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GENERAL HIGHER-ORDER DYNAMIC OPIAL INEQUALITIES WITH APPLICATIONS

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ABSTRACT. In this paper, we present some new generalizations of dynamic Opial-type inequalities of higher order on time scales. The results contain as special cases many of the results currently given in literature. As an application, we apply these inequalities together with a Hardy-type inequality on time scales to establish some lower bounds of the distance between zeros of a solution and/or its derivatives for a fourth-order dynamic equation.

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1. Introduction

In 1960, Olech [15] extended an inequality of Opial [16] and proved that if $f \in \mathrm{C}^1([0,h],\mathbb{R})$ with $h>0$ satisfies $f(0)=0$, then

$$\int_0^h |f(t)f'(t)|\,\mathrm{d}t \le \frac{h}{2}\int_0^h (f'(t))^2\mathrm{d}t. \tag{1.1}$$

In 1995, Agarwal and Pang published an entire monograph [6] devoted to (1.1), its extensions, its discrete counterparts, and its applications. In 2001, Bohner and Kaymakçalan [8] (see also [11, Theorem 6.23]) initiated the study of dynamic versions of (1.1) and proved that if $\mathbb{T}$ is an arbitrary time scale and $f \in \mathrm{C}^1_{\mathrm{rd}}([0,h]_{\mathbb{T}},\mathbb{R})$ with $h>0$ satisfies $f(0)=0$, then

$$\int_0^h \left|(f^2)^\Delta(t)\right|\Delta t \le h\int_0^h (f^\Delta(t))^2\Delta t. \tag{1.2}$$

The monograph [5, Chapter 3] contains a summary of the results between 2001 and 2014 that generalized (1.2). Using a novel technique introduced in 2015 in [9] and pursued in [10] (see also [2]), the recent paper [1] proves the following general dynamic inequality of Opial type.

Theorem 1.1 (See [1, Theorem 3])**.** *Assume that $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,*

$$r, s \in \mathrm{C}_{\mathrm{rd}}([a,b]_{\mathbb{T}},(0,\infty)), \quad \text{and} \quad f \in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R}).$$

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Let $\alpha \geq 1$, $\beta \geq 0$, and $k > \beta + 1$. If $f(a) = 0$, then

$$\int_a^b s(t) \left|(f^\alpha)^\Delta(t)(f^\Delta(t))^\beta\right| \Delta t \leq K \left\{ \int_a^b r(t) \left|f^\Delta(t)\right|^k \Delta t \right\}^{\frac{\alpha+\beta}{k}},$$

where

$$K = c \left\{ \int_a^b (s(t))^{\frac{k}{k-\beta-1}} \frac{\left(R^{\frac{k\alpha-\alpha-\beta}{k-\beta-1}}\right)^\Delta(t)}{(r(t))^{\frac{k\beta}{(k-1)(k-\beta-1)}}} \Delta t \right\}^{\frac{k-\beta-1}{k}}$$

with

$$c = \alpha \left(\frac{k-\beta-1}{k\alpha-\alpha-\beta}\right)^{\frac{k-\beta-1}{k}} \left(\frac{\beta+1}{\alpha+\beta}\right)^{\frac{\beta+1}{k}}$$

and

$$R(t) = \int_a^t \frac{\Delta\tau}{(r(\tau))^{\frac{1}{k-1}}}.$$

The purpose of this paper is to study Opial-type inequalities that involve not just derivatives of order one but derivatives of higher order of the function f. We offer several generalizations of the following recent result from [10] and apply them to the study of fourth-order dynamic equations on time scales.

Theorem 1.2 (See [10, Theorem 4.1])**.** *Let $n \in \mathbb{N}$. Assume that $a \in \mathbb{T}$, $b \in (a, \infty)_\mathbb{T}$,*

$$r, s \in \mathrm{C_{rd}}([a,b]_\mathbb{T}, (0,\infty)), \quad \text{and} \quad f \in \mathrm{C}^n_{\mathrm{rd}}([a,b]_\mathbb{T}, \mathbb{R}).$$

Let $\alpha > 1$ and $\beta \geq 0$. If $f^{\Delta^i}(a) = 0$ for all $0 \leq i \leq n-1$, then

$$\int_a^b s(t) \left|(f^\alpha)^\Delta(t)(f^{\Delta^n}(t))^\beta \frac{f^{\Delta^n}(t)}{f^\Delta(t)}\right| \Delta t \leq K \int_a^b r(t) \left|f^{\Delta^n}(t)\right|^{\alpha+\beta} \Delta t,$$

where

$$K = \frac{\alpha(\beta+1)^{\frac{\beta+1}{\alpha+\beta}}}{\alpha+\beta} \left\{ \int_a^b (s(t))^{\frac{\alpha+\beta}{\alpha-1}} \frac{(R^{\alpha+\beta})^\Delta(t)}{(r(t))^{\frac{\beta+1}{\alpha-1}} R^\Delta(t)} \Delta t \right\}^{\frac{\alpha-1}{\alpha+\beta}}$$

with

$$R(t) = \int_a^t \frac{(h_{n-1}(t,\sigma(\tau)))^{\frac{\alpha+\beta}{\alpha+\beta-1}}}{(r(\tau))^{\frac{1}{\alpha+\beta-1}}} \Delta\tau.$$

Note that some related results which are contained and/or improved in our study are given by Anastassiou in [7, Proposition 2.9], Saker in [19, Theorem 2.9], Srivastava, Tseng, Tseng, and Lo in [21, Theorem 4], and Wong, Lian, Yeh, and Yu in [22, Theorem 2.2]. In addition, our presented results contain as special cases some results given in [1, 4, 8–10, 13] and improve the inequalities given in [17, 18, 20].

