h dW_h(t). \tag{2}$$

Integrating both sides of the Equation (2) from s to t, we obtain the explicit solution

$$H(t) = H(s)\exp\left\{\left(\mu_h - \frac{1}{2}\sigma_h^2\right)(t-s) + \sigma_h(W_h(t) - W_h(s))\right\},\ 0 \le s \le t \le T.$$

Thus, the conditional mean of $H(t)$ given $\mathcal{F}_s$ is

$$E\left[H(t)|\mathcal{F}_s\right] = H(s)e^{\mu_h(t-s)},\ 0 \le s \le t \le T. \tag{3}$$

In particular, the mean of $H(t)$ is

$$E\left[H(t)\right] = H_0 e^{\mu_h t},\ 0 \le t \le T. \tag{4}$$

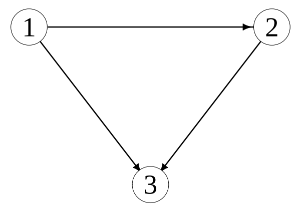

FIGURE 1. The three state Markov model.

2.2. **Interest Rate Model.** We next assume that the instantaneous short-rate dynamics evolves as an Ornstein-Uhlenbeck process with constant coefficients (*i.e.*, the Vasicek model (Vasicek, 1977)). Specifically, in the (complete) filtered probability space $(\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_t\}_{t=0}^T)$, the interest rate process is governed by the following stochastic differential equation

$$dr(t) = \alpha_r(\mu_r - r(t))dt + \sigma_r dW_r(t),\ 0 \le t \le T,\ r(0) = r_0, \tag{5}$$

where $r_0, \alpha_r, \mu_r, \sigma_r$ are positive constants, and $\{W_r(t),\ 0 \le t \le T\}$ is a $\mathcal{P}$-standard Brownian motion, independent of $\{W_h(t),\ 0 \le t \le T\}$.

Applying Itô's formula to $e^{\alpha_r t} r(t)$, we obtain

$$d(e^{\alpha_r t} r(t)) = e^{\alpha_r t}\alpha_r\mu_r dt + e^{\alpha_r t}\sigma_r dW_r(t). \tag{6}$$

Integrating both sides of Equation (6) over (s,t], we arrive at

$$r(t) = e^{-\alpha_r(t-s)} r(s) + \mu_r(1 - e^{-\alpha_r(t-s)}) + \sigma_r \int_s^t e^{-\alpha_r(t-u)} dW_r(u),\ 0 \le s \le t \le T. \tag{7}$$

With some trivial computations, we have

$$\begin{aligned} &E\left[\exp\left(-\int_0^t r(s)ds\right)\right] \\ &= \exp\left[\left(\frac{\sigma_r^2}{2\alpha_r^2} - \mu_r\right)t + \frac{1}{\alpha_r}(\mu_r - r_0)(1 - e^{-\alpha_r t}) + \frac{\sigma_r^2}{4\alpha_r^3}\left[1 - (2 - e^{-\alpha_r t})^2\right]\right]. \end{aligned} \tag{8}$$

For the derivation of Eq. (8) we refer to Norberg (2004).

2.3. **Temporally Inhomogeneous Markov Chain Model.** We designate $t = 0$ to be the time at which a policy combining reverse mortgage with long-term care is signed. We assume that the process $\{\xi_t,\ t \ge 0\}$ is a temporally inhomogeneous Markov chain taking values in the state space $S = \{1, 2, 3\}$, where $\{\xi_t,\ t \ge 0\}$ is defined on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_t\}_{t=0}^T)$, independent of $\{W_h(t),\ 0 \le t \le T\}$ and $\{W_r(t),\ 0 \le t \le T\}$. Here, the states $1, 2$, and 3 respectively represent that the insured is *not-disabled, disabled, and dead.* Clearly, the states 1 and 2 are transient and the state 3 is absorbing. Herein and after, ξ_t denotes the status of the insured at time $t \in [0, T)$. Then, the events $\{\xi_t = 1\}$, $\{\xi_t = 2\}$ and $\{\xi_t = 3\}$

respectively represent that the insured is not-disabled, disabled, and dead at time t. We disregard the possibility of recovery from disability to health. Hence, the directed line from 2 to 1 does not exist in Figure 1. The transition probabilities and intensities are denoted respectively by

$$p_{ij}(s,t) = P(\xi_t = j | \xi_s = i), \quad s \le t, \ i,j \in S,$$
$$\lambda_{ij}(t) = \lim_{h \to 0} \frac{p_{ij}(t,t+h)}{h}, \quad i \ne j.$$

With the Chapman-Kolmogorov equation, the transition probabilities and intensities satisfy the following system of differential equations:

$$\frac{d}{dt}p_{11}(s,t) = -p_{11}(s,t)[\lambda_{12}(t) + \lambda_{13}(t)], \tag{9}$$
$$\frac{d}{dt}p_{12}(s,t) = p_{11}(s,t)\lambda_{12}(t) - p_{12}(s,t)\lambda_{23}(t), \tag{10}$$
$$\frac{d}{dt}p_{13}(s,t) = p_{11}(s,t)\lambda_{13}(t) + p_{12}(s,t)\lambda_{23}(t),$$
$$\frac{d}{dt}p_{22}(s,t) = -p_{22}(s,t)\lambda_{23}(t), \tag{11}$$
$$\frac{d}{dt}p_{23}(s,t) = p_{22}(s,t)\lambda_{23}(t).$$

Using the boundary conditions $p_{ii}(s,s) = 1$ $(i = 1,2)$ and $p_{ij}(s,s) = 0$ $(i \ne j)$ to Equations (9) and (11), we obtain

$$p_{11}(s,t) = \exp\left[-\int_s^t (\lambda_{12}(u) + \lambda_{13}(u))\, du\right], \tag{12}$$
$$p_{22}(s,t) = \exp\left[-\int_s^t \lambda_{23}(u)\, du\right]. \tag{13}$$

We solve explicitly for $p_{12}(s,t)$ in two steps using Equation (10). First solve

$$\frac{d}{dt}p_{12}(s,t) = -p_{12}(s,t)\lambda_{23}(t),$$

and then utilize the method of variation of constants to solve Equation (10). We then obtain

$$p_{12}(s,t) = \int_s^t p_{11}(s,u)\lambda_{12}(u)p_{22}(u,t)\, du.$$

It is clear that

$$p_{13}(s,t) = 1 - p_{11}(s,t) - p_{12}(s,t),$$
$$p_{23}(s,t) = 1 - p_{22}(s,t).$$

Let L be the limit of age in the Life Table, x_0 be the age of the insured at the start of the contract $(t = 0)$, and $T \doteq L - x_0$ be the insured's maximum future lifetime. Let τ_i, $(i = 1,2)$, denote the sojourn time in state i. Since the states 1 and 2 are strictly transient, τ_1 will be the time of first exit from state 1, and $\tau_1 + \tau_2$ is the time that the insured dies. The following Proposition 1 presents the density function for τ_1 and the joint density function for (τ_1, τ_2).

