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IMPLICIT DIFFERENTIAL INCLUSIONS WITH ACYCLIC RIGHT-HAND SIDES: AN ESSENTIAL FIXED POINTS APPROACH

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ABSTRACT. Effective criteria are given for the solvability of initial as well as boundary value problems to implicit ordinary differential inclusions whose right-hand sides are governed by compact acyclic maps. Cauchy and periodic implicit problems are also considered on proximate retracts. Our new approach is based on the application of the topological essential fixed point theory. Implicit problems for partial differential inclusions are only indicated.

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1. INTRODUCTION

The aim of the present paper is to develop a technique for the solvability of implicit problems associated with differential inclusions whose right-hand sides are governed by compact acyclic mappings. It is for the first time when multivalued maps with such a general regularity are considered with this respect. The main tool for this investigation is the appropriate topological theory about essential fixed points of compact acyclic maps on absolute neighbourhood retracts.

For ordinary differential equations, implicit initial value problems were studied e.g. in [8, 23, 26] and implicit boundary value problems e.g. in [6, 7, 14, 16, 20, 22, 27, 29, 32, 33, 35, 36, 37, 38, 39, 45]. For ordinary differential inclusions, implicit initial value problems were studied e.g. in [5, 43] and implicit boundary value problems, as far as we know, only in [15], where the highest (second-order) derivative occurs however "only" separately in a single-valued continuous function. Many of these results will be therefore extended and some of them also generalized in the sense that the sets of values of multivalued right-hand sides of given inclusions are not necessarily convex.

Our paper is organized as follows. After recalling some auxiliary technicalities in Preliminaries, the topological theory for essential fixed points of compact acyclic

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maps on arbitrary absolute neighbourhood retracts will be briefly presented. On this basis, in Section 4, implicit initial value problems and, in Section 5, implicit boundary value problems will be examined. In Section 6, both initial as well as boundary value problems will be investigated on proximate retracts. Finally, a possible application of our method to implicit problems for partial differential inclusions will be indicated.

2. **PRELIMINARIES**

In this paper, all topological spaces are assumed to be metric and all single-valued mappings are assumed to be continuous.

Let us recall that a space X is an *absolute neighbourhood retract* (written $X \in$ ANR) if, for every space Y and every closed subset $A \subset Y$, each map $f\colon A \to X$ is extendable over some open neighbourhood U of A in Y. A space X is called an *absolute retract* (written $X \in$ AR) if each $f\colon A \to X$ is extendable over Y (for more details, see [10]). Evidently, if $X \in$ AR, then $X \in$ ANR.

A closet subset X of the n-dimensional Euclidean space $\mathbb{R}^n$ is called a *proximate retract* (written $X \in$ PR) if there exists an open neighbourhood U of X in $\mathbb{R}^n$ and a (continuous) retraction map $r\colon U \to X$ such that

$$\forall y \in U\colon \|y - r(y)\| = \operatorname{dist}(y, X) = \inf\{\|x - y\| \mid x \in X\}. \tag{2.1}$$

Evidently if $X \in$ PR, then $X \in$ ANR. It is well known that any closed, convex set $X \subset \mathbb{R}^n$ or any closed set with a $C^{1,1}$-boundary are proximate retracts (see e.g. [9]). For more details, see e.g. [28, 40].

A space X is called an *R_δ-set* if there exists a decreasing sequence $\{X_n\}$ of compact absolute retracts such that $X = \bigcap_n X_n$. Any compact absolute retract is obviously an R_δ-set.

In the sequel, we shall use the Čech homology theory with compact carriers and the coefficients in the field of rational numbers $\mathbb{Q}$ (see e.g. [1, 28]).

A nonempty compact set X is called *acyclic*, provided

$$H_n(X) = \begin{cases} 0, & n > 0, \\ \mathbb{Q}, & n = 0, \end{cases}$$

where H_n denotes the n-dimensional Čech homology functor with rational coefficients.

A map $p\colon Y \to X$ is called a *Vietoris map*, provided

(i) p is *proper*, i.e. $p^{-1}(K)$ is a compact set, for every compact $K \subset X$,

(ii) for every $x \in X$, the given set $p^{-1}(x)$ is an acyclic set.

By a *multivalued mapping* $\varphi\colon X \multimap Y$, we shall understand, as usually, that $\varphi\colon X \to 2^Y \setminus \{\emptyset\}$.

Definition 2.1. Let $\varphi\colon X \multimap Y$ be a multivalued mapping with closed values.

(i) φ is called *upper semicontinuous* (u.s.c.) if, for every open set $U \subset Y$, the set

$$\varphi^{-1}(U) := \{x \in X \mid \varphi(x) \subset U\},$$

is an open subset of X;

(ii) φ is called *lower semicontinuous* (l.s.c.) if, for every open set $U \subset Y$, the set

$$\varphi_+^{-1}(U) := \{x \in X \mid \varphi(x) \cap U \neq \emptyset\},$$

is an open subset of X.

A multivalued map $\varphi\colon X \multimap Y$ is called a *compact map* if the closure

$$\overline{\varphi(X)} := \overline{\{y \in Y \mid y \in \varphi(x), x \in X\}}$$

of the set $\varphi(X)$ is a compact subset of Y.

For a given $\varphi\colon X \multimap Y$, we let

$$\Gamma_\varphi := \{(x,y) \in X \times Y \mid y \in \varphi(x)\},$$

and

$$\begin{aligned} p_\varphi\colon \Gamma_\varphi \to X, &\quad p_\varphi(x,y) = x, \\ q_\varphi\colon \Gamma_\varphi \to Y, &\quad q_\varphi(x,y) = y. \end{aligned}$$

Then Γ_φ is called the *graph* of φ and p_φ, q_φ are called the *natural projections* associated with φ.

Lemma 2.2. *Let $\varphi\colon X \multimap Y$ be a compact map with closed values. The map φ is u.s.c. if and only if the graph Γ_φ of φ is a closed subset of $X \times Y$.*

Lemma 2.3. *Let $\varphi\colon X \multimap Y$ be a multivalued map such that, for every open subset $A \subset Y$, the set $\varphi_+^{-1}(A)$ is an open subset of X. We let $\overline{\varphi}\colon X \multimap Y$ for the mapping defined as follows:*

$$\overline{\varphi}(x) := \overline{\varphi(x)}, \text{ where } \overline{\varphi(x)} \text{ denotes the closure of } \varphi(x) \text{ in } Y.$$

Then $\overline{\varphi}$ is an l.s.c. mapping with closed values.

For the proofs of Lemma 2.2, Lemma 2.3, and for more details concerning multivalued mappings, see e.g. [1, 28].

3. ESSENTIAL FIXED POINTS OF ACYCLIC MAPPINGS

Let us recall the notion of an acyclic map (see e.g. [1, 21, 28]).

Definition 3.1. A u.s.c. map $\varphi\colon X \multimap Y$ is called *acyclic*, provided $\varphi(x)$ is an acyclic set, for every $x \in X$.

Observe that if $\varphi\colon X \multimap Y$ is a compact acyclic map, then the natural projection $p_\varphi\colon \Gamma_\varphi \to X$ is a Vietoris map.