This paper is organized as follows. In Section 2, we briefly present the basic time scales calculus definitions and results that are used in the sequel. Sections 3 and 4 contain the statements and proofs of the two main results as well as several corollaries and remarks. In Section 5, we employ the inequalities from Sections 3 and 4 together with some new Hardy-type inequalities [14] in order to establish lower bounds of the distance between zeros of a solution and/or its derivatives for a fourth-order dynamic equation on time scales.

2. **Time Scales Preliminaries**

In this section, we briefly present some basic definitions and results concerning the delta calculus on time scales that we use in this article. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. We define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ for $t \in \mathbb{T}$. For any function $f : \mathbb{T} \to \mathbb{R}$, we put $f^\sigma = f \circ \sigma$. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous, denoted by $f \in \mathrm{C_{rd}}$, if it is continuous at each right-dense point (i.e., $\sigma(t) = t$) and there exists a finite left-sided limit at all left-dense points (i.e., $\rho(t) = t$, where the backward jump ρ is defined in a similar way as the forward jump σ). For the definition of the delta derivative and the delta integral, we refer to [11, 12]. If $f \in \mathrm{C}^1(\mathbb{R}, \mathbb{R})$ and $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable, then the *time scales chain rule*, see [11, Theorem 1.90], states that

$$(f \circ g)^\Delta = g^\Delta \int_0^1 f'(hg^\sigma + (1-h)g^\Delta)\mathrm{d}h,$$

and a special case, which we use in Sections 3 and 4 of this paper, is given by

$$(f^\gamma)^\Delta = \gamma f^\Delta \int_0^1 (hf^\sigma + (1-h)f)^{\gamma-1}\,\mathrm{d}h \quad \text{for} \quad \gamma \in \mathbb{R}.$$

One of our main tools for the proofs given in Sections 3 and 4 is the *time scales Hölder inequality*, see [11, Theorem 6.13], which says

$$\int_a^b |f(t)g(t)|\,\Delta t \le \left\{\int_a^b |f(t)|^\gamma\,\Delta t\right\}^{\frac{1}{\gamma}} \left\{\int_a^b |g(t)|^\nu\,\Delta t\right\}^{\frac{1}{\nu}},$$

where $a, b \in \mathbb{T}$, $f, g \in \mathrm{C_{rd}}([a,b]_\mathbb{T}, \mathbb{R})$, $\gamma > 1$, and $\nu = \gamma/(\gamma-1)$. In our main results in Sections 3 and 4, we use the time scales monomials h_k, and we refer to their definition given in [3, 11]. In Section 3, we use the *time scales Taylor formula*, see [11, Theorem 1.113], i.e.,

$$f(t) = \sum_{k=0}^{n-1} h_i(t,a)f^{\Delta^i}(a) + \int_a^t h_{n-1}(t,\sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau$$

for $a \in \mathbb{T}$ and a function $f \in \mathrm{C}^n_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ with $n \in \mathbb{N}$, more precisely, its special case

$$f(t) = \int_a^t h_{n-1}(t,\sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau \tag{2.1}$$

that holds provided $f^{\Delta^i}(a) = 0$ for all $0 \le i \le n-1$, and replacing n by $n-m$ with $1 \le m \le n-1$ and f by f^{Δ^m},

$$f^{\Delta^m}(t) = \int_a^t h_{n-m-1}(t,\sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau, \tag{2.2}$$

which holds provided $f^{\Delta^{i+m}}(a) = 0$ for all $0 \le i \le n - m - 1$, and this last formula is used in Section 4. Finally, in Section 5, we also employ the *time scales integration by parts formula* (see [11, Theorem 1.77 (vi)])

$$\int_a^b f(t)g^\Delta(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t.$$

3. **Higher-Order Opial Inequalities**

Our first main result reduces to Theorem 1.2 when $k = \alpha + \beta$, just like Theorem 1.1 reduces to [9, Theorem 3.1] when $k = \alpha + \beta$.

Theorem 3.1. *Let $n \in \mathbb{N}$. Assume that $a \in \mathbb{T}$, $b \in (a, \infty)_\mathbb{T}$,*

$$r, s \in \mathrm{C_{rd}}([a, b]_\mathbb{T}, (0, \infty)), \quad \text{and} \quad f \in \mathrm{C}^n_{\mathrm{rd}}([a, b]_\mathbb{T}, \mathbb{R}).$$

Let $\alpha \ge 1$, $\beta \ge 0$, and $k > \beta + 1$. If $f^{\Delta^i}(a) = 0$ for all $0 \le i \le n - 1$, then

$$\int_a^b s(t) \left| (f^\alpha)^\Delta(t)(f^{\Delta^n}(t))^\beta \frac{f^{\Delta^n}(t)}{f^\Delta(t)} \right| \Delta t \le K \left\{ \int_a^b r(t) \left| f^{\Delta^n}(t) \right|^k \Delta t \right\}^{\frac{\alpha+\beta}{k}},$$

where

$$K = c \left\{ \int_a^b (s(t))^{\frac{k}{k-\beta-1}} \frac{\left(R^{\frac{k\alpha-\beta-\alpha}{k-\beta-1}} \right)^\Delta (t)}{(r(t))^{\frac{\beta+1}{k-\beta-1}} R^\Delta(t)} \Delta t \right\}^{\frac{k-\beta-1}{k}}$$

with

$$c = \alpha \left(\frac{k - \beta - 1}{k\alpha - \beta - 1} \right)^{\frac{k-\beta-1}{k}} \left(\frac{\beta + 1}{\alpha + \beta} \right)^{\frac{\beta+1}{k}}$$

and

$$R(t) = \int_a^t \frac{(h_{n-1}(t, \sigma(\tau)))^{\frac{k}{k-1}}}{(r(\tau))^{\frac{1}{k-1}}} \Delta\tau.$$