Proposition 1 *Assume that the time-inhomogeneous Markov chain* $\{\xi_t,\ t \geq 0\}$ *is a separable process. Then,* **(1)** *the density function for* τ_1 *is*

$$f_{\tau_1}(x) = \begin{cases} \exp\left[-\int_0^x (\lambda_{12}(u) + \lambda_{13}(u))du\right] (\lambda_{12}(x) + \lambda_{13}(x))\,,\ 0 < x < T, \\ 0,\ \textit{otherwise}, \end{cases} \tag{14}$$

(2) *the conditional density for* τ_2 *given* $\{\tau_1 = x\}$, *is*

$$f_{\tau_2|\tau_1 = x}(y) = \begin{cases} \exp\left[-\int_x^{x+y} \lambda_{23}(u)du\right] \lambda_{23}(x+y),\ 0 < y < T - x, \\ 0,\ \textit{otherwise}, \end{cases} \tag{15}$$

and **(3)** *the joint probability density function for* (τ_1, τ_2) *is given by*

$$f(x,y) = \begin{cases} p_{11}(0,x)p_{22}(x,x+y)\left[\lambda_{12}(x) + \lambda_{13}(x)\right]\lambda_{23}(x+y),\ (x,y) \in D, \\ 0,\ \textit{otherwise}, \end{cases} \tag{16}$$

where

$$\begin{aligned} p_{11}(0,x) &= \exp\left[-\int_0^x (\lambda_{12}(u) + \lambda_{13}(u))du\right] \\ p_{22}(x, x+y) &= \exp\left[-\int_x^{x+y} \lambda_{23}(u)du\right], \end{aligned}$$

are as in Equation (12) *and Equation* (13), and the set D is given by

$$D := \{(x,y) | 0 < x < T, 0 < y < T - x\}.$$

Proof: From the separability assumption and Markov property we obtain, for any $t > 0$, $s > 0$, $i = 1, 2$, and the separability set $R = \{\frac{k}{2^n}, k, n = 0, 1 \ldots\}$, that

$$\begin{aligned} &P(\xi_{s+u} = i,\ 0 \leq u \leq t \mid \xi_s = i) \\ &\qquad = \lim_{n\to\infty} P(\xi_{s+\frac{kt}{2^n}} = i,\ 0 \leq k \leq 2^n \mid \xi_s = i) \\ &\qquad = \lim_{n\to\infty} \prod_{k=0}^{2^n-1} P(\xi_{s+\frac{(k+1)t}{2^n}} = i \mid \xi_{s+\frac{kt}{2^n}} = i). \end{aligned} \tag{17}$$

Setting $i = 1$, $s = 0$, $t = x$, and $0 < x < T$ in Equation (17), we obtain from Equation (12) that

$$\begin{aligned} &P(\xi_u = 1, 0 \leq u \leq x \mid \xi_0 = 1) \\ &\quad = \lim_{n\to\infty} \prod_{k=0}^{2^n-1} P(\xi_{\frac{(k+1)x}{2^n}} = 1 \mid \xi_{\frac{kx}{2^n}} = 1) \\ &\quad = \lim_{n\to\infty} \prod_{k=0}^{2^n-1} \exp\left[-\int_{\frac{kx}{2^n}}^{\frac{(k+1)x}{2^n}} (\lambda_{12}(u) + \lambda_{13}(u))du\right] \\ &\quad = \exp\left[-\int_0^x (\lambda_{12}(u) + \lambda_{13}(u))du\right] \end{aligned}$$

$$= p_{11}(0, x). \tag{18}$$

From Relations (12), (18), and

$$\begin{aligned}
\{\tau_1 > x\} &\Leftrightarrow \{\xi_u = 1,\ 0 \le \mu \le x \mid \xi_0 = 1\}, \\
F_{\tau_1}(x) &= P(\tau_1 \le x) = 1 - P(\tau_1 > x),
\end{aligned}$$

we obtain the density function (14) for τ_1. This proves Part 1.

Taking $i = 2$, $s = \tau_1 = x$, $t = y$, and $0 < y < T - \tau_1$ in Equation (17) and using Equation (13), we have

$$\begin{aligned}
&P(\xi_{\tau_1+u} = 2,\ 0 \le u \le y \mid \xi_{\tau_1} = 2, \tau_1 = x) \\
&\quad = \lim_{n\to\infty} \prod_{k=0}^{2^n-1} P(\xi_{x+\frac{(k+1)y}{2^n}} = 2 \mid \xi_{x+\frac{ky}{2^n}} = 2) \\
&\quad = \lim_{n\to\infty} \prod_{k=0}^{2^n-1} \exp\left[-\int_{x+\frac{ky}{2^n}}^{x+\frac{(k+1)y}{2^n}} \lambda_{23}(u)du\right] \\
&\quad = \exp\left[-\int_x^{x+y} \lambda_{23}(u)du\right] \\
&\quad = p_{22}(x, x+y). \qquad (19)
\end{aligned}$$

Now, Relation (15) results from Relations (13), (19), and

$$\begin{aligned}
\{\tau_2 > y \mid \tau_1 = x\} &\Leftrightarrow \{\xi_{\tau_1+u} = 2, 0 \le u \le y \mid \xi_{\tau_1} = 2, \tau_1 = x\}, \\
F_{\tau_2|\tau_1=x}(y) &= P(\tau_2 \le y \mid \tau_1 = x) = 1 - P(\tau_2 > y \mid \tau_1 = x).
\end{aligned}$$

Also, the joint probability density function for (τ_1, τ_2) is

$$f(x, y) = f_{\tau_1}(x) f_{\tau_2|\tau_1=x}(y \mid x).$$

Remarks 1 (1.) From Relations (14) and (15), we notice that the sojourn times τ_1 and τ_2 do not follow the exponential distribution. This is different from that of the time-homogeneous Markov chain case.

(2.) From Relations (18) and (19), we get

$$\begin{aligned}
P(\xi_u = 1,\ 0 \le u \le t \mid \xi_0 = 1) &= P(\xi_t = 1 \mid \xi_0 = 1), \\
P(\xi_{\tau_1+u} = 2,\ 0 \le u \le t \mid \xi_{\tau_1} = 2) &= P(\xi_{\tau_1+t} = 2 \mid \xi_{\tau_1} = 2).
\end{aligned}$$

This implies that the occupancy probabilities for states 1 and 2 coincide with their respective transition probabilities.

3. PRICING THE INSURANCE PRODUCT INTEGRATING REVERSE MORTGAGE WITH LONG-TERM CARE

In this section, we first design an insurance product integrating reverse mortgage with long-term care. We then build the actuarial pricing model for the combined product based on the principle of balance between expected gain and expected payment. We also present the closed-form solutions of pricing models under a certain assumptions. Finally, we analyze the monotonicity of the lump sum, annuity, and annuity payment factors with respect to the parameters of home price and interest rate models.

3.1. **Integrating Reverse Mortgage with Long Term Care.** The bundled 'reverse mortgage - long term care' contract that we design has the following basic features:

(I) The insurer starts the payments of annuity to the insured at the beginning of the year following the signing of the contract. The annuity payment is terminated upon the death of the insured. More precisely, had the insured survived through the k-th year, $(k \geq 1)$, the insurer would have paid the annuity sums $A_1,\ A_2, \ldots, A_k$ to the insured at the beginning of the second, third, $\ldots$, $(k+1)$-st years, respectively.

(II) At time τ_1, the insurer will take over the insured's mortgaged property, sell it in the market, and keep all the proceeds from the sale of the mortgaged property. If the insured became disabled, the insured would be requested to move into the nursing home to benefit from the long term care service. If the insured died, the integrated 'reverse mortgage - long term care' contract would automatically terminate.