Using the Vietoris mapping theorem (see e.g. [1, Theorem I.4.8], [28, Theorem I.8.7]), we are able to define the induced linear map $\varphi_*\colon H(X) \to H(Y)$ by putting:

$$\varphi_* := (q_\varphi)_* \circ (p_\varphi)_*^{-1}, \tag{3.1}$$

and consequently, for $\varphi\colon X \multimap Y$, we define the *generalized Lefschetz number* $\Lambda(\varphi)$ of φ by letting

$$\Lambda(\varphi) := \Lambda(\varphi_*), \tag{3.2}$$

provided $\Lambda(\varphi_*)$ is well defined.

It is well known (see e.g. [1, 28]) that if $X \in \mathrm{ANR}$, then $\Lambda(\varphi)$ is well defined.

For a map $\varphi\colon X \multimap Y$, by $\mathrm{Fix}(\varphi)$, we denote the set of all *fixed points* of φ, i.e.

$$\mathrm{Fix}(\varphi) := \{x \in X \mid x \in \varphi(x)\}.$$

Note that if φ is a compact map, then $\mathrm{Fix}(\varphi)$ is a (possibly empty) compact set.

Let $\varphi, \psi\colon X \multimap Y$ be two compact acyclic mappings. We say that φ and ψ are *homotopic* (written $\varphi \sim \psi$), provided there exists a compact acyclic map $\chi\colon X \times [0,1] \multimap X$ such that $\chi(x,0) = \varphi(x)$ and $\chi(x,1) = \psi(x)$, for every $x \in X$.

Assume that $X \in \mathrm{ANR}$, U is an open subset of X and ∂U denotes the boundary of U in X.

We let:

$$A_U(X) := \{\varphi\colon X \multimap X \mid \varphi \text{ is compact acyclic and } \partial U \cap \mathrm{Fix}(\varphi) = \emptyset\}.$$

For $\varphi, \psi \in A_U(X)$, we say that φ and ψ are *homotopic* if there exists a compact acyclic homotopy $\chi\colon X \times [0,1] \multimap X$ linking φ and ψ such that:

$$\mathrm{Fix}(\chi(\cdot,t)) \cap \partial U = \emptyset, \text{ for every } t \in [0,1].$$

The symbol $\mathbb{Z}$ denotes, as usually, the set of integers. For any $\varphi \in A_U(X)$, by $\mathrm{Ind}(\varphi, U) \in \mathbb{Z}$, we shall denote the *fixed point index* of φ with respect to U, provided the following properties are satisfied:

(P1) (Existence) If $\mathrm{Ind}(\varphi, U) \neq 0$, then $\mathrm{Fix}(\varphi) \cap U \neq \emptyset$.

(P2) (Excision) If $\{x \in U \mid x \in \varphi(x)\} \subset V \subset U$, then $\mathrm{Ind}(\varphi, U) = \mathrm{Ind}(\varphi, V)$, where $V \subset X$ is an open subset.
(P3) (Additivity) Let U_1, U_2 be two open subsets of X such that $U = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$ and $\mathrm{Fix}(\varphi) \cap (\overline{U} \setminus (U_1 \cup U_2)) = \emptyset$, then $\mathrm{Ind}(\varphi, U) = \mathrm{Ind}(\varphi, U_1) + \mathrm{Ind}(\varphi, U_2)$.
(P4) (Homotopy) If $\varphi, \psi \in A_U(X)$ are homotopic, then $\mathrm{Ind}(\varphi, U) = \mathrm{Ind}(\psi, U)$.
(P5) (Normalization) If $U = X$, then $\mathrm{Ind}(\varphi, U) = \Lambda(\varphi)$.

Remark 3.2. In [24, 25], the fixed point index is defined for finite compositions of compact acyclic maps. In [1, 28], it is defined for so called admissible maps. In particular, any compact acyclic map is admissible. In [21] (cf. also [28]), it is defined for continuous (both l.s.c. and u.s.c.) $(1-n)$-acyclic maps. A map $\varphi\colon X \multimap X$ is $(1-n)$-*acyclic*, provided, for every $x \in X$, we have $\varphi(x)$ to be acyclic or $\varphi(x)$ to consist of n-acyclic components. Using the approximation methods, the fixed point index can be also defined for compact mappings with R_δ-values on compact ANR-spaces (see e.g. [1, 28] and the references therein).

Until the end of this paper, we shall assume that $X \in \mathrm{ANR}$ and $\varphi\colon X \multimap X$ is a compact acyclic map.

Definition 3.3. Let $x_0 \in \mathrm{Fix}(\varphi)$. We say that x_0 is an *essential fixed point* of φ if, for every open neighbourhood U of x_0 in X, there exists an open subset $V \subset U$ such that:

(i) $x_0 \in V$,
(ii) $\partial V \cap \mathrm{Fix}(\varphi) = \emptyset$,
(iii) $\mathrm{Ind}(\varphi, V) \neq 0$.

We let:

$$\mathrm{Ess}(\varphi) := \{x \in \mathrm{Fix}(\varphi) \mid x \text{ is essential}\}.$$

Observe that if $x_0 \in \mathrm{Fix}(\varphi)$ is an isolated fixed point of φ and there exists an open subset $V \subset X$ such that $x_0 \in V$, $\mathrm{Fix}(\varphi) \cap \overline{V} = \{x_0\}$ and $\mathrm{Ind}(\varphi, V) \neq 0$, then $x_0 \in \mathrm{Ess}(\varphi)$.

Let us recall the following useful lemma (see e.g. [2, 3, 8, 28]).

Lemma 3.4. *Let A be a compact subset of X such that* $\dim A = 0$*, where* $\dim A$ *denotes the topological dimension of A. Then, for every $x \in A$ and for every open neighbourhood U of x in X, there exists an open neighbourhood $V \subset U$ of x in X such that $\partial V \cap A = \emptyset$.*

The following proposition is a multivalued version of [3, Theorem 4] and [8, Proposition 2.1].

Proposition 3.5. *Let $\varphi\colon X \multimap X$ be as above, i.e. $X \in \mathrm{ANR}$ and φ be a compact acyclic mapping. Assume furthermore that:*

(i) *there exists an open subset $V \subset X$ such that $\mathrm{Fix}(\varphi) \cap \partial V = \emptyset$ and $\mathrm{Ind}(\varphi, V) \neq 0$,*
(ii) *$\dim \mathrm{Fix}(\varphi) = 0$.*

Then $\mathrm{Ess}(\varphi) \cap V \neq \emptyset$.

Applying Lemma 3.4, jointly with the properties (P1)–(P5) of the fixed point index, the proof of Proposition 3.5 is quite analogous to those in [3, Theorem 4] and [8, Proposition 2.1].

Remark 3.6. Replacing condition (i) in Proposition 3.5 by $\Lambda(\varphi) \neq 0$ (in particular, for $X \in \mathrm{AR}$, we have automatically $\Lambda(\varphi) = 1$), we obviously get that $\mathrm{Ess}(\varphi) \neq \emptyset$, provided $\dim \mathrm{Fix}(\varphi) = 0$.