Proof. Define

$$g(t) := \int_a^t r(\tau) \left| f^{\Delta^n}(\tau) \right|^k \Delta\tau.$$

Then $g(a) = 0$,

$$g^\Delta = r \left| f^{\Delta^n} \right|^k \quad \text{so that} \quad \left| f^{\Delta^n} \right| = \left(\frac{g^\Delta}{r} \right)^{\frac{1}{k}},$$

and, by the time scales Taylor formula, more precisely, (2.1),

$$\begin{aligned} |f(t)| &= \left| \int_a^t \frac{h_{n-1}(t, \sigma(\tau))}{(r(\tau))^{\frac{1}{k}}} (r(\tau))^{\frac{1}{k}} f^{\Delta^n}(\tau)\Delta\tau \right| \\ &\le \int_a^t \frac{h_{n-1}(t, \sigma(\tau))}{(r(\tau))^{\frac{1}{k}}} (r(\tau))^{\frac{1}{k}} \left| f^{\Delta^n}(\tau) \right| \Delta\tau \\ &\le \left\{ \int_a^t \frac{(h_{n-1}(t, \sigma(\tau)))^{\frac{k}{k-1}}}{(r(\tau))^{\frac{1}{k-1}}} \Delta\tau \right\}^{\frac{k-1}{k}} \left\{ \int_a^t r(\tau) \left| f^{\Delta^n}(\tau) \right|^k \Delta\tau \right\}^{\frac{1}{k}} \end{aligned}$$

$$= (R(t))^{\frac{k-1}{k}} (g(t))^{\frac{1}{k}},$$

where we have used the time scales Hölder inequality with conjugate exponents $\frac{k}{k-1}$ and $k > 1$. Thus, for $h \in [0,1]$, we obtain, exactly as in the proof of [10, Theorem 3],

$$\begin{aligned}
&|hf^\sigma + (1-h)f| \le h\,|f^\sigma| + (1-h)\,|f| \\
&\le h\,(R^\sigma)^{\frac{k-1}{k}}\,(g^\sigma)^{\frac{1}{k}} + (1-h)R^{\frac{k-1}{k}} g^{\frac{1}{k}} \\
&= (hR^\sigma)^{\frac{k-1}{k}}\,(hg^\sigma)^{\frac{1}{k}} + ((1-h)R)^{\frac{k-1}{k}}\,((1-h)g)^{\frac{1}{k}} \\
&\le (hR^\sigma + (1-h)R)^{\frac{k-1}{k}}\,(hg^\sigma + (1-h)g)^{\frac{1}{k}},
\end{aligned}$$

where we have used the classical Hölder inequality for sums with conjugate exponents $\frac{k}{k-1}$ and $k > 1$. Hence, again exactly as in the proof of [10, Theorem 3],

$$\begin{aligned}
&\left|\int_0^1 (hf^\sigma + (1-h)f)^{\alpha-1}\,\mathrm{d}h\right| \\
&\le \int_0^1 |hf^\sigma + (1-h)f|^{\alpha-1}\,\mathrm{d}h \\
&\le \int_0^1 (hR^\sigma + (1-h)R)^{\frac{(k-1)(\alpha-1)}{k}}\,(hg^\sigma + (1-h)g)^{\frac{\alpha-1}{k}}\,\mathrm{d}h \\
&\le \left\{\int_0^1 (hR^\sigma + (1-h)R)^{\frac{(k-1)(\alpha-1)}{k-\beta-1}}\,\mathrm{d}h\right\}^{\frac{k-\beta-1}{k}} \\
&\quad \times \left\{\int_0^1 (hg^\sigma + (1-h)g)^{\frac{\alpha-1}{\beta+1}}\,\mathrm{d}h\right\}^{\frac{\beta+1}{k}},
\end{aligned}$$

where we have used the classical Hölder inequality for integrals with conjugate exponents $\frac{k}{k-\beta-1}$ and $\frac{k}{\beta+1} > 1$. Therefore, using the time scales chain rule three times, we get

$$\begin{aligned}
&\left|(f^\alpha)^\Delta \left(f^{\Delta^n}\right)^\beta \frac{f^{\Delta^n}}{f^\Delta}\right| = \alpha\left|f^{\Delta^n}\right|^{\beta+1}\left|\int_0^1 (hf^\sigma + (1-h)f)^{\alpha-1}\,\mathrm{d}h\right| \\
&= \frac{\alpha(g^\Delta)^{\frac{\beta+1}{k}}}{r^{\frac{\beta+1}{k}}}\left|\int_0^1 (hf^\sigma + (1-h)f)^{\alpha-1}\,\mathrm{d}h\right| \\
&\le \frac{\alpha(g^\Delta)^{\frac{\beta+1}{k}}}{r^{\frac{\beta+1}{k}}}\left\{\int_0^1 (hR^\sigma + (1-h)R)^{\frac{(k-1)(\alpha-1)}{k-\beta-1}}\,\mathrm{d}h\right\}^{\frac{k-\beta-1}{k}} \\
&\quad \times \left\{\int_0^1 (hg^\sigma + (1-h)g)^{\frac{\alpha-1}{\beta+1}}\,\mathrm{d}h\right\}^{\frac{\beta+1}{k}} \\
&= \frac{\alpha\left(g^\Delta\right)^{\frac{\beta+1}{k}}}{r^{\frac{\beta+1}{k}}}
\end{aligned}$$

$$\times \frac{\left\{\left(\frac{(k-1)(\alpha-1)}{k-\beta-1}+1\right) R^{\Delta} \int_0^1 \left(hR^{\sigma}+(1-h)R\right)^{\frac{(k-1)(\alpha-1)}{k-\beta-1}} \mathrm{d}h\right\}^{\frac{k-\beta-1}{k}}}{\left(\frac{k\alpha-\beta-\alpha}{k-\beta-1}\right)^{\frac{k-\beta-1}{k}} \left(R^{\Delta}\right)^{\frac{k-\beta-1}{k}}}$$