The Feature (I) above implies that the insurer will pay annuity to the insured from the start of the year following the signing of the contract, and this payment continues until the insured dies. Moreover, the cash flow going to the insured need not be a fixed amount. For instance, let $\tau_1 = 2.3$ years and $\tau_2 = 1.6$ years. This means that the insured first lives in his/her home and claims two cash payments $A_1,\ A_2$, one at the beginning of the second year and the other at the beginning of the third year of the contract, respectively. At time $\tau_1 = 2.3$, the insured leaves home (*i.e.* leaves State 1) and moves into the nursing home (*i.e.* State 2). The insured lives in the nursing home for 1.6 years and claims the one-time cash payment of A_3 at the beginning of fourth year.

The Feature (II) implies the following. When the insured leaves home (State 1) to enter into the nursing home (State 2) or dies (State 3), the insurer will take over the insured's mortgaged property and sell it. The cash that is acquired from the sale of the insured's house is used to repay loan (including annuity and accumulated interests) that the insured owes to the insurer. Thus, it provides capital reserve for the insurer in future. Compared with the taking-over of the mortgaged property after the insured dies, the terms of the above contract integrating reverse mortgage with long term care stipulates that the insured's mortgaged home equity is assumed by the insurer when the insured either moves into the nursing home or dies (whichever happens first).

3.2. **Actuarial Pricing Model.** We assume that we are in the perfectly competitive market. We price the 'reverse mortgage - long term care' contract by the principle of balance between expected gain and expected payment (*i.e.* the expected discounted present value of future sale of the pledged property is the same as the expected discounted present value of cash flow that the insurer pays).

At time τ_1, ($\tau_1 \geq 0$), the insurer takes over the insured's mortgaged property, and sells it at time $\tau_1 + t_0$ where $t_0 \geq 0$ is the delay time between the insurer taking over the mortgaged property and the sale of the mortgaged property. We assume that t_0 is fixed and not a random variable. Then the expectation of discounted present value of the sale price of the morgaged property (i.e., the insurer's expected gain) is

$$E\left[H(\tau_1 + t_0)v_{\tau_1+t_0}\right], \tag{20}$$

where $H(t)$ is the value of the mortgaged property at time t given by the stochastic differential equation (1), and v_t is the discount factor at time t given by the Equation

$$v_t := \exp\left(-\int_0^t r(s)ds\right), \tag{21}$$

and $r(s)$ is the interest rate given by the stochastic differential Equation (5).

The expectation of discounted present value of the insured's annuities (i.e., the insurer's expected payment) is

$$E\left[\mathbf{1}_{\{1\leq\tau_1<T\}}\sum_{i=1}^{[\tau_1]}A_i v_i + \mathbf{1}_{\{\xi_0=1,\xi_{\tau_1}=2\}}\mathbf{1}_{S_0\bigcap\overline{S}_1}\sum_{i=[\tau_1]+1}^{[\tau_1+\tau_2]}A_i v_i\right], \tag{22}$$

where $\mathbf{1}_{S_0\bigcap\overline{S}_1}$ is the indicator function of the event $S_0\bigcap\overline{S}_1$ in which

$$S_0 := \{0 \leq \tau_1 < T, 0 \leq \tau_2 < T, 0 \leq \tau_1 + \tau_2 < T\},$$

$$S_1 := \bigcup_{i=0}^{T-1}\{[\tau_1] = i, [\tau_2] = 0, [\tau_1 + \tau_2] = i\},$$

$\overline{S}_1$ being the complementary set of S_1, and the function $[x]$ returns the largest integer not greater than x.

Then, the principle of balance between expected gain and expected payment yields

(23)
$$E\left[H(\tau_1+t_0)v_{\tau_1+t_0}\right] = E\left[\mathbf{1}_{\{1\le\tau_1<T\}}\sum_{i=1}^{[\tau_1]}A_iv_i + \mathbf{1}_{\{\xi_0=1,\xi_{\tau_1}=2\}}\mathbf{1}_{S_0\bigcap\overline{S}_1}\sum_{i=[\tau_1]+1}^{[\tau_1+\tau_2]}A_iv_i\right].$$

We shall now proceed to derive the explicit solution for annuities. Though the explicit solution of annuity payment is difficult to obtain from the Equation (23), we can obtain the explicit solution under suitable conditions.

Proposition 2 *Assume that the dynamics of home price follows the Black-Scholes model given by the Equation* (1)*, and the instantaneous short interest rate is governed by the Equation* (5)*. Then the annuity payments* A_k $(k=1,2,\ldots,T-1)$ *satisfy the following pricing equation*

$$(24)\quad \int_0^T G(x+t_0)F(x+t_0)f_{\tau_1}(x)dx = \sum_{k=1}^{T-1}A_kF(k)P_1(k) + \sum_{i=0}^{T-2}\sum_{k=i+1}^{T-1}A_kF(k)P_2(i,k),$$

where $f_{\tau_1}(x)$ *is given by* (14)*, and*

$$(25)\qquad G(x+t_0) = H_0\ \exp[\mu_h(x+t_0)],$$

$$\begin{aligned}F(x+t_0) \;&=\; \exp\left\{\left(\frac{\sigma_r^2}{2\alpha_r^2}-\mu_r\right)(x+t_0) + \frac{1}{\alpha_r}(\mu_r-r_0)\left[1-e^{-\alpha_r(x+t_0)}\right]\right.\\ &\left.\qquad +\frac{\sigma_r^2}{4\alpha_r^3}\left[1-(2-e^{-\alpha_r(x+t_0)})^2\right]\right\},\end{aligned}\qquad(26)$$

(27)
$$F(k) = \exp\left\{\left(\frac{\sigma_r^2}{2\alpha_r^2}-\mu_r\right)k + \frac{1}{\alpha_r}(\mu_r-r_0)(1-e^{-\alpha_rk}) + \frac{\sigma_r^2}{4\alpha_r^3}\left[1-(2-e^{-\alpha_rk})^2\right]\right\},$$

$$(28)\quad P_1(k) = \exp\left\{-\int_0^k\left[\lambda_{12}(u)+\lambda_{13}(u)\right]du\right\} - \exp\left\{-\int_0^T\left[\lambda_{12}(u)+\lambda_{13}(u)\right]du\right\},$$

$$(29)\qquad P_2(i,k) = \int_i^{i+1}\exp\left[-\int_x^k\lambda_{23}(u)du\right]p_{12}(0,x)f_{\tau_1}(x)dx,$$

$$(30)\qquad p_{12}(0,x) = \int_0^x p_{11}(0,u)\lambda_{12}(u)p_{22}(u,x)\ du.$$

Here, $p_{11}(0,x)$ *and* $p_{22}(x,T)$ *are given by Equation* (12) *and Equation* (13)*, respectively.*

Proof: Let $f_{\tau_1}(x)$ denote the probability density function for τ_1. Noting that $H(t), r(t)$ and τ_1 are independent, we get

$$E\left[H(\tau_1+t_0)v_{\tau_1+t_0}\right]$$

$$
\begin{aligned}
&= E\left[H(\tau_1+t_0)\exp\left(-\int_0^{\tau_1+t_0} r(s)ds\right)\right] \\
&= \int_0^T E\left[H(x+t_0)\right] E\left[\exp\left(-\int_0^{x+t_0} r(s)ds\right)\right] f_{\tau_1}(x)dx \\
&= \int_0^T G(x+t_0)F(x+t_0)f_{\tau_1}(x)dx, \qquad (31)
\end{aligned}
$$

The Equations (4) and (8), characterize $G(x+t_0)$, and $F(x+t_0)$, respectively, as corresponding Equations (25) and (26). From the independence of $r(t)$ and τ_1, we have

$$
\begin{aligned}
E\left[\mathbf{1}_{\{1\le\tau_1<T\}}\sum_{k=1}^{[\tau_1]} A_k v_k\right] &= E\left[\sum_{i=1}^{T-1}\sum_{k=1}^{i} A_k v_k \mathbf{1}_{\{[\tau_1]=i\}}\right] \\
&= \sum_{i=1}^{T-1}\sum_{k=1}^{i} A_k E\left[v_k\right] P([\tau_1]=i) \\
&= \sum_{k=1}^{T-1} A_k F(k) P(k\le\tau_1<T) \\
&= \sum_{k=1}^{T-1} A_k F(k) P_1(k) \qquad (32)
\end{aligned}
$$

where $F(k)$ is characterized as (27) by Equation (8). Recalling that the probability density function for τ_1 is given by the Relation (14), we get the Equation (28).