Assume that $\varphi\colon Y \times X \multimap X$ is a compact acyclic map. For every $y \in Y$, by $\varphi_y\colon X \multimap X$, we shall understand the map defined as follows:

$$\varphi_y(x) := \varphi(y, x), \text{ for every } x \in X.$$

Evidently, φ_y is also a compact acyclic map. Let us assume that the Lefschetz number $\Lambda(\varphi_y)$ of φ_y is different from zero, for every $y \in Y$. Consequently, we get that $\mathrm{Fix}(\varphi_y) \neq \emptyset$ is a compact set, for every $y \in Y$. We define the map $\psi\colon Y \multimap X$ by the formula:

$$\psi(y) := \mathrm{Fix}(\varphi_y), \text{ for every } y \in Y.$$

We can prove the following lemma.

Lemma 3.7. *Under the above assumptions, the map $\psi\colon Y \multimap X$ is compact and u.s.c.*

Proof. Since $\psi(Y) \subset \varphi(Y \times X) \subset K \subset X$, where K is a compact set, in view of Lemma 2.2, it is sufficient to show that the graph Γ_ψ of ψ is a closed subset of $Y \times X$.

Consider a sequence $\{(y_n, x_n) \in \Gamma_\psi\}$ such that $\lim_{n\to\infty}(y_n, x_n) = (y, x)$, where lim is considered with respect to the max-metric in $Y \times X$. We need to prove that $(y, x) \in \Gamma_\psi$.

Since $(y_n, x_n) \in \Gamma_\psi$, we get that $(y_n, x_n, x_n) \in \Gamma_\varphi$. Thus, $(y, x, x) \in \Gamma_\varphi$, and consequently $(y, x) \in \Gamma_\psi$, which completes the proof. □

Now, let us assume still that

$$\dim \mathrm{Fix}(\varphi_y) = 0, \text{ for every } y \in Y.$$

In view of Proposition 3.5 (cf. also Remark 3.6), we obtain that $\mathrm{Ess}(\varphi_y) \neq \emptyset$, for every $y \in Y$. It allows us to define the map $\eta\colon Y \multimap X$, $\eta(y) := \mathrm{Ess}(\varphi_y)$. Evidently, $\eta(y) \subset \psi(y)$, for every $y \in Y$.

We claim the following:

Lemma 3.8. *Let Y be a locally arc-wise connected space. The map $\eta\colon Y \multimap X$, $\eta(y) := \mathrm{Ess}(\varphi_y)$, has the following property:*

$$\{y \in Y \mid \eta(y) \cap U \neq \emptyset\} \text{ is an open subset of } Y, \text{ for every open } U \subset X.$$

Proof. Let $y_0 \in Y$ be a point such that $\eta(y_0) \cap U \neq \emptyset$. Assume, furthermore, that $x_0 \in \eta(y_0) \cap U$. Then there exists an open neighbourhood V of x_0 in X such that $V \subset U$ and $\mathrm{Ind}(\varphi_{y_0}, V) \neq 0$. Since ψ is a u.s.c. mapping and Y is locally arc-wise connected, there exists an open arc-wise connected subset $W \subset Y$ such that $y_0 \in W$ and, for every $y \in W$, we have:

$$\mathrm{Fix}(\varphi_y) \cap \partial V = \emptyset. \tag{3.3}$$

Let $y \in W$ and $\sigma\colon [0,1]$ be an arc linking y_0 with y. We define a homotopy $\chi\colon V \times [0,1] \multimap X$ by putting

$$\chi(x,t) := \varphi(\sigma(t), x).$$

It follows from (3.3) that χ is a well defined homotopy joining φ_{y_0} with φ_y, and so we get $\mathrm{Ind}(\varphi_{y_0}, V) = \mathrm{Ind}(\varphi_y, V) \neq 0$. Thus, the claim follows from Proposition 3.5. □

4. IMPLICIT INITIAL VALUE PROBLEMS

In this section, we will apply the results of the foregoing section to Cauchy (initial value) problems for implicit ordinary differential inclusions.

In order to employ the joint consequence of Lemma 2.3 and Lemma 3.8, the following theorem due to A. Bressan [11, Theorem 4], [12, Theorem 3], which we state here in the form of proposition, will be useful.

Proposition 4.1. *Let $\eta\colon [0,1] \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a bounded l.s.c. mapping with compact values. Then the Cauchy (initial value) problem*

$$\begin{cases} x'(t) \in \eta(t, x(t)), \ \text{a.e. on } [0,1], \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{4.1}$$

admits, for every $x_0 \in \mathbb{R}^n$, a Carathéodory solution $x(\cdot)$ on $[0,1]$, i.e. there exists an absolutely continuous functions $x\colon [0,1] \to \mathbb{R}^n$ (written $x \in \mathrm{AC}([0,1], \mathbb{R}^n)$) such that (4.1) is satisfied, for a.a. $t \in [0,1]$.

Along the lines of Section 3, we put $Y = [0,1] \times \mathbb{R}^n$ and $X = \mathbb{R}^n$. Let $\varphi\colon [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a compact acyclic map.

Let us consider the following implicit differential inclusion:

$$x'(t) \in \varphi(t, x(t), x'(t)), \tag{4.2}$$

whose solutions $x\colon [0,1] \to \mathbb{R}^n$ are also understood in the Carathéodory sense, i.e. absolutely continuous functions satisfying (4.2), for almost all $t \in [0,1]$. For every $(t,y) \in [0,1] \times \mathbb{R}^n$, by (cf. Section 3):

$$\varphi_{(t,y)}\colon \mathbb{R}^n \multimap \mathbb{R}^n,$$

we shall mean the map defined as follows:

$$\varphi_{(t,y)}(x) := \varphi(t,y,x), \quad \text{for every } x \in \mathbb{R}^n.$$

Evidently, $\varphi_{(t,y)}$ is also a compact acyclic map which implies that $\mathrm{Fix}(\varphi_{(t,y)})$ is a nonempty compact subset of $\mathbb{R}^n$.

Defining the map (cf. Section 3) $\psi\colon [0,1] \times \mathbb{R}^n \multimap \mathbb{R}^n$ by the formula:

$$\psi(t,y) := \mathrm{Fix}(\varphi_{(t,y)}), \quad \text{for every } (t,y) \in [0,1] \times \mathbb{R}^n,$$

it follows from Lemma 3.7 that ψ is a compact u.s.c. mapping.

Now, we shall associate with (4.2) the following differential inclusion:

$$x'(t) \in \psi(t, x(t)). \tag{4.3}$$

Denoting by $S(\varphi)$ the solution set of the Cauchy problem related to (4.2), with $x(0) = x_0$, and by $S(\psi)$ the set of all solutions of the Cauchy problem related to (4.3), with $x(0) = x_0$, we have $S(\varphi) = S(\psi)$.