$$\times \frac{\left\{\left(\frac{\alpha-1}{\beta+1}+1\right) g^{\Delta} \int_0^1 \left(hg^{\sigma}+(1-h)g\right)^{\frac{\alpha-1}{\beta+1}} \mathrm{d}h\right\}^{\frac{\beta+1}{k}}}{\left(\frac{\alpha+\beta}{\beta+1}\right)^{\frac{\beta+1}{k}} \left(g^{\Delta}\right)^{\frac{\beta+1}{k}}}$$

$$= \frac{c}{r^{\frac{\beta+1}{k}} \left(R^{\Delta}\right)^{\frac{k-\beta-1}{k}}} \left\{\left(R^{\frac{k\alpha-\alpha-\beta}{k-\beta-1}}\right)^{\Delta}\right\}^{\frac{k-\beta-1}{k}} \left\{\left(g^{\frac{\alpha+\beta}{\beta+1}}\right)^{\Delta}\right\}^{\frac{\beta+1}{k}},$$

and thus finally

$$\begin{aligned}
&\int_a^b s(t) \left|\left(f^{\alpha}\right)^{\Delta}(t) \left(f^{\Delta^n}(t)\right)^{\beta} \frac{f^{\Delta^n}(t)}{f^{\Delta}(t)}\right| \Delta t \\
&\le c \int_a^b s(t) \frac{\left\{\left(R^{\frac{k\alpha-\alpha-\beta}{k-\beta-1}}\right)^{\Delta}(t)\right\}^{\frac{k-\beta-1}{k}}}{(r(t))^{\frac{\beta+1}{k}} \left(R^{\Delta}(t)\right)^{\frac{k-\beta-1}{k}}} \left\{\left(g^{\frac{\alpha+\beta}{\beta+1}}\right)^{\Delta}(t)\right\}^{\frac{\beta+1}{k}} \Delta t \\
&\le c \left\{\int_a^b (s(t))^{\frac{k}{k-\beta-1}} \frac{\left(R^{\frac{k\alpha-\alpha-\beta}{k-\beta-1}}\right)^{\Delta}(t)}{(r(t))^{\frac{\beta+1}{k-\beta-1}} R^{\Delta}(t)} \Delta t\right\}^{\frac{k-\beta-1}{k}} \\
&\quad \times \left\{\int_a^b \left(g^{\frac{\alpha+\beta}{\beta+1}}\right)^{\Delta}(t) \Delta t\right\}^{\frac{\beta+1}{k}} \\
&= K \left\{g^{\frac{\alpha+\beta}{\beta+1}}(b)\right\}^{\frac{\beta+1}{k}} \\
&= K (g(b))^{\frac{\alpha+\beta}{k}},
\end{aligned}$$

where we have used one last time the time scales Hölder inequality with conjugate exponents $\frac{k}{k-\beta-1}$ and $\frac{k}{\beta+1} > 1$. The proof is complete. ☐

Remark 3.2. As a special case, when $k = \alpha + \beta$, we see that Theorem 3.1 reduces to Theorem 1.2.

The next result follows from Theorem 3.1 by choosing $\beta = 0$.

Corollary 3.3. *Let $n \in \mathbb{N}$. Assume that $a \in \mathbb{T}$, $b \in (a, \infty)_{\mathbb{T}}$,*

$$r, s \in \mathrm{C_{rd}}([a,b]_{\mathbb{T}}, (0,\infty)), \quad \textit{and} \quad f \in \mathrm{C}^n_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}).$$

Let $\alpha \ge 1$ and $k > 1$. If $f^{\Delta^i}(a) = 0$ for all $0 \le i \le n-1$, then

$$\int_a^b s(t) \left|\left(f^{\alpha}\right)^{\Delta}(t) \frac{f^{\Delta^n}(t)}{f^{\Delta}(t)}\right| \Delta t \le K \left\{\int_a^b r(t) \left|f^{\Delta^n}(t)\right|^k \Delta t\right\}^{\frac{\alpha}{k}},$$

where

$$K = \left\{ \int_a^b (s(t))^{\frac{k}{k-1}} \frac{(R^\alpha)^\Delta(t)}{(r(t))^{\frac{1}{k-1}} R^\Delta(t)} \Delta t \right\}^{\frac{k-1}{k}}$$

with

$$R(t) = \int_a^t \frac{(h_{n-1}(t,\sigma(\tau)))^{\frac{k}{k-1}}}{(r(\tau))^{\frac{1}{k-1}}} \Delta\tau.$$

The next result follows from Corollary 3.3 by choosing $k = \alpha$.

Corollary 3.4. *Let $n \in \mathbb{N}$. Assume that $a \in \mathbb{T}$, $b \in (a,\infty)_\mathbb{T}$,*

$$r, s \in \mathrm{C_{rd}}([a,b]_\mathbb{T}, (0,\infty)), \quad \textit{and} \quad f \in \mathrm{C}^n_{\mathrm{rd}}([a,b]_\mathbb{T}, \mathbb{R}).$$

Let $\alpha > 1$. If $f^{\Delta^i}(a) = 0$ for all $0 \le i \le n-1$, then

$$\int_a^b s(t) \left| (f^\alpha)^\Delta(t) \frac{f^{\Delta^n}(t)}{f^\Delta(t)} \right| \Delta t \le K \int_a^b r(t) \left| f^{\Delta^n}(t) \right|^\alpha \Delta t,$$

where

$$K = \left\{ \int_a^b (s(t))^{\frac{\alpha}{\alpha-1}} \frac{(R^\alpha)^\Delta(t)}{(r(t))^{\frac{1}{\alpha-1}} R^\Delta(t)} \Delta t \right\}^{\frac{\alpha-1}{\alpha}}$$

with

$$R(t) = \int_a^t \frac{(h_{n-1}(t,\sigma(\tau)))^{\frac{\alpha}{\alpha-1}}}{(r(\tau))^{\frac{1}{\alpha-1}}} \Delta\tau.$$

The next result follows from Corollary 3.4 by choosing $\alpha = 2$.