Noting that $r(t)$ and (τ_1,τ_2,ξ_t) are independent, we have

$$
\begin{aligned}
&E\left[\mathbf{1}_{S_0\bigcap\overline{S}_1}\mathbf{1}_{\{\xi_0=1,\xi_{\tau_1}=2\}}\sum_{l=[\tau_1]+1}^{[\tau_1+\tau_2]} A_l v_l\right] \\
&= E\left[\sum_{i=0}^{T-2}\sum_{k=i+1}^{T-1}\sum_{l=i+1}^{k} A_l v_l \mathbf{1}_{\{[\tau_1]=i,[\tau_1+\tau_2]=k\}}\mathbf{1}_{\{\xi_0=1,\xi_{\tau_1}=2\}}\right] \\
&= E\left[\sum_{i=0}^{T-2}\sum_{k=i+1}^{T-1} A_k v_k \mathbf{1}_{\{[\tau_1]=i,\tau_1+\tau_2\ge k\}}\mathbf{1}_{\{\xi_0=1,\xi_{\tau_1}=2\}}\right] \\
&= \sum_{i=0}^{T-2}\sum_{k=i+1}^{T-1} A_k E(v_k) E\left[\mathbf{1}_{\{i\le\tau_1<i+1,\tau_1+\tau_2\ge k\}}\mathbf{1}_{\{\xi_0=1,\xi_{\tau_1}=2\}}\right] \\
&= \sum_{i=0}^{T-2}\sum_{k=i+1}^{T-1} A_k F(k)\int_i^{i+1}\exp\left[-\int_x^k \lambda_{23}(u)du\right] p_{12}(0,x) f_{\tau_1}(x)dx. \qquad (33)
\end{aligned}
$$

Thus, Proposition 2 is proved.

The following Propositions 3, 4 and 5 are special cases of the Proposition 2. They present the pricing formulas for the growing (decreasing) perpetuity annuity, the state annuity, and the level annuity, respectively.

Proposition 3 *The payments for the growing (decreasing) perpetuity annuity are characterized as follows. At the beginning of period* $(k+1)$*, the annuity payment is* $A_k := A_0 + d \cdot k,\ k = 1, 2, \ldots, n$ *with* A_0 *and* d *positive constants (as the insured is alive). Here,* A_0 *and* d *are determined by the simultaneous equations*

$$A_0 = \frac{\widetilde{G} - (\widetilde{F}_3 + \widetilde{F}_4) \cdot d}{\widetilde{F}_1 + \widetilde{F}_2}, \tag{34}$$

$$d = \frac{\widetilde{G} - (\widetilde{F}_1 + \widetilde{F}_2) \cdot A_0}{\widetilde{F}_3 + \widetilde{F}_4}, \tag{35}$$

where

$$\widetilde{G} := \int_0^T G(x+t_0)F(x+t_0)f_{\tau_1}(x)dx, \tag{36}$$

$$\widetilde{F}_1 := \sum_{k=1}^{T-1} F(k)P_1(k), \qquad \widetilde{F}_2 := \sum_{i=0}^{T-2}\sum_{k=i+1}^{T-1} F(k)P_2(i,k), \tag{37}$$

$$\widetilde{F}_3 := \sum_{k=1}^{T-1} kF(k)P_1(k), \quad \widetilde{F}_4 := \sum_{i=0}^{T-2}\sum_{k=i+1}^{T-1} kF(k)P_2(i,k) \tag{38}$$

and $G(x+t_0), F(x+t_0), F(k), P_1(k)$ *and* $P_2(i,k)$ *are given by Relations* (25)–(29)*, respectively.*

Proposition 4 *In the case of state annuity, let* B *stand for the annuity payment when the insured continues to live at home and let* λB $(\lambda > 0)$ *as the payment after the insured enters the nursing home, i.e.*

$$A_k = \begin{cases} B,\ 1 \le k \le [\tau_1], \\ \lambda B,\ [\tau_1] + 1 \le k < T. \end{cases} \tag{39}$$

Then

$$B = \frac{\widetilde{G}}{\widetilde{F}_1 + \lambda \cdot \widetilde{F}_2}, \tag{40}$$

where $\widetilde{G}$*,* $\widetilde{F}_1$ *and* $\widetilde{F}_2$ *are the same as those in Proposition* 3*.*

Proposition 5 *For the level annuity, the fixed amount* A *of annuity is paid during the whole insurance period and is given by*

$$A = \frac{\widetilde{G}}{\widetilde{F}_1 + \widetilde{F}_2}, \tag{41}$$

where $\widetilde{G}$*,* $\widetilde{F}_1$ *and* $\widetilde{F}_2$ *are the same as those in Proposition* 3*.*

The expected discounted present value of housing price $\widetilde{G}$ is the average lump sum that the insured can claim at time $t = 0$; we shall call it the *lump sum* herein and hereafter. It is easy to see that $\widetilde{F}_1$, $\widetilde{F}_2$, $\widetilde{F}_3$ and $\widetilde{F}_4$ can affect the amount of each annuity payment, and therefore we shall call them the *annuity payment factors.* The following Propositions 6–8 analyze how the annuity payments, lump sum, and annuity payment factors vary with the parameters (μ_h, σ_h, t_0, r_0, μ_r, σ_r) involved in the home pricing model and the interest rate model.

Proposition 6 *The annuity payment A, the lump sum $\widetilde{G}$, and the annuity payment factors $\widetilde{F}_i$ $(i = 1, 2, 3, 4)$ have the following properties:*

1. Parameter μ_h: **(a)** *The lump sum $\widetilde{G}$ is an increasing function of the mean return of house price μ_h.* **(b)** *The annuity payment factors $\widetilde{F}_1$, $\widetilde{F}_2$, $\widetilde{F}_3$ and $\widetilde{F}_4$ are totally independent of μ_h.* **(c)** *The A_0 and d in Proposition 3, the annuity payment B in Proposition 4, and the level annuity payment A in Proposition 5 are increasing functions of μ_h.*

2. Parameter σ_h: **(a)** *The lump sum $\widetilde{G}$ and the annuity payment factors $\widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_3, \widetilde{F}_4$ are completely independent of the volatility σ_h of the house price.* **(b)** *The A_0 and d in Proposition 3, the annuity payment B in Proposition 4, the level annuity payment A in Proposition 5 also do not dependent on σ_h.*

Proof: Noting that $F(x + t_0)$ and $f_{\tau_1}(x)$ do not depend on μ_h, the partial derivative of the integrand in the definition of $\widetilde{G}$, (see Relation (36)), is

$$\frac{\partial \left[G(x+t_0)F(x+t_0)f_{\tau_1}(x)\right]}{\partial \mu_h} = G(x+t_0)(x+t_0)F(x+t_0)f_{\tau_1}(x),$$

Since $G(x + t_0) > 0, F(x + t_0) > 0, f_{\tau_1}(x) \geq 0$ and $t_0 \geq 0$, we have

$$\frac{\partial \left[G(x+t_0)F(x+t_0)f_{\tau_1}(x)\right]}{\partial \mu_h} \geq 0, \quad (0 \leq x \leq T).$$

This implies that the lump sum $\widetilde{G}$ is the increasing function of μ_h.