Let us still assume that

$$\dim \mathrm{Fix}(\varphi_{(t,x)}) = 0, \quad \text{for every } (t,x) \in [0,1] \times \mathbb{R}^n. \tag{4.4}$$

In view of Proposition 3.5, we obtain that $\mathrm{Ess}(\varphi_{(t,y)}) \neq \emptyset$. Consequently, we can define a multivalued map $\eta\colon [0,1] \times \mathbb{R}^n \multimap \mathbb{R}^n$ as follows:

$$\eta(t,y) := \overline{\mathrm{Ess}(\varphi_{(t,y)})}, \quad \text{for every } (t,y) \in [0,1] \times \mathbb{R}^n.$$

By means of Lemma 2.3 and Lemma 3.8, we immediately get:

Proposition 4.2. *The map $\eta\colon [0,1] \times \mathbb{R}^n \multimap \mathbb{R}^n$ is l.s.c. with compact values.*

As a direct consequence of Proposition 4.1 and Proposition 4.2, we also get:

Proposition 4.3. *The set*

$$S(\eta) := \{x \in \mathrm{AC}([0,1], \mathbb{R}^n) \mid x'(t) \in \eta(t, x(t)), \ \text{a.e. on } [0,1],\ x(0) = x_0\} \neq \emptyset,$$

for every $x_0 \in \mathbb{R}^n$, where $\mathrm{AC}([0,1], \mathbb{R}^n)$ denotes the space of absolutely continuous functions $x\colon [0,1] \to \mathbb{R}^n$.

After all, we have

$$S(\eta) \subset S(\psi) = S(\varphi). \tag{4.5}$$

Summing up, we can state the first theorem:

Theorem 4.4. *Let $\varphi\colon [0,1]\times\mathbb{R}^n\times\mathbb{R}^n \multimap \mathbb{R}^n$ be a compact acyclic mapping. Then the set $S(\varphi)$ of Carathéodory solutions of every Cauchy problem related to* (4.2) *is nonempty, i.e. $S(\varphi)\neq\emptyset$, provided* (4.4) *holds.*

Now, taking $Y=[0,1]\times\mathbb{R}^{kn}$, $X=\mathbb{R}^n$, we will consider the following implicit differential inclusion:

$$x^{(k)}(t)\in\varphi(t,x(t),x'(t),\dots,x^{(k)}(t)), \tag{4.6}$$

where $\varphi\colon Y\times X\multimap X$ is a compact acyclic map.

By $S(\varphi)$, we shall denote the set of all Carathéodory solutions of the Cauchy problem related to (4.6), i.e. $x\in \mathrm{AC}^{(k-1)}([0,1],\mathbb{R}^n)$, where $\mathrm{AC}^{(k-1)}([0,1],\mathbb{R}^n)$ denotes the space of functions $x\colon[0,1]\to\mathbb{R}^n$, whose $(k-1)$-th derivative is absolutely continuous, satisfying (4.6) almost everywhere, together with $x^{(j)}(0)=x_j\in\mathbb{R}^n$, $j=0,1,\dots,k-1$.

Similarly as for (4.2), letting:

$$\psi\colon Y\multimap X,\quad \psi(y)=\mathrm{Fix}(\varphi(y,\cdot)),$$

we can define the following differential inclusion:

$$x^{(k)}(t)\in\psi(t,x(t),x'(t),\dots,x^{(k-1)}(t)), \tag{4.7}$$

whose set of solutions $x\in\mathrm{AC}^{(k-1)}([0,1],\mathbb{R}^n)$, satisfying $x^{(j)}(0)=x_j$, $j=0,1,\dots,k-1$, will be denoted by $S(\psi)$. Assuming still

$$\dim\mathrm{Fix}(\varphi(y,\cdot))=0,\quad \text{for every } y\in Y, \tag{4.8}$$

we can define the map

$$\eta\colon Y\multimap X,\quad \eta(y)=\overline{\mathrm{Ess}(\varphi(y,\cdot))}.$$

By the analogous arguments as above, we arrive at:

$$S(\eta)\subset S(\psi)=S(\varphi). \tag{4.9}$$

In order to get $S(\eta)\neq\emptyset$, for the set of all solutions of the following Cauchy problem:

$$\begin{cases} x^{(k)}(t)\in\eta(t,x(t),x'(t),\dots,x^{(k-1)}(t)),\\ x^{(j)}(0)=x_j\in\mathbb{R}^n,\ j=0,1,\dots,k-1,\end{cases} \tag{4.10}$$

it is enough to modify Proposition 4.1 as follows:

Proposition 4.5. *Let $\eta\colon[0,1]\times\mathbb{R}^{kn}\multimap\mathbb{R}^n$ be a bounded l.s.c. mapping with compact values. Then the Cauchy problem* (4.10) *admits, for all $x_j\in\mathbb{R}^n$, $j=0,1,\dots,k-1$, a Carathéodory solution on $[0,1]$.*

Proof. Since η is by the hypothesis bounded, there exists a suitable constant, say M, such that $\|x^{(k)}(t)\| \leq M$ holds a.e. on $[0,1]$, for every solution $x(\cdot)$ of (4.10), and subsequently

$$\max_{t\in[0,1]} \|x^{(j)}(t)\| \leq \sum_{l=j}^{k-1} \|x_l\| + M := M_j, \quad j = 0,1,\dots,k-1. \tag{4.11}$$

Problem (4.10) can be equivalently transformed into the form

$$\begin{cases} \boldsymbol{x}'(t) \in \boldsymbol{\eta}(t,\boldsymbol{x}(t)), \ \text{a.e. on } [0,1], \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{R}^{kn}, \end{cases} \tag{4.12}$$

where

$$\begin{aligned} \boldsymbol{x}(t) &= \big(x_0(t), x_1(t), \dots, x_{(k-1)}(t)\big), \\ \boldsymbol{x}_0 &= (x_0, x_1, \dots, x_{k-1}), \\ \boldsymbol{\eta}(t,\boldsymbol{x}(t)) &= \big(x_1(t), \dots, x_{k-1}(t), \eta(t, x_0(t), \dots, x_{k-1}(t))\big)^T. \end{aligned}$$

In view of (4.11), it can be still equivalently rewritten into the form

$$\begin{cases} \boldsymbol{x}'(t) \in \boldsymbol{\eta}^*(t,\boldsymbol{x}(t)), \ \text{a.e. on } [0,1], \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{R}^{kn}, \end{cases} \tag{4.13}$$

where

$$\boldsymbol{\eta}^*(t, y_0, y_1, \dots, y_{k-1}) = \big(y_1^*, \dots, y_{k-1}^*, \eta^*(t, y_0, y_1, \dots, y_{k-1})\big)^T,$$

and

$$y_j^* = \begin{cases} y_j, & \text{for } \|y_j\| \leq M_j, \\ r_j, & \text{where } \|r_j\| = M_j, \ \text{for } y_j = r_j t, \ t \geq 1, \end{cases}$$

$j = 0,1,\dots,k-1$,

$$\eta^* = \begin{cases} \eta, & \text{for } \|\eta\| \leq M, \\ r, & \text{where } \|r\| = M, \ \text{for } \eta = rt, \ t \geq 1. \end{cases}$$

Since the mapping $\boldsymbol{\eta}^* \colon [0,1] \times \mathbb{R}^{kn} \multimap \mathbb{R}^{kn}$ is evidently bounded l.s.c. with compact values, the claim follows as a corollary of Proposition 4.1. $\square$

In view of (4.9) and Proposition 4.5 implying $S(\eta) \neq \emptyset$, we are ready to formulate the related theorem for higher order inclusions.