Corollary 3.5. *Let $n \in \mathbb{N}$. Assume that $a \in \mathbb{T}$, $b \in (a,\infty)_\mathbb{T}$,*

$$r, s \in \mathrm{C_{rd}}([a,b]_\mathbb{T}, (0,\infty)), \quad \textit{and} \quad f \in \mathrm{C}^n_{\mathrm{rd}}([a,b]_\mathbb{T}, \mathbb{R}).$$

If $f^{\Delta^i}(a) = 0$ for all $0 \le i \le n-1$, then

$$\int_a^b s(t) \left| (f(t) + f(\sigma(t))) f^{\Delta^n}(t) \right| \Delta t \le K \int_a^b r(t) \left(f^{\Delta^n}(t) \right)^2 \Delta t,$$

where

$$K = \sqrt{\int_a^b (s(t))^2 \frac{R(t) + R(\sigma(t))}{r(t)} \Delta t}$$

with

$$R(t) = \int_a^t \frac{(h_{n-1}(t,\sigma(\tau)))^2}{r(\tau)} \Delta\tau.$$

The next result follows from Corollary 3.3 by choosing $n = 1$ (see also [1, Corollary 1]).

Corollary 3.6. *Assume that* $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,

$$r, s \in \mathrm{C}_{\mathrm{rd}}([a,b]_{\mathbb{T}}, (0,\infty)), \quad \textit{and} \quad f \in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}).$$

Let $\alpha \geq 1$ *and* $k > 1$. *If* $f(a) = 0$, *then*

$$\int_a^b s(t) \left|(f^\alpha)^\Delta(t)\right| \Delta t \leq K \left\{ \int_a^b r(t) \left|f^\Delta(t)\right|^k \Delta t \right\}^{\frac{\alpha}{k}},$$

where

$$K = \left\{ \int_a^b (s(t))^{\frac{k}{k-1}} \left(R^\alpha\right)^\Delta (t) \Delta t \right\}^{\frac{k-1}{k}}$$

with

$$R(t) = \int_a^t \frac{\Delta\tau}{(r(\tau))^{\frac{1}{k-1}}}.$$

The next result follows from Corollary 3.6 by choosing $k = \alpha$ (see also [1, Corollary 2] and [10, Corollary 3.2]).

Corollary 3.7. *Assume that* $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,

$$r, s \in \mathrm{C}_{\mathrm{rd}}([a,b]_{\mathbb{T}}, (0,\infty)), \quad \textit{and} \quad f \in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}).$$

Let $\alpha > 1$. *If* $f(a) = 0$, *then*

$$\int_a^b s(t) \left|(f^\alpha)^\Delta(t)\right| \Delta t \leq K \int_a^b r(t) \left|f^\Delta(t)\right|^\alpha \Delta t,$$

where

$$K = \left\{ \int_a^b (s(t))^{\frac{\alpha}{\alpha-1}} \left(R^\alpha\right)^\Delta (t) \Delta t \right\}^{\frac{\alpha-1}{\alpha}}$$

with

$$R(t) = \int_a^t \frac{\Delta\tau}{(r(\tau))^{\frac{1}{\alpha-1}}}.$$

Finally, if we put $s = r$ in Theorem 3.1, then we obtain the following result.

Corollary 3.8. *Let* $n \in \mathbb{N}$. *Assume that* $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,

$$r \in \mathrm{C}_{\mathrm{rd}}([a,b]_{\mathbb{T}}, (0,\infty)), \quad \textit{and} \quad f \in \mathrm{C}^n_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}).$$

Let $\alpha \geq 1$, $\beta \geq 0$, *and* $k > \beta + 1$. *If* $f^{\Delta^i}(a) = 0$ *for all* $0 \leq i \leq n-1$, *then*

$$\int_a^b r(t) \left|(f^\alpha)^\Delta(t) (f^{\Delta^n}(t))^\beta \frac{f^{\Delta^n}(t)}{f^\Delta(t)}\right| \Delta t \leq K \left\{ \int_a^b r(t) \left|f^{\Delta^n}(t)\right|^k \Delta t \right\}^{\frac{\alpha+\beta}{k}},$$

where

$$K = c \left\{ \int_a^b r(t) \frac{\left(R^{\frac{k\alpha-\beta-\alpha}{k-\beta-1}}\right)^\Delta (t)}{R^\Delta(t)} \Delta t \right\}^{\frac{k-\beta-1}{k}}$$

with

$$c=\alpha\left(\frac{k-\beta-1}{k\alpha-\beta-1}\right)^{\frac{k-\beta-1}{k}}\left(\frac{\beta+1}{\beta+\alpha}\right)^{\frac{\beta+1}{k}}$$

and

$$R(t)=\int_a^t\frac{(h_{n-1}(t,\sigma(\tau)))^{\frac{k}{k-1}}}{(r(\tau))^{\frac{1}{k-1}}}\Delta\tau.$$

Remark 3.9. Theorem 3.1 contains the results proved in the recent papers [1, Theorem 1.3], [10, Theorem 3.1 and Theorem 4.1], [4, Theorem 3.1], [19, Theorem 2.9] and improves the inequalities in [20, Theorem 2.1] and [18, Theorem 2.1] by removing the term which contains the graininess function. Moreover, Corollary 3.5 contains the results proved in [8, Theorem 1.3], [13, Theorem 3.1] and improves the inequalities in [17, Theorem 1].

4. **Different Higher-Order Opial Inequalities**

In this section, we prove a new weighted Opial-type inequality with different higher-order derivatives.