From the definitions of the annuity payment factors $\widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_3, \widetilde{F}_4$ (see Relations (37) and (38)), we note that these annuity payment factors are independent of μ_h.

Furthermore, from the Equations (34), (35), (40) and (41), we have the A_0 and d in Proposition 3, the annuity payment B in Proposition 4, and the level annuity payment A in Proposition 5 are increasing functions of μ_h. Thus, Part 1 of the proposition is proved.

From the definitions of $\widetilde{G}$, $\widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_3$ and $\widetilde{F}_4$ it is easy to see that they have nothing to do with σ_h. Part 2 of the proposition is now obtained from the Equations (34), (35), (40) and (41).

Proposition 7 Delay time t_0 between taking over of the mortgaged property and the sale of that property:

(a) *The annuity payment factors $\widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_3, \widetilde{F}_4$ do not depend on t_0.*

(b) *Now set*

$$z_1 = -\frac{\alpha_r^2}{\sigma_r^2}\left[\mu_r - r_0 - \frac{\sigma_r^2}{\alpha_r^2} + \sqrt{(\mu_r - r_0)^2 - \frac{2\sigma_r^2}{\alpha_r^2}(\mu_h - r_0)}\,\right], \tag{42}$$

and

$$z_2 = -\frac{\alpha_r^2}{\sigma_r^2}\left[\mu_r - r_0 - \frac{\sigma_r^2}{\alpha_r^2} - \sqrt{(\mu_r - r_0)^2 - \frac{2\sigma_r^2}{\alpha_r^2}(\mu_h - r_0)}\,\right]. \tag{43}$$

(b-1) *In the case of any one of the following conditions*

$$\begin{aligned}(\mu_r - r_0)^2 &\leq \frac{2\sigma_r^2}{\alpha_r^2}(\mu_h - r_0),\\ (\mu_r - r_0)^2 &\geq \frac{2\sigma_r^2}{\alpha_r^2}(\mu_h - r_0),\ \alpha_r > 0, z_1 \geq 1,\\ (\mu_r - r_0)^2 &\geq \frac{2\sigma_r^2}{\alpha_r^2}(\mu_h - r_0),\ \alpha_r > 0,\ z_2 \leq 0,\end{aligned}$$

the lump sum $\widetilde{G}$ is an increasing function of t_0. Also, the A_0 and d in Proposition 3, the annuity payment B in Proposition 4, and the level annuity payment A in Proposition 5 are increasing functions of t_0.

(b-2) *If*

$$(\mu_r - r_0)^2 \geq \frac{2\sigma_r^2}{\alpha_r^2}(\mu_h - r_0), \quad \alpha_r > 0,\ z_1 \leq 0,\ z_2 \geq 1, \tag{44}$$

holds then the lump sum $\widetilde{G}$ is a decreasing function of t_0. The A_0 and d in Proposition 3, the annuity payment B in Proposition 4, and the level annuity payment A in proposition 5 are decreasing functions of t_0.

Proof: Putting

$$g_1(z) := \frac{\sigma_r^2}{2\alpha_r^2}z^2 + \left(\mu_r - r_0 - \frac{\sigma_r^2}{\alpha_r^2}\right)z + \left(\mu_h + \frac{\sigma_r^2}{2\alpha_r^2} - \mu_r\right), \quad (-\infty < z < +\infty),$$

the minmium of $g_1(z)$ is

$$\left[\frac{2\sigma_r^2}{\alpha_r^2}(\mu_h - r_0) - (\mu_r - r_0)^2\right]\frac{\alpha_r^2}{2\sigma_r^2}.$$

Noting that

$$\begin{aligned}\frac{\partial G(x+t_0)}{\partial t_0} &= G(x+t_0)\mu_h,\\ \frac{\partial F(x+t_0)}{\partial t_0} &= F(x+t_0)\left[\frac{\sigma_r^2}{2\alpha_r^2} - \mu_r + \left(\mu_r - r_0 - \frac{\sigma_r^2}{\alpha_r^2}\right)\exp(-\alpha_r(x+t_0))\right.\\ &\qquad \left. + \frac{\sigma_r^2}{2\alpha_r^2}\exp(-2\alpha_r(x+t_0))\right],\end{aligned}$$

we have

$$\frac{\partial\left[G(x+t_0)F(x+t_0)f_{\tau_1}(x)\right]}{\partial t_0} = g_1(e^{-\alpha_r(x+t_0)})F(x+t_0)G(x+t_0)f_{\tau_1}(x),$$

where

$$g_1(e^{-\alpha_r(x+t_0)}) = \mu_h + \frac{\sigma_r^2}{2\alpha_r^2} - \mu_r + \left(\mu_r - r_0 - \frac{\sigma_r^2}{\alpha_r^2}\right)e^{-\alpha_r(x+t_0)} + \frac{\sigma_r^2}{2\alpha_r^2}e^{-2\alpha_r(x+t_0)}.$$

If the condition $\frac{2\sigma_r^2}{\alpha_r^2}(\mu_h - r_0) - (\mu_r - r_0)^2 \geq 0$ holds, we then have $g_1(z) \geq 0$, and thus $\widetilde{G}$ is the increasing function of t_0.

Recall the definitions of z_1 and z_2 given above by the Relations (42) and (43), respectively. Now, if the condition $\frac{2\sigma_r^2(\mu_h - r_0)}{\alpha_r^2} - (\mu_r - r_0)^2 \leq 0$ holds, then $g_1(z_i) = 0,\ i = 1, 2$. Moreover, it is obvious that $0 < e^{-\alpha_r(x+t_0)} \leq 1$ in case of $\alpha_r > 0$ and $x + t_0 \geq 0$. Thus the lump sum $\widetilde{G}$ is an decreasing function of t_0 if the condition (44) holds. One similarly obtains the rest of the proposition.

The following Proposition 8 analyzes how the lump sum and annuity payment factors vary with the parameters involved in the interest rate model.