Theorem 4.6. *Let $\varphi \colon [0,1] \times \mathbb{R}^{kn} \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a compact acyclic mapping. Then the set $S(\varphi)$ of Carathéodory solutions of every Cauchy problem related to* (4.6) *is nonempty, i.e. $S(\varphi) \neq \emptyset$, provided* (4.8) *holds.*

5. IMPLICIT BOUNDARY VALUE PROBLEMS

We can consider more generally the implicit differential inclusions with a constraint $C \subset \mathrm{AC}([0,1], \mathbb{R}^n)$, namely

$$\begin{cases} x'(t) \in \varphi(t, x(t), x'(t)), \text{ for a.a. } t \in [0,1], \\ x \in C, \end{cases} \tag{5.1}$$

where $\varphi\colon [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a compact acyclic mapping such that

$$\dim \operatorname{Fix} \varphi(t, x, \cdot) = 0, \text{ for every } (t, x) \in [0,1] \times \mathbb{R}^n, \tag{5.2}$$

and C can be a set of initial conditions or boundary conditions or functional conditions, etc.

By a *solution* $x\colon [0,1] \to \mathbb{R}^n$ of (5.1), we mean the Carathéodory solution, i.e. an absolutely continuous function which satisfies (5.1) almost everywhere.

In view of the arguments presented in Section 3, we can also consider the explicit problems

$$\begin{cases} x'(t) \in \psi(t, x(t)), \text{ for a.a. } t \in [0,1], \\ x \in C, \end{cases} \tag{5.3}$$

where $\psi(t, x) := \operatorname{Fix}(\varphi(t, x, \cdot))$, for every $(t, x) \in [0,1] \times \mathbb{R}^n$, and

$$\begin{cases} x'(t) \in \eta(t, x(t)), \text{ for a.a. } t \in [0,1], \\ x \in C, \end{cases} \tag{5.4}$$

where $\eta(t, x) := \overline{\operatorname{Ess}(\varphi(t, x, \cdot))}$, for every $(t, x) \in [0,1] \times \mathbb{R}^n$.

Denoting by $S_C(\varphi)$, $S_C(\psi)$, $S_C(\eta)$ the respective solution sets of problems (5.1), (5.2), (5.3), we get quite analogously as in Section 3 the analogy of (4.5), i.e.

$$S_C(\eta) \subset S_C(\psi) = S_C(\varphi). \tag{5.5}$$

These relations can be still completed by means of the following theorem due to A. Bressan [12, Theorem 6] which we state here in the form of proposition.

Proposition 5.1. *Let $\eta\colon [0,1] \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a bounded l.s.c. mapping with closed values. Then there exists a bounded u.s.c. mapping with compact, convex values $\hat{\eta}\colon [0,1] \times \mathbb{R}^n \multimap \mathbb{R}^n$ such that every Carathéodory solution of $x'(t) \in \hat{\eta}(t, x(t))$ is also a solution of $x'(t) \in \eta(t, x(t))$*

Remark 5.2. In fact, the mapping $\hat{\eta}$ is a convex and compact valued bounded u.s.c. regularization of a single-valued directionally continuous selection of η. For the definitions and more details, see [11, 12].

Hence, considering the problem involving such a mapping $\hat{\eta}$, namely

$$\begin{cases} x'(t) \in \hat{\eta}(t, x(t)), \text{ for a.a. } t \in [0,1], \\ x \in C, \end{cases} \tag{5.6}$$

and denoting by $S_C(\hat{\eta})$ the set of all solutions to (5.6), relations (5.4) can be completed, by virtue of Proposition 5.1, into

$$S_C(\hat{\eta}) \subset S_C(\eta) \subset S_C(\psi) = S_C(\varphi). \tag{5.7}$$

In order to have $S_C(\hat{\eta}) \neq \emptyset$, let us specify C to be the boundary condition

$$Lx = 0,$$

where $L\colon \mathrm{AC}([0,1], \mathbb{R}^n) \to \mathbb{R}^n$ is a linear bounded operator, and assume that the homogeneous problem

$$\begin{cases} x'(t) = 0, \ t \in [0,1], \\ Lx = 0, \end{cases} \tag{5.8}$$

has only the trivial solution $0 \in \mathbb{R}^n$. It is well known (see e.g. [1, Example III.5.37], [30, Theorem 3]) that then the boundary value problem

$$\begin{cases} x'(t) \in \hat{\eta}(t, x(t)), \text{ for a.a. } t \in [0,1], \\ Lx = 0, \end{cases} \tag{5.9}$$

admits a Carathéodory solution, i.e. $S_L(\hat{\eta}) \neq \emptyset$ holds for the solution set $S_L(\hat{\eta})$ of (5.9).

Therefore, in view of (5.7), we can formulate the following theorem.

Theorem 5.3. *Let $\varphi\colon [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a compact acyclic mapping satisfying* (5.2). *Assume, furthermore, that the homogeneous problem* (5.8), *where $L\colon \mathrm{AC}([0,1], \mathbb{R}^n) \to \mathbb{R}^n$ is a given linear bounded operator, has only the trivial solution $0 \in \mathbb{R}^n$. Then the implicit inclusion $x'(t) \in \varphi(t, x(t), x'(t))$ admits on the interval $[0,1]$ a Carathéodory solution $x(\cdot)$ such that $Lx(\cdot) = 0$.*

Remark 5.4. As an illustrative example of the application of Theorem 5.3, we can consider the Cauchy–Nicoletti problem, i.e. (5.1), where $C = \{\mathrm{AC}([0,1], \mathbb{R}^n) \mid x = (x_1, \dots, x_n),\ x_i(t_i) = 0,\ t_i \in [0,1],\ i = 1, \dots, n\}$, or the Cauchy problem, i.e. (5.1), where $C = \{\mathrm{AC}([0,1], \mathbb{R}^n) \mid x(0) = 0\}$.

Similarly as in Section 4, higher order problems can be often suitably transformed to the first order problems for which Theorem 5.3 already applies. For instance, consider the implicit Picard problem

$$\begin{cases} x''(t) \in \varphi(t, x(t), x'(t), x''(t)), \text{ for a.a. } t \in [0,1], \\ x(0) = x(1) = 0, \end{cases} \tag{5.10}$$

where $\varphi\colon [0,1]\times\mathbb{R}^{2n}\times\mathbb{R}^n \multimap \mathbb{R}^n$ is a compact acyclic mapping such that

$$\dim \operatorname{Fix}\varphi(t,x,y,\cdot)=0,\ \text{for every } (t,x,y)\in[0,1]\times\mathbb{R}^{2n}. \tag{5.11}$$

By a *solution* $x\colon [0,1]\to\mathbb{R}^n$ of (5.10), we understand a function whose first derivative is absolutely continuous, i.e. $x\in \mathrm{AC}^{(1)}([0,1],\mathbb{R}^n)$, which satisfies (5.10), for almost all $t\in[0,1]$.