Theorem 4.1. *Let $n\in\mathbb{N}$. Assume that $a\in\mathbb{T}$, $b\in(a,\infty)_{\mathbb{T}}$,*

$$r,s\in \mathrm{C_{rd}}([a,b]_{\mathbb{T}},(0,\infty)),\quad \textit{and}\quad f\in \mathrm{C^n_{rd}}([a,b]_{\mathbb{T}},\mathbb{R}).$$

Let $\alpha,\beta>0$ and $k>\max\{1,\beta\}$. If $0\le m\le n-1$ and $f^{\Delta^{m+i}}(a)=0$ for all $0\le i\le n-m-1$, then

$$\int_a^b s(t)\left|f^{\Delta^m}(t)\right|^{\alpha}\left|f^{\Delta^n}(t)\right|^{\beta}\Delta t\le K\left\{\int_a^b r(t)\left|f^{\Delta^n}(t)\right|^{k}\Delta t\right\}^{\frac{\alpha+\beta}{k}},$$

where

$$K=\left(\frac{\beta}{\alpha+\beta}\right)^{\frac{\beta}{k}}\left\{\int_a^b (s(t))^{\frac{k}{k-\beta}}\frac{(R(t))^{\frac{\alpha(k-1)}{k-\beta}}}{(r(t))^{\frac{\beta}{k-\beta}}}\Delta t\right\}^{\frac{k-\beta}{k}}$$

with

$$R(t)=\int_a^t\frac{(h_{n-m-1}(t,\sigma(\tau)))^{\frac{k}{k-1}}}{(r(\tau))^{\frac{1}{k-1}}}\Delta\tau.$$

Proof. Define

$$g(t):=\int_a^t r(\tau)\left|f^{\Delta^n}(\tau)\right|^k\Delta\tau.$$

Then $g(a)=0$,

$$g^{\Delta}=r\left|f^{\Delta^n}\right|^k\quad\text{so that}\quad\left|f^{\Delta^n}\right|=\left(\frac{g^{\Delta}}{r}\right)^{\frac{1}{k}},$$

and, by the time scales Taylor formula, more precisely, (2.2),

$$\left|f^{\Delta^m}(t)\right|=\left|\int_a^t\frac{h_{n-m-1}(t,\sigma(\tau))}{(r(\tau))^{\frac{1}{k}}}(r(\tau))^{\frac{1}{k}}f^{\Delta^n}(\tau)\Delta\tau\right|$$

$$
\begin{aligned}
&\leq \int_a^t \frac{h_{n-m-1}(t,\sigma(\tau))}{(r(\tau))^{\frac{1}{k}}}(r(\tau))^{\frac{1}{k}} \left|f^{\Delta^n}(\tau)\right| \Delta\tau \\
&\leq \left\{\int_a^t \frac{(h_{n-m-1}(t,\sigma(\tau)))^{\frac{k}{k-1}}}{(r(\tau))^{\frac{1}{k-1}}}\Delta\tau\right\}^{\frac{k-1}{k}} \left\{\int_a^t r(\tau)\left|f^{\Delta^n}(\tau)\right|^k \Delta\tau\right\}^{\frac{1}{k}} \\
&= (R(t))^{\frac{k-1}{k}}(g(t))^{\frac{1}{k}},
\end{aligned}
$$

where we have used the time scales Hölder inequality with conjugate exponents $\frac{k}{k-1}$ and $k > 1$. Altogether, we have

$$
\left|f^{\Delta^m}\right|^\alpha \left|f^{\Delta^n}\right|^\beta \leq R^{\frac{(k-1)\alpha}{k}} g^{\frac{\alpha}{k}} \left(\frac{g^\Delta}{r}\right)^{\frac{\beta}{k}}
$$

and thus finally

$$
\begin{aligned}
&\int_a^b s(t)\left|f^{\Delta^m}(t)\right|^\alpha \left|f^{\Delta^n}(t)\right|^\beta \Delta t \\
&\leq \int_a^b s(t)\frac{(R(t))^{\frac{\alpha(k-1)}{k}}}{(r(t))^{\frac{\beta}{k}}}(g(t))^{\frac{\alpha}{k}}\left(g^\Delta(t)\right)^{\frac{\beta}{k}} \Delta t \\
&\leq \left\{\int_a^b (s(t))^{\frac{k}{k-\beta}} \frac{(R(t))^{\frac{\alpha(k-1)}{k-\beta}}}{(r(t))^{\frac{\beta}{k-\beta}}}\Delta t\right\}^{\frac{k-\beta}{k}} \\
&\quad \times \left\{\int_a^b (g(t))^{\frac{\alpha}{\beta}} g^\Delta(t)\Delta t\right\}^{\frac{\beta}{k}} \\
&\leq \left\{\int_a^b (s(t))^{\frac{k}{k-\beta}} \frac{(R(t))^{\frac{\alpha(k-1)}{k-\beta}}}{(r(t))^{\frac{\beta}{k-\beta}}}\Delta t\right\}^{\frac{k-\beta}{k}} \\
&\quad \times \left\{\int_a^b \frac{\beta}{\alpha+\beta}\left(g^{\frac{\alpha}{\beta}+1}\right)^\Delta (t)\Delta t\right\}^{\frac{\beta}{k}} \\
&= \left(\frac{\beta}{\alpha+\beta}\right)^{\frac{\beta}{k}} \left\{\int_a^b (s(t))^{\frac{k}{k-\beta}} \frac{(R(t))^{\frac{\alpha(k-1)}{k-\beta}}}{(r(t))^{\frac{\beta}{k-\beta}}}\Delta t\right\}^{\frac{k-\beta}{k}} (g(b))^{\frac{\alpha+\beta}{k}},
\end{aligned}
$$

where we have used first one last time the time scales Hölder inequality with conjugate exponents $\frac{k}{k-\beta}$ and $\frac{k}{\beta} > 1$ and then the increasing character of g together with the time scales chain rule, i.e.,

$$
\begin{aligned}
\left(g^{\frac{\alpha+\beta}{\beta}}\right)^\Delta &= \frac{\alpha+\beta}{\beta} g^\Delta \int_0^1 \left(hg^\sigma + (1-h)g\right)^{\frac{\alpha}{\beta}} \mathrm{d}h \\
&\geq \frac{\alpha+\beta}{\beta} g^\Delta \int_0^1 \left(hg + (1-h)g\right)^{\frac{\alpha}{\beta}} \mathrm{d}h \\
&= \frac{\alpha+\beta}{\beta} g^\Delta g^{\frac{\alpha}{\beta}}.
\end{aligned}
$$

The proof is complete. $\square$

Remark 4.2. Theorem 4.1 improves [19, Theorem 2.10], as the occurring parameters there are more restricted. In the same way, when $m = 0$, Theorem 4.1 improves [19, Theorem 2.9].