Proposition 8 *The lump sum $\widetilde{G}$ and annuity payment factors $\widetilde{F}_i$ $(i = 1, 2, 3, 4)$ have the following properties:*

1. Parameter r_0: *If $\alpha_r \neq 0$, then $\widetilde{G}, \widetilde{F}_1, \widetilde{F}_2,\ \widetilde{F}_3$ and $\widetilde{F}_4$ are decreasing functions of r_0.*

2. Parameter μ_r: *If $\alpha_r > 0$, then $\widetilde{G}, \widetilde{F}_1, \widetilde{F}_2,\ \widetilde{F}_3$ and $\widetilde{F}_4$ are decreasing functions of μ_r. If the opposite case $\alpha_r < 0$ holds, then $\widetilde{G}, \widetilde{F}_i,\ i = 1, 2, 3, 4,$ are increasing functions of μ_r.*

3. Parameter σ_r: *If $\alpha_r \neq 0,\ \sigma_r > 0$, then $\widetilde{G}, \widetilde{F}_i,\ i = 1, 2, 3, 4,$ are increasing functions of σ_r.*

Proof: Noting that both $G(x + t_0)$ and $f_{\tau_1}(x)$ are totally independent of r_0, the partial derivative w.r.t r_0 of the integrand in the definition of $\widetilde{G}$ is

$$\frac{\partial\left[G(x+t_0)F(x+t_0)f_{\tau_1}(x)\right]}{\partial r_0} = -\frac{1}{\alpha_r}\left[1 - e^{-\alpha_r(x+t_0)}\right]G(x+t_0)F(x+t_0)f_{\tau_1}(x).$$

The partial derivatives w.r.t r_0 of the annuity payment factors $\widetilde{F}_i,\ i = 1, 2, 3, 4$, are

$$\begin{aligned}
\frac{\partial \widetilde{F}_1}{\partial r_0} &= -\frac{1}{\alpha_r}\sum_{k=1}^{T-1}(1 - e^{-\alpha_r k})F(k)P_1(k), \\
\frac{\partial \widetilde{F}_2}{\partial r_0} &= -\frac{1}{\alpha_r}\sum_{i=0}^{T-2}\sum_{k=i+1}^{T-1}(1 - e^{-\alpha_r k})F(k)P_2(i,k), \\
\frac{\partial \widetilde{F}_3}{\partial r_0} &= -\frac{1}{\alpha_r}\sum_{k=1}^{T-1}(1 - e^{-\alpha_r k})kF(k)P_1(k), \\
\frac{\partial \widetilde{F}_4}{\partial r_0} &= -\frac{1}{\alpha_r}\sum_{i=0}^{T-2}\sum_{k=i+1}^{T-1}(1 - e^{-\alpha_r k})kF(k)P_2(i,k).
\end{aligned}$$

Since $\frac{1}{\alpha_r}(1 - e^{-\alpha_r z}) \geq 0$ whenever $\alpha_r \neq 0$ and $z \geq 0$, we obtain Part 1 of the Proposition.

Define

$$g_2(z) := -z + \frac{1}{\alpha_r}\left(1 - e^{-\alpha_r z}\right).$$

Since $G(x+t_0)$ and $f_{\tau_1}(x)$ are free from μ_r, the partial derivative w.r.t. μ_r of the integrand in the definition of $\widetilde{G}$ is

$$\frac{\partial\left[G(x+t_0)F(x+t_0)f_{\tau_1}(x)\right]}{\partial \mu_r} = G(x+t_0)F(x+t_0)f_{\tau_1}(x)g_2(x+t_0).$$

The partial derivatives w.r.t μ_r of the annuity payment factors are

$$\begin{aligned}
\frac{\partial \widetilde{F}_1}{\partial \mu_r} &= \sum_{k=1}^{T-1} F(k)g_2(k)P_1(k), \quad & \frac{\partial \widetilde{F}_2}{\partial \mu_r} &= \sum_{i=0}^{T-2}\sum_{k=i+1}^{T-1} F(k)g_2(k)P_2(i,k),\\
\frac{\partial \widetilde{F}_3}{\partial \mu_r} &= \sum_{k=1}^{T-1} kF(k)g_2(k)P_1(k), \quad & \frac{\partial \widetilde{F}_4}{\partial \mu_r} &= \sum_{i=0}^{T-2}\sum_{k=i+1}^{T-1} kF(k)g_2(k)P_2(i,k).
\end{aligned}$$

Note that $g_2(z) \leq 0$ in case of $\alpha_r > 0$, $z \geq 0$, and $g_2(z) \geq 0$ in case of $\alpha_r < 0$, $z \geq 0$. We thus obtain Part 2.

Define

$$g_3(z) := z + \left[1-(2-e^{-\alpha_r z})^2\right]\frac{1}{2\alpha_r}.$$

Because $G(x+t_0)$ and $f_{\tau_1}(x)$ do not depend on σ_r, the partial derivative w.r.t σ_r of the integrand in the definition of $\widetilde{G}$ is

$$\frac{\partial\left[G(x+t_0)F(x+t_0)f_{\tau_1}(x)\right]}{\partial \sigma_r} = \frac{\sigma_r}{\alpha_r^2}G(x+t_0)F(x+t_0)f_{\tau_1}(x)g_3(x+t_0).$$

The partial derivatives w.r.t σ_r of the annuity payment factors are

$$\begin{aligned}
\frac{\partial \widetilde{F}_1}{\partial \sigma_r} &= \frac{\sigma_r}{\alpha_r^2}\sum_{k=1}^{T-1} F(k)g_3(k)P_1(k), \quad & \frac{\partial \widetilde{F}_2}{\partial \sigma_r} &= \frac{\sigma_r}{\alpha_r^2}\sum_{i=0}^{T-2}\sum_{k=i+1}^{T-1} F(k)g_3(k)P_2(i,k),\\
\frac{\partial \widetilde{F}_3}{\partial \sigma_r} &= \frac{\sigma_r}{\alpha_r^2}\sum_{k=1}^{T-1} kF(k)g_3(k)P_1(k), \quad & \frac{\partial \widetilde{F}_4}{\partial \sigma_r} &= \frac{\sigma_r}{\alpha_r^2}\sum_{i=0}^{T-2}\sum_{k=i+1}^{T-1} kF(k)g_3(k)P_2(i,k).
\end{aligned}$$

We have

$$\frac{dg_3(z)}{dz} = \left(e^{-\alpha_r z}-1\right)^2 > 0.$$

Consequently, $g_3(z) \geq g_3(0) = 0$ in case of $z \in [0,+\infty)$. This proves Part 3.

Remarks 2: Obviously, the following properties are also true:

1. The A_0 in Proposition 3 is a decreasing function of d, and in turn, d is a decreasing function of A_0.

2. The annuity payment B in Proposition 4 is a decreasing function of λ.

4. NUMERICAL RESULTS

We devote this section to the numerical analysis of the annuity for an insurance product integrating reverse mortgage with long-term care. To keep the computation simple, we only consider the case of a single insured. We will illustrate that the impacts of risks involving the home price, interest rate, and the longevity on the annuity payment, the expected discounted present value of total annuity, and the annuity payment factors.

We take the parameters of the standard case with the following values:
• at the initial time, the home price is $H_0 = 100$;
• the annual mean return rate of home price $\mu_h = 0.04$;
• the delay time of selling the mortgaged house $t_0 = 0$ (that is, as soon as the insured leaves the State 1, the house is sold off);
• the initial interest rate $r_0 = 0.04$;
• the mean reversion level of interest rate $\mu_r = 0.06$;
• the volatility rate of interest rate $\sigma_r = 0.01$;
• the mean reversion speed of interest rate $\alpha_r = 0.25$;
• the limiting age $L = 110$;
• the age of the insured at time 0 is $x_0 = 65$;
• the incremental creep of the growing (decreasing) perpetuity annuity $d = 0$ (this means that the perpetuity annuity degenerates into the level annuity);
• the proportion $\lambda = 1$ (it implies that the state annuity is simplified to the level annuity);
• the transition intensities of three-state Markov model are given by

$$\begin{aligned} \lambda_{12}(t) &= 0.0004 + 10^{0.06(x_0+t)-5.46}, \\ \lambda_{13}(t) &= \lambda_{23}(t) = 0.0005 + 10^{0.038(x_0+t)-4.12}. \end{aligned} \tag{45}$$

The model is employed to value the premiums of the disability policy by the Danish companies (Ramlau-Hansen, 2001). The $\lambda_{13}(t) = \lambda_{23}(t)$ implies that the mortality for active lives and for disabled lives are not discriminated. With these parametric values,
(1) the level annuity $A = 13.003$,
(2) the lump sum $\widetilde{G} = 90.252$,
(3) the annuity payment factors $\widetilde{F}_1 = 6.033$ and $\widetilde{F}_2 = 0.908$.