In order to apply Theorem 5.3, one can readily check that there exists a suitable constant M, implied by the boundedness of φ, such that

$$\|x''(t)\|\le M \text{ holds a.e. on } [0,1], \quad \text{and } \max_{t\in[0,1]}\|x(t)\|\le M,\ \max_{t\in[0,1]}\|x'(t)\|\le M, \tag{5.12}$$

for every solution $x(\cdot)$ of (5.10).

Therefore, problem (5.10) can be equivalently rewritten into the first order truncated problem (cf. (5.12)):

$$\begin{cases} \boldsymbol{x}'(t)\in\boldsymbol{\varphi}^*(t,\boldsymbol{x}(t),\boldsymbol{x}'(t)),\ \text{a.e. on } [0,1],\\ x_0(0)=x_0(1)=0, \end{cases} \tag{5.13}$$

where

$$\boldsymbol{x}(t)=(x_0(t),x_1(t)),\ \boldsymbol{x}'(t)=(x_1(t),x_1'(t)),\ \boldsymbol{\varphi}^*(t,y_0,y_1,y_2)=(y_1^*,\varphi^*(t,y_0,y_1,y_2))^T,$$

and

$$y_j^*=\begin{cases} y_j, & \text{for } \|y_j\|\le M,\\ r_j, & \text{where } \|r_j\|=M,\ \text{for } y_j=r_jt,\ t\ge1,\ j=0,1,2, \end{cases}$$

$$\eta^*=\begin{cases} \varphi, & \text{for } \|\varphi\|\le M,\\ r, & \text{where } \|r\|=M,\ \text{for } \varphi=rt,\ t\ge1. \end{cases}$$

Since $\boldsymbol{\varphi}^*\colon [0,1]\times\mathbb{R}^{2n}\times\mathbb{R}^{2n}\multimap\mathbb{R}^{2n}$ is evidently a compact acyclic mapping satisfying (cf. (5.11))

$$\dim\operatorname{Fix}\varphi^*(t,y_0,y_1,\cdot)=0,\ \text{for every } (t,y_0,y_1)\in[0,1]\times\mathbb{R}^{2n},$$

Theorem 5.3 implies the existence of a solution of (5.13) as well as of (5.10).

We can conclude this section by the following theorem.

Theorem 5.5. *Let* $\varphi\colon [0,1]\times\mathbb{R}^{2n}\times\mathbb{R}^n\multimap\mathbb{R}^n$ *be a compact acyclic mapping satisfying* (5.11). *Then the implicit Picard problem* (5.10) *admits a Carathéodory solution.*

Remark 5.6. We could also proceed analogously, step by step, as in Section 4, when applying Proposition 5.1.

6. IMPLICIT PROBLEMS ON PROXIMATE RETRACTS

Differential inclusions on proximate retracts were studied e.g. in [9, 26, 40]. In this section, initial as well as boundary value problems will be considered for implicit differential inclusions on proximate retracts.

Let $K \subset \mathbb{R}^n$ be a compact proximate retract. Let us recall that a subset $T_K(x) \subset \mathbb{R}^n$, $x \in K$, defined by

$$T_K(x) := \left\{ y \in \mathbb{R}^n \;\middle|\; \liminf_{t \to 0+} \frac{\operatorname{dist}(x + ty, K)}{t} = 0 \right\},$$

is called the *Bouligand tangent cone* to K at x.

Taking this time $Y = [0,1] \times K$, $X = \mathbb{R}^n$, and $\varphi\colon [0,1] \times K \times \mathbb{R}^n \multimap \mathbb{R}^n$ to be a compact acyclic map which satisfies the condition

$$\varphi(t,x,y) \subset T_K(x), \text{ for every } (t,x,y) \in [0,1] \times K \times \mathbb{R}^n, \tag{6.1}$$

the following Cauchy problem for implicit differential inclusions will be at first under our consideration:

$$x'(t) \in \varphi(t, x(t), x'(t)) \text{ and } x(0) = x_0 \in K. \tag{6.2}$$

By a *solution* $x\colon [0,1] \to K$ of (6.2), we mean again an absolutely continuous function which satisfies (6.2), for almost all $t \in [0,1]$.

The symbol $S(\varphi; K)$ will be reserved for the set of all solutions of (6.2).

By the hypothesis, we have guaranteed that $\operatorname{Fix}(\varphi(t,x,\cdot))$ is a nonempty compact set, for every $(t,x) \in [0,1] \times K$. Thus (cf. Section 4), we can consider the map $\psi(t,x)\colon [0,1] \times K \multimap \mathbb{R}$ defined by:

$$\psi(t,x) := \operatorname{Fix}(\varphi(t,x,\cdot)), \text{ for every } (t,x) \in [0,1] \times K,$$

and the associated Cauchy problem:

$$x'(t) \in \psi(t, x(t)) \text{ and } x(0) = x_0 \in K. \tag{6.3}$$

Let the symbol $S(\psi; K)$ denote the set of all solutions of (6.3).

Assuming still

$$\dim \operatorname{Fix}(\varphi(t,x,\cdot)) = 0, \quad \text{for every } (t,x) \in [0,1] \times K, \tag{6.4}$$

we can define the map $\eta\colon [0,1] \times K \multimap \mathbb{R}^n$,

$$\eta(t,x) := \overline{\operatorname{Ess}(\varphi(t,x,\cdot))}, \quad \text{for every } (t,x) \in [0,1] \times K.$$

We also let $S(\eta; K)$ to be the set of solutions of the following Cauchy problem:

$$x'(t) \in \eta(t, x(t)) \text{ and } x(0) = x_0 \in K. \tag{6.5}$$

Observe that the mappings ψ and η obviously satisfy the conditions

$$\psi(t,x) \subset T_K(x), \quad \text{for every } (t,x) \in [0,1] \times K, \tag{6.6}$$

and

$$\eta(t,x) \subset T_K(x), \quad \text{for every } (t,x) \in [0,1] \times K. \tag{6.7}$$

Moreover, it was proved in Section 3 that ψ is a u.s.c. mapping with compact values and η is an l.s.c. mapping with compact values such that $S(\eta;K) \subset S(\psi;K) = S(\varphi;K)$.

Now, we will embed the existence problems for (6.3) and (6.5) to the respective problems for differential inclusions in $\mathbb{R}^n$.

Since $K \in \mathrm{PR}$, there exist an open subset U of $\mathbb{R}^n$ such that $K \subset U$ and a (continuous) retraction $r\colon U \to K$ satisfying (2.1).