The next result follows from Theorem 4.1 by choosing $\alpha = \beta = 1$.

Corollary 4.3. *Let $n \in \mathbb{N}$. Assume that $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,*

$$r, s \in \mathrm{C_{rd}}([a,b]_{\mathbb{T}}, (0,\infty)), \quad \text{and} \quad f \in \mathrm{C}^n_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}).$$

Let $k > 1$. If $0 \le m \le n-1$ and $f^{\Delta^{m+i}}(a) = 0$ for all $0 \le i \le n-m-1$, then

$$\int_a^b s(t) \left| f^{\Delta^m}(t) \right| \left| f^{\Delta^n}(t) \right| \Delta t \le K \left\{ \int_a^b r(t) \left| f^{\Delta^n}(t) \right|^k \Delta t \right\}^{\frac{2}{k}},$$

where

$$K = \frac{1}{\sqrt[k]{2}} \left\{ \int_a^b (s(t))^{\frac{k}{k-1}} \frac{R(t)}{(r(t))^{\frac{1}{k-1}}} \Delta t \right\}^{\frac{k-1}{k}}$$

with

$$R(t) = \int_a^t \frac{(h_{n-m-1}(t,\sigma(\tau)))^{\frac{k}{k-1}}}{(r(\tau))^{\frac{1}{k-1}}} \Delta\tau.$$

The next result follows from Corollary 4.3 by choosing $k = 2$.

Corollary 4.4. *Let $n \in \mathbb{N}$. Assume that $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,*

$$r, s \in \mathrm{C_{rd}}([a,b]_{\mathbb{T}}, (0,\infty)), \quad \text{and} \quad f \in \mathrm{C}^n_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}).$$

If $0 \le m \le n-1$ and $f^{\Delta^{m+i}}(a) = 0$ for all $0 \le i \le n-m-1$, then

$$\int_a^b s(t) \left| f^{\Delta^m}(t) \right| \left| f^{\Delta^n}(t) \right| \Delta t \le K \int_a^b r(t) \left(f^{\Delta^n}(t) \right)^2 \Delta t,$$

where

$$K = \sqrt{\frac{1}{2} \int_a^b (s(t))^2 \frac{R(t)}{r(t)} \Delta t}$$

with

$$R(t) = \int_a^t \frac{(h_{n-m-1}(t,\sigma(\tau)))^2}{r(\tau)} \Delta\tau.$$

The next result follows from Corollary 4.3 by choosing $r = s = 1$.

Corollary 4.5. *Let $n \in \mathbb{N}$. Assume that $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$, and*

$$f \in \mathrm{C}^n_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}).$$

Let $k > 1$. If $0 \le m \le n-1$ and $f^{\Delta^{m+i}}(a) = 0$ for all $0 \le i \le n-m-1$, then

$$\int_a^b \left| f^{\Delta^m}(t) \right| \left| f^{\Delta^n}(t) \right| \Delta t \le K \left\{ \int_a^b \left| f^{\Delta^n}(t) \right|^k \Delta t \right\}^{\frac{2}{k}},$$

where

$$K = \frac{1}{\sqrt[k]{2}} \left\{ \int_a^b R(t) \Delta t \right\}^{\frac{k-1}{k}}$$

with

$$R(t) = \int_a^t (h_{n-m-1}(t, \sigma(\tau)))^{\frac{k}{k-1}} \Delta \tau.$$

Setting $k = \alpha + \beta$ in Theorem 4.1 yields the following result.

Corollary 4.6. *Let $n \in \mathbb{N}$. Assume that $a \in \mathbb{T}$, $b \in (a, \infty)_{\mathbb{T}}$,*

$$r, s \in \mathrm{C}_{\mathrm{rd}}([a, b]_{\mathbb{T}}, (0, \infty)), \quad \text{and} \quad f \in \mathrm{C}^n_{\mathrm{rd}}([a, b]_{\mathbb{T}}, \mathbb{R}).$$

Let $\alpha, \beta > 0$ with $\alpha + \beta > 1$. If $0 \le m \le n-1$ and $f^{\Delta^{m+i}}(a) = 0$ for all $0 \le i \le n-m-1$, then

$$\int_a^b s(t) \left| f^{\Delta^m}(t) \right|^{\alpha} \left| f^{\Delta^n}(t) \right|^{\beta} \Delta t \le K \int_a^b r(t) \left| f^{\Delta^n}(t) \right|^{\alpha+\beta} \Delta t,$$

where

$$K = \left(\frac{\beta}{\alpha + \beta} \right)^{\frac{\beta}{\alpha+\beta}} \left\{ \int_a^b (s(t))^{\frac{\alpha+\beta}{\alpha}} \frac{(R(t))^{\alpha+\beta-1}}{(r(t))^{\frac{\beta}{\alpha}}} \Delta t \right\}^{\frac{\alpha}{\alpha+\beta}}$$

with

$$R(t) = \int_a^t \frac{(h_{n-m-1}(t, \sigma(\tau)))^{\frac{\alpha+\beta}{\alpha+\beta-1}}}{(r(\tau))^{\frac{1}{\alpha+\beta-1}}} \Delta \tau.$$

Finally, setting $s = r$ in Theorem 4.1 yields the following result.