4.1. **Sensitivity Analysis for Parameters of Home Price.**

We start the numerical analysis of how the average rate of return of the home price impacts the annuity, lump sum, and annuity payment factors, while we keep fixed the other parametric values given above.

TABLE 1. Impacts of the average return rate of house price

μ_h	0.02	0.04	0.06	0.08	0.1	0.12	0.14	0.16
A	11.107	13.003	15.378	18.379	22.206	27.130	33.522	41.892
$\widetilde{G}$	77.092	90.252	106.735	127.565	154.127	188.301	232.662	290.761

Table 1 shows how the mean return of home price μ_h affect the annuity, lump sum and the annuity payment factors, while other parameters are kept fixed. Here we note the following.

(A) The higher the mean return of the home price, the greater the lump sum and annuity are; however, the annuity payment factors remain constant, $\widetilde{F}_1 = 6.033$ and $\widetilde{F}_2 = 0.908$ (as they are not affected by μ_h, which coincides with the conclusion in Proposition 6).

The essence of this insurance product is to exchange the profit from selling the mortgaged house with the insured's annuity until his/her death. The higher mean return of the home price contributes to increased profit from the sale of the mortgaged house in future, and similar is the case with lump sum and annuity.

When μ_h increases from 0.02 to 0.16, **(a)** the lump sum leaps up from 77.092 to 290.761, with the the average rate of change $\frac{290.761-77.092}{0.16-0.02} = 1526.207$; and **(b)** the annuity rises from 11.107 to 41.892, with the average rate of change $\frac{41.892-11.107}{0.16-0.02} = 219.893$.

Compared with others parameters of the home price and interest rate model, the mean return of house price has the dominating influence on both the lump sum and annuity.

Next we vary the time delay t_0 (while keeping other parameters fixed as above) and analyze how it affects the annuity, lump sum, and annuity payment factors.

TABLE 2. Impacts of the delay time of selling house

t_0	0	0.25	0.5	0.75	1	1.25	1.5	1.75
A	13.003	12.907	12.808	12.707	12.605	12.501	12.396	12.290
$\widetilde{G}$	90.252	89.584	88.898	88.198	87.486	86.764	86.035	85.301

Table 2 contains the corresponding annuity, lump sums and annuity payment factors while we vary the delay time t_0, (with other parameters kept fixed). We now note:

(B) the larger the delay time in selling the house (the larger t_0), smaller the lump sum and annuity, while the annuity payment factors remain constant, (not influenced by t_0). This is in accord with Proposition 7(b-2) in the case of $(\mu_r - r_0)^2 - \frac{2\sigma_r^2}{\alpha_r^2}(\mu_h - r_0) = 4 \times 10^{-4} \geq 0$, $\alpha_r = 0.25 > 0$, $z_1 = -24 \leq 0$, and $z_2 = 1 \geq 1$. However, it should be noted that if we changed some parameters, the lump sum might also increase with the increase of t_0 (refer to Proposition 7(b-1)).

The average rate of change of the lump sum and annuity are $\frac{90.252-85.301}{1.75-0} = 2.829$ and $\frac{13.003-12.290}{1.75-0} = 0.407$. This indicates that the lump sum and annuity remain stable as the delay time $t_0 < 1.75$ years.

4.2. **Sensitivity Analysis for Parameters of Interest Rate.** This subsection provides the numerical analysis of how the parameters of interest rate model impacts the annuity A, the lump sum $\widetilde{G}$, and the annuity factors $\widetilde{F}_i,\ i = 1, 2$; again, we keep the other parametric values fixed.

Table 3. Impacts of the initial interest rate

r_0	0.02	0.04	0.06	0.08	0.1	0.12	0.14	0.16
A	13.092	13.003	12.915	12.828	12.741	12.655	12.570	12.486
$\widetilde{G}$	95.882	90.252	84.986	80.060	75.451	71.138	67.102	63.322
$\widetilde{F}_1$	6.345	6.033	5.738	5.459	5.197	4.949	4.714	4.492
$\widetilde{F}_2$	0.979	0.908	0.842	0.782	0.725	0.673	0.624	0.579

Now we note from Table 3:

(C) The lump sum $\widetilde{G}$ and annuity payment factors $\widetilde{F}_i, i = 1, 2$, are decreasing as the initial interest rate r_0 increases (while other parameters are kept fixed). This agrees with our Proposition 8, and the annuity A is also decreasing.

With the explicit solution of interest rate in Equation (7), we know that the higher initial interest rate means that higher level of average interest rate, which contributes to the lower level of average discounted factor of interest rate, and that in turn results in the lower lump sum and annuity payment factors. This implies that the higher risk of interest rate produces lower annuity.

The average change rate of the lump sum and annuity is respectively $\frac{95.882-63.322}{0.16-0.02} = 232.571$ and $\frac{13.092-12.486}{0.16-0.02} = 4.329$, which illustrates that the slight fluctuations of initial interest rate leads to the large fluctuations of the lump sum, while the slight fluctuations of the annuity ensues. It implies that, while the annuity is stable for the fluctuations of initial interest rate r_0, the lump sum is sensitive for the fluctuations.

The Table 4 provides the numerical values caused by the impact of the average reversion level μ_r of the interest rate. Here we note:

(D) the lump sum $\widetilde{G}$ and annuity payment factors $\widetilde{F}_i, i = 1, 2$, decrease with the increase of average reversion level μ_r of interest rate. This conclusion is theoretically supported by the Proposition 8.

From the mean of discounted factor given by Equation (7), it is obvious that the greater μ_r is smaller the discounted factor. Thus, the lump sum decreases from

Table 4. Impacts of the average reversion level of interest rate

μ_r	0.02	0.04	0.06	0.08	0.1	0.12	0.14	0.16
A	13.672	13.342	13.003	12.669	12.346	12.039	11.751	11.482
$\widetilde{G}$	112.386	100.334	90.252	81.760	74.560	68.414	63.135	58.573
$\widetilde{F}_1$	6.788	6.387	6.033	5.717	5.436	5.183	4.954	4.748
$\widetilde{F}_2$	1.432	1.133	0.908	0.736	0.604	0.500	0.419	0.353

112.386 to 58.573 as μ_r increases from 0.02 to 0.16, where the average change rate of the lump sum and annuity is respectively $\frac{112.386-58.573}{0.16-0.02} = 384.379$ and $\frac{13.672-11.482}{0.16-0.02} = 15.643$, exerting a significant influence on both the lump sum and annuity, second in importance only to the average housing price returns.

Table 5 given below provides the impact of the volatility of interest rate on A, $\widetilde{G}$, and $\widetilde{F}_i,\ i = 1, 2$.