Let $\alpha\colon \mathbb{R}^n \to [0,1]$ be the Urysohn function for K, i.e. $\alpha(x) = 1$, for every $x \in K$, $\alpha(x) = 0$, for every $x \in \mathbb{R}^n \setminus U$, and α is continuous. We are able to define two mappings (extensions) $\tilde{\psi}, \tilde{\eta}\colon [0,1] \times \mathbb{R}^n \multimap \mathbb{R}^n$ induced respectively by ψ and η as follows:

$$\tilde{\psi}(t,x) = \begin{cases} \alpha(x) \cdot \psi(t, r(x)), & \text{for } x \in U, \\ 0, & \text{for } x \in \mathbb{R}^n \setminus U, \end{cases}$$

$$\tilde{\eta}(t,x) = \begin{cases} \alpha(x) \cdot \eta(t, r(x)), & \text{for } x \in U, \\ 0, & \text{for } x \in \mathbb{R}^n \setminus U. \end{cases}$$

Lemma 6.1. *(i) $\tilde{\psi}$ is a compact u.s.c. mapping, and (ii) $\tilde{\eta}$ is a compact l.s.c. mapping with closed values.*

For the proof, see [9, 40].

Proposition 6.2. *We have $S(\psi;K) = S(\tilde{\psi};K)$ and $S(\eta;K) = S(\tilde{\eta};K)$ and, in view of the above arguments and Lemma* 6.1*, the desired relations can be sum up as follows:*

$$S(\eta;K) = S(\tilde{\eta};K) \subset S(\psi;K) = S(\tilde{\psi};K) = S(\varphi;K). \tag{6.8}$$

For the proof, see [9, Proposition 9.2 and Remark 9.3] and [26, Lemma 2.4].

Since $S(\tilde{\eta};K) \neq \emptyset$, according to Proposition 4.1, we arrive (in view of (6.8)) at $S(\varphi;K) \neq \emptyset$. An alternative proof can be obtained by means of [26, Theorem 3.1] whose all assumptions are satisfied for the maps ψ and η. In fact, since ψ and η are compact maps with closed values satisfying conditions (6.6) and (6.7), all solutions of (6.3) and (6.5) are bounded and, in particular, they are integrably bounded. Moreover, ψ is compact u.s.c. and η is compact l.s.c. Thus, Theorem 3.1 in [26] can be applied for both ψ and η, and so we arrive at the relations $S(\eta;K) = S(\tilde{\eta};K) \neq \emptyset$ and $S(\psi;K) = S(\tilde{\psi};K) \neq \emptyset$, and subsequently (see (6.8)) $S(\varphi;K) \neq \emptyset$.

We are ready to formulate the related theorem.

Theorem 6.3. *Let $\varphi\colon [0,1] \in \mathbb{R}\times\mathbb{R}^n \multimap \mathbb{R}^n$ be a compact acyclic mapping such that* (6.1) *is satisfied for a compact proximate retract $K\subset\mathbb{R}^n$. Then the set $S(\varphi;K)$ of Carathéodory solutions of* (6.2) *is nonempty, i.e. $S(\varphi;K)\neq\emptyset$, provided condition* (6.4) *holds.*

Now, we will consider the periodic problem for implicit differential inclusions on a special proximate retract.

Hence, let $K\subset\mathbb{R}^n$ be a compact, convex subset of $\mathbb{R}^n$ and $\varphi\colon\mathbb{R}\times K\times\mathbb{R}^n\multimap\mathbb{R}^n$ be a compact acyclic map which satisfies condition (6.1). Assume, furthermore, that φ is 1-periodic in the first variable on the whole line, i.e. that $\varphi(t,x,y)\equiv\varphi(t+1,x,y)$.

The problem under our consideration reads as follows:

$$\tag{6.9}\begin{cases} x'(t)\in\varphi(t,x(t),x'(t)), \text{ for a.a. } t\in[0,1],\\ x(t)\in K, \text{ for every } t\in[0,1], \text{x(0)=x(1)}.\end{cases}$$

By a 1-*periodic solution* $x\colon\mathbb{R}\to K$ of the inclusion $x'\in\varphi(t,x,x')$, we mean a 1-periodic prolongation of a solution of (6.9), i.e. of an absolutely continuous function which satisfies (6.9), for almost all $t\in[0,1]$.

Besides (6.9), consider also the periodic problems

$$\tag{6.10}\begin{cases} x'(t)\in\psi(t,x(t)), \text{ for a.a. } t\in[0,1],\\ x(t)\in K, \text{ for every } t\in[0,1], \text{x(0)=x(1)},\end{cases}$$

where $\psi(t,x):=\operatorname{Fix}(\varphi(t,x,\cdot))$, for every $(t,x)\in[0,1]\times K$, and

$$\tag{6.11}\begin{cases} x'(t)\in\eta(t,x(t)), \text{ for a.a. } t\in[0,1],\\ x(t)\in K, \text{ for every } t\in[0,1], \text{x(0)=x(1)},\end{cases}$$

where $\eta(t,x):=\overline{\operatorname{Ess}(\varphi(t,x,\cdot))}$, for every $(t,x)\in[0,1]\times K$, provided condition (6.4) takes place (see Proposition 3.5).

Since the maps ψ and η satisfy conditions (6.6) and (6.7), we have again that (as in Section 3) ψ is a u.s.c. mapping with compact values and η is an l.s.c. mapping with compact values such that

$$\tag{6.12} S_P(\eta,K)\subset S_P(\psi,K)=S_P(\varphi,K),$$

where $S_P(\eta,K)$, $S_P(\psi,K)$, $S_P(\varphi,K)$ denote the solution sets of (6.9), (6.10), (6.11), respectively.

In order to get $S_P(\eta;K)\neq\emptyset$, we will apply the following theorem of A. Bressan [12, Theorem 7] which we state here in the form of proposition.

Proposition 6.4. *Let* $\eta\colon \mathbb{R}\times K \multimap \mathbb{R}^n$*, where* $K \subset \mathbb{R}^n$ *is a compact, convex subset of* $\mathbb{R}^n$*, be a bounded l.s.c. mapping with compact values such that* $\eta(t,x,y) \equiv \eta(t+1,x,y)$*. If condition* (6.7) *holds, then the differential inclusion* $x'(t) \in \eta(t,x(t))$ *admits a* 1*-periodic solution with values in* K*.*

Since φ is by the hypothesis compact, it is in particular bounded, and so must be ψ and η. Thus, all the assumptions of Proposition 6.4 are satisfied, by which $S_P(\eta;K) \neq \emptyset$, as claimed. In view of (6.12), we can give therefore the last theorem.

Theorem 6.5. *Let* $K \subset \mathbb{R}^n$ *be a compact, convex subset of* $\mathbb{R}^n$ *and* $\varphi\colon \mathbb{R}\times K\times \mathbb{R}^n \multimap \mathbb{R}^n$ *be a compact, acyclic mapping which satisfies condition* (6.1)*. Assume, furthermore, that* $\varphi(t,x,y) \equiv \varphi(t+1,x,y)$*. Then the implicit differential inclusion* $x'(t) \in \varphi(t,x(t),x'(t))$ *admits a* 1*-periodic Carathéodory solution with values in* K*, i.e.* $S_P(\varphi;K) \neq \emptyset$*, provided condition* (6.4) *holds.*

7. CONCLUDING REMARKS AND COMMENTS

As already pointed out in Introduction, the main novelty of our paper consists in a consideration of implicit inclusions with acyclic right-hand sides, i.e. u.s.c. maps with acyclic values. Let us recall that the class of acyclic sets (i.e. sets which are homologically equivalent to one point spaces) contains, besides standard convex sets, more general absolute retracts, R_δ-sets and contractible sets (i.e. sets which are homotopically equivalent to one point spaces). Therefore, there are many non-convex valued right-hand sides for which our theorems apply. On the other hand, nontrivial examples of acyclic sets which are not contractible, i.e. unless they have a trivial fundamental group, are rather sophisticated like the punctured Poincaré homology sphere or the Alexander horned sphere. In this light, for those who are not familiar with the algebraic topology, the values of right-hand sides of inclusions under our consideration are mainly compact and contractible.

We could also see that, besides the standard regularity (u.s.c.) assumptions imposed on the right-hand sides of given inclusions, the only additional restriction was the zero dimensionality of the sets of fixed points (see (4.4), (4.8), (5.2), (5.11), (6.1)) which might not be easy to verify in general. Nevertheless, especially for single-valued maps, there are plenty of situations, where even the finiteness of the sets of fixed points can be easily visible.

Due to this tendentious (for the sake of transparentness) singlicity of assumptions, the obtained existence criteria are obviously not best possible. Using the a priori estimates technique, the assumed explicit compactness and, in particular, boundedness of maps on the right-hand sides of given inclusions, can be significantly relaxed. Moreover, the maps need not be considered on the Cartesian products of the whole Euclidean spaces, but only on their ANR-subsets, including the last variable in which the problems become implicit. In this case, we can apply the Lefschetz number or the fixed point index, as defined in Section 3, in order to guarantee the existence of fixed points. Our approach would also apply, on the basis of the theory developed by A. Bressan and A. Cortesi [13], to implicit problems in Banach spaces. Another possibility would consist in replacing the acyclic maps by the class of directionally u.s.c. multivalued maps, considered by ourselves in [4], but provided the zero dimensionality (of fixed points) condition is valid for their u.s.c. regularization (see [4, Definition 3.3]).

Implicit problems were also studied for partial differential equations and inclusions (see e.g. [17, 18, 19, 34, 42, 44], and the references therein), but as far as we know not for inclusions with acyclic right-hand sides. We will therefore finally indicate, how our approach could fit to them, especially for hyperbolic inclusions.

Hence, letting $Y = [0,1] \times [0,1] \times \mathbb{R}^{3n}$, $X = \mathbb{R}^n$ and $\varphi\colon Y \times X \multimap X$ to be a compact acyclic map, let us consider the Darboux problem, namely

$$\begin{cases} \frac{\partial^2 u(t,s)}{\partial t \partial s} \in \varphi\left(t, s, u(t,s), \frac{\partial u(t,s)}{\partial t}, \frac{\partial u(t,s)}{\partial s}, \frac{\partial^2 u(t,s)}{\partial t \partial s}\right), \\ \qquad \text{for a.a. } (t,s) \in [0,1] \times [0,1], \\ u(t,0) = \lambda(t,0), \text{ for } t \in [0,1], \ u(0,s) = \lambda(0,s), \text{ for } s \in [0,1], \end{cases} \tag{7.1}$$

where $\lambda\colon [0,1] \times [0,1] \to \mathbb{R}^n$ is a given (single-valued) continuous function.

By a *solution* $u\colon [0,1] \times [0,1] \to \mathbb{R}^n$ of (7.1), we understood a function such that $u(t,\cdot), u(\cdot,s) \in \mathrm{AC}([0,1],\mathbb{R}^n)$ and $\frac{\partial^2 u(t,s)}{\partial t \partial s}\colon [0,1]^2 \to \mathbb{R}^2$ is integrable, which satisfies (7.1), for almost all $(t,s) \in [0,1] \times [0,1]$.

Assuming that

$$\dim \operatorname{Fix} \varphi(t, s, u_0, u_1, u_2) = 0$$

holds, for all $(t, s, u_0, u_1, u_2) \in [0,1]^2 \to \mathbb{R}^{3n}$, the solvability of (7.1) can be easily reduced, by the same arguments as for ordinary differential inclusions, to the one for inclusions whose right-hand sides ψ and η can be defined quite analogously as in the ordinary cases above. Thus,

$$S_D(\eta) \subset S_D(\psi) = S_D(\varphi), \tag{7.2}$$

where $S_D(\eta)$, $S_D(\psi)$, $S_D(\varphi)$ denote the solution sets of the respective Darboux problems.

Furthermore, the directionally continuous selection η of an l.s.c. mapping ψ can be regularized into the convex and compact valued bounded mapping $\hat{\eta}\colon Y \multimap \mathbb{R}^{3n}$, for which the associated Darboux problem

$$\begin{cases} \frac{\partial^2 u(t,s)}{\partial t \partial s} \in \hat{\eta}\left(t, s, u(t,s), \frac{\partial u(t,s)}{\partial t}, \frac{\partial u(t,s)}{\partial s}\right), \text{ for a.a. } (t,s) \in [0,1] \times [0,1], \\ u(t,0) = \lambda(t,0), \text{ for } t \in [0,1], \ u(0,s) = \lambda(0,s), \text{ for } s \in [0,1], \end{cases} \tag{7.3}$$

has a solution (see e.g. [41, Theorem V.1.2]).

On the other hand, because of the absence of the "partial" analogy to Proposition 5.1, we are not sure whether or not $S_D(\hat{\eta}) \subset S_D(\eta)$ holds as in (5.7), where $S_D(\hat{\eta})$ denotes the set of solutions to (7.3). In the positive case, there would exist a solution of (7.1).

For elliptic differential inclusions, the situation is similar. Letting $Y = B_n(0,1) \times \mathbb{R}^{2n}$, $X = \mathbb{R}^n$, where $B_n(0,1)$ denotes the unit closed ball in $\mathbb{R}^n$, and $\varphi\colon Y \times \mathbb{R}^n \multimap \mathbb{R}^n$ to be a compact acyclic mapping, let us consider the Dirichlet problem, namely

$$
\begin{cases} \Delta u(z) \in \varphi(z, u(z), \nabla u(z), \Delta u(z)), \\ u(z) = 0, \ \text{for } z \in \partial B_n(0,1) := \{z \in \mathbb{R}^n \mid \|z\| = 1\}, \end{cases} \tag{7.4}
$$

where $\nabla u(z) := \sum_{i=1}^n \frac{\partial u(z)}{\partial z_i}$, $z = (z_1, \ldots, z_n)$ and $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial z_i^2}$ stands for the Laplace operator.

By a *solution* $u\colon B_n(0,1) \to \mathbb{R}^n$ of (7.4), we understood this time a function from a suitable Sobolev space (e.g. $W^{1,2}(B_n(0,1))$).

We can proceed quite analogously as in the hyperbolic case, obtaining inclusions (7.2), for the respective sets of solutions. Nevertheless, apart from a different regularity of solutions, the same obstruction related to the validity of the inclusion $S_D(\hat{\eta}) \subset S_D(\eta)$ remains as an open problem.

The same is true for implicit problems associated with parabolic differential inclusions.

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