Table 5. Impacts of the volatility of interest rate

σ_r	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08
A	13.003	13.036	13.092	13.169	13.268	13.388	13.526	13.680
$\widetilde{G}$	90.252	91.094	92.531	94.616	97.430	101.088	105.746	111.623
$\widetilde{F}_1$	6.033	6.060	6.105	6.171	6.259	6.370	6.508	6.678
$\widetilde{F}_2$	0.908	0.928	0.962	1.014	1.085	1.181	1.310	1.482

(E) Table 5 reveals that the lump sum $\widetilde{G}$ and annuity payment factors $\widetilde{F}_i, i = 1, 2$, increase as the volatility of interest rate σ_r increases, and this is consistent with our Proposition 8.

The higher volatility rate contributes to the higher average level of the discounted factor. Hence, the lump sum and annuity payment factors become greater. It appears to be unreasonable that the lump sum increases with the increase of volatility rate, which is probably caused by the flaws of pricing model itself. On the other hand, the annuity decreases with the increase of σ_r, and this is reasonable.

(F) The higher volatility rate generates higher market risk. Thus the insurer will certainly pay the smaller annuity to the insured in order to avoid the higher risk. Therefore, we do not advise that the model be used to pricing the lump sum of the product of reverse mortgage integrated with the long-term care; however, it is suitable for pricing the annuity. The average change rate of the lump sum and annuity is respectively $\frac{111.623-90.252}{0.08-0.01} = 305.3$ and $\frac{13.680-13.003}{0.08-0.01} = 9.671$, indicating that the volatility of interest rate has an important effect on both the lump sum and annuity.

We next consider the impact of the reversion speed α_r of the interest rate on A, $\widetilde{G}$, and $\widetilde{F}_i$, $i = 1, 2$.

TABLE 6. Impacts of the reversion speed of interest rate

α_r	0.05	0.25	0.5	0.75	1	1.25	1.5	1.75
A	13.245	13.003	12.934	12.915	12.909	12.906	12.904	12.904
$\widetilde{G}$	97.375	90.252	87.838	86.873	86.362	86.046	85.832	85.678
$\widetilde{F}_1$	6.288	6.033	5.919	5.866	5.836	5.817	5.804	5.794
$\widetilde{F}_2$	1.064	0.908	0.872	0.860	0.854	0.850	0.848	0.846

(G) From Table 6, it is clear that the annuity, the lump sum and the annuity payment factors decrease with the increasing of the reversion speed α_r of interest rate as other parameters take fixed values. The average change rate of the lump sum and annuity are $\frac{97.375-85.678}{1.75-0.05} = 6.881$ and $\frac{13.245-12.904}{1.75-0.05} = 0.201$, respectively. This implies that the reversion speed slightly affects the lump sum and annuity as α_r takes value between 0 and 1.75.

4.3. **Sensitivity Analysis for the Age of the Insured.** In this subsection we discuss the impact on A, $\widetilde{G}$, and $\widetilde{F}_i$, $i = 1, 2.$ by the limiting age and initial age, impact on A_0 by the incremental creep d, and the impact on B by the proportion λ.

TABLE 7. Impacts of the limited age

L	95	100	105	110	115	120
A	13.0172	13.0044	13.0033	13.0033	13.0033	13.0033
$\widetilde{G}$	90.2518	90.2518	90.2518	90.2518	90.2518	90.2518
$\widetilde{F}_1$	6.0325	6.0325	6.0325	6.0325	6.0325	6.0325
$\widetilde{F}_2$	0.9007	0.9075	0.9081	0.9081	0.9081	0.9081

(H) Table 7 reveals the following: When the age limit (survival) is over 95, the lump sum $\widetilde{G}$ and the first annuity payment factor $\widetilde{F}_1$ almost stay unchanged. If the age limit is over 110, then the annuity A and the second annuity payment factor $\widetilde{F}_2$ almost stay unchanged. Thus, it seems reasonable to assume that the longevity is 110.

(I) Table 8 illustrates that as the initial age x_0 of the insured increases, the lump sum $\widetilde{G}$ and annuity A are increasing, while annuity payment factors $\widetilde{F}_i, i = 1, 2$, show a decreasing trend. Meanwhile, the decreasing speed of annuity payment factor $\widetilde{F}_2$ is significantly lower than that of annuity payment factor $\widetilde{F}_1$. This shows that the annuity payment factor $\widetilde{F}_2$ plays more and more important role in calculating the annuity with the increase of the initial age of the insured.

TABLE 8. Impacts of the initial age

x_0	50	55	60	65	70	75	80	85
A	6.717	8.162	10.161	13.003	17.147	23.339	32.819	47.680
$\widetilde{G}$	75.395	80.739	85.780	90.252	93.907	96.592	98.323	99.280
$\widetilde{F}_1$	10.776	9.310	7.702	6.033	4.415	2.972	1.802	0.951
$\widetilde{F}_2$	0.448	0.583	0.740	0.908	1.061	1.167	1.194	1.131

TABLE 9. Impacts of the incremental creep

d	0	0.5	1	1.5	2	2.5	3	3.5
A_0	13.003	9.757	6.511	3.265	0.019	-3.227	-6.473	-9.719

(J) For the growing (decreasing) perpetuity annuity in Proposition 3, Table 9 shows that A_0 decreases with the increase of the incremental creep d, and this is in agreement with Remarks 2. The lump sum and the annuity payment factors remain constant, $\widetilde{G} = 90.252$, $\widetilde{F}_1 = 6.033$, $\widetilde{F}_2 = 0.908$, $\widetilde{F}_3 = 31.867$ and $\widetilde{F}_4 = 13.192$.

TABLE 10. Impacts of the proportion

λ	0	0.5	1	1.5	2	2.5	3	3.5
B	14.961	13.914	13.003	12.205	11.499	10.870	10.306	9.798

(K) For the state annuity in Proposition 4, Table 10 shows that the annuity payment B decreases with the increase of the proportion λ, and this is in accord with Remarks 2. Moreover, the lump sum and annuity payment factors remain constant, $\widetilde{G} = 90.252$, $\widetilde{F}_1 = 6.033$ and $\widetilde{F}_2 = 0.908$.

5. **CONCLUSION**

The product integrating reverse mortgage with long-term care mainly involves several risk factors such as the home price risk, interest rate risk, disability risk, and life expectancy risk. We employ a three-state temporally inhomogeneous Markov chain model to describe the disability risk and life expectancy risk. This gives a unified and rigorous approach for the combined product reverse mortgage and long-term care insurance. We use the Black-Scholes model to describe the dynamics of the home price and the Ornstein-Uhlenbeck process for that of interest rate. This paper builds a pricing model for the lifetime annuity of the combined product, derives the closed-form solutions of the growing (decreasing) perpetuity annuity, the state annuity, and the level annuity. We then discuss the impact of the parameters associated with the home price and interest rate over monotonicity of the lump sum, annuity, and annuity payment factors. We present a numerical analysis of the lump sum, the annuity, and annuity payment factors, and analyze their sensitivity to the said parameters. The result shows that the average return of home price has a major influence on the annuity and the lump sum. Next to the average return of home price, the mean reversion level and volatility of interest rate play a dominant role. Initial interest rate affects the lump sum in a significant way, while affecting the annuity only slightly. Meanwhile, when the delay time of selling the house is in the range $0 \le t_0 \le 1.75$ and the reversion speed of interest rate in the range $0 < \alpha_r \le 1.75$, they hardly exert any effect on the lump sum and annuity.

Having noticed in this work, both theoretically and numerically, the greater sensitivity of the annuity and lump sum to the parameters of average return of home price, the mean reversion level and volatility of interest rate, we continue this problem analyzing these properties in terms of switching Markov chains.

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