Corollary 4.7. *Let $n \in \mathbb{N}$. Assume that $a \in \mathbb{T}$, $b \in (a, \infty)_{\mathbb{T}}$,*

$$r \in \mathrm{C}_{\mathrm{rd}}([a, b]_{\mathbb{T}}, (0, \infty)), \quad \text{and} \quad f \in \mathrm{C}^n_{\mathrm{rd}}([a, b]_{\mathbb{T}}, \mathbb{R}).$$

Let $\alpha, \beta > 0$ and $k > \max\{1, \beta\}$. If $0 \le m \le n-1$ and $f^{\Delta^{m+i}}(a) = 0$ for all $0 \le i \le n-m-1$, then

$$\int_a^b r(t) \left| f^{\Delta^m}(t) \right|^{\alpha} \left| f^{\Delta^n}(t) \right|^{\beta} \Delta t \le K \left\{ \int_a^b r(t) \left| f^{\Delta^n}(t) \right|^{k} \Delta t \right\}^{\frac{\alpha+\beta}{k}},$$

where

$$K = \left(\frac{\beta}{\alpha + \beta} \right)^{\frac{\beta}{k}} \left\{ \int_a^b r(t) (R(t))^{\frac{\alpha(k-1)}{k-\beta}} \Delta t \right\}^{\frac{k-\beta}{k}}$$

with

$$R(t) = \int_a^t \frac{(h_{n-m-1}(t, \sigma(\tau)))^{\frac{k}{k-1}}}{(r(\tau))^{\frac{1}{k-1}}} \Delta \tau.$$

5. **Applications**

In this section, we consider the fourth-order dynamic equation

$$\left(rf^{\Delta^3}\right)^{\Delta}(t)-\left(pf^{\Delta}\right)^{\Delta}(t)+q(t)f^{\sigma}(t)=0,\quad t\in[a,b]_{\mathbb{T}} \tag{5.1}$$

on an arbitrary time scale $\mathbb{T}$.

Theorem 5.1. *Assume that $a\in\mathbb{T}$, $b\in(a,\infty)_{\mathbb{T}}$,*

$$p,q\in \mathrm{C_{rd}}([a,b]_{\mathbb{T}},\mathbb{R}),\ r\in \mathrm{C_{rd}}([a,b]_{\mathbb{T}},(0,\infty)),\quad \textit{and}\quad f\in \mathrm{C^3_{rd}}([a,b]_{\mathbb{T}},\mathbb{R}).$$

If f is a nontrivial and nonnegative solution of (5.1) such that f^{Δ^3} is nonnegative and

$$f(a)=f^{\Delta}(a)=f^{\Delta\Delta}(a)=f^{\Delta}(b)=f^{\Delta^3}(b)=0,$$

then

$$K_1+K_2+4L^2K_3\geq 1,$$

where

$$K_1=\sqrt{\int_a^b\frac{(Q(t))^2(R_1(t)+R_1(\sigma(t))}{r(t)}\Delta t},\quad K_2=\sqrt{\frac{1}{2}\int_a^b\frac{(p(t))^2R_2(t)}{r(t)}\Delta t},$$

$$K_3=\sqrt{\frac{1}{2}\int_a^b\frac{(Q(t))^2R_3(t)}{r(t)}\Delta t},\quad \textit{and}\quad L=\sup_{t\in[a,b)_{\mathbb{T}}}\sqrt{\int_a^{\sigma(t)}r(\tau)\Delta\tau\int_t^b\frac{\Delta\tau}{r(\tau)}}$$

with

$$R_1(t)=\int_a^t\frac{(h_2(t,\sigma(\tau)))^2}{r(\tau)}\Delta\tau,\quad R_2(t)=\int_a^t\frac{(t-\sigma(\tau))^2}{r(\tau)}\Delta\tau,$$

$$R_3(t)=\int_a^b\frac{\Delta\tau}{r(\tau)},\quad \textit{and}\quad Q(t)=\int_t^b q(\tau)\Delta\tau.$$

Proof. First note that Corollary 3.6 (with $n=3$) yields

$$\begin{aligned}\int_a^b|Q(t)|\,|f(\sigma(t))|\left|f^{\Delta^3}(t)\right|\Delta t&\leq\int_a^b|Q(t)|\left|(f(t)+f(\sigma(t)))f^{\Delta^3}(t)\right|\Delta t\\&\leq K_1\int_a^b r(t)\left(f^{\Delta^3}(t)\right)^2\Delta t,\end{aligned}$$

Corollary 4.4 (with $m=1$ and $n=3$) gives

$$\int_a^b|p(t)|\left|f^{\Delta}(t)\right|\left|f^{\Delta^3}(t)\right|\Delta t\leq K_2\int_a^b r(t)\left(f^{\Delta^3}(t)\right)^2\Delta t,$$

and again Corollary 4.4 (with $m=1$ and $n=2$) yields

$$\begin{aligned}\int_a^b|Q(t)|\left|f^{\Delta}(t)\right|\left|f^{\Delta\Delta}(t)\right|\Delta t&\leq K_3\int_a^b r(t)\left(f^{\Delta\Delta}(t)\right)^2\Delta t\\&\leq 4L^2K_3\int_a^b r(t)\left(f^{\Delta^3}(t)\right)^2\Delta t,\end{aligned}$$

where the last inequality follows from the recently developed Hardy-type inequality [14, Theorem 3.2] by replacing f, u, v, q, and p in [14, Theorem 3.2] by f^{Δ^3}, r, r, 2, and 2, respectively. Using these three estimates, together with three applications of the time scales integration by parts rule, we obtain

$$\begin{aligned}
&\int_a^b r(t)\left(f^{\Delta^3}(t)\right)^2\Delta t = \int_a^b \left(rf^{\Delta^3}\right)(t)f^{\Delta^3}(t)\Delta t\\
&= \left(rf^{\Delta^3}\right)(b)f^{\Delta\Delta}(b) - \left(rf^{\Delta^3}\right)(a)f^{\Delta\Delta}(a)\\
&\quad - \int_a^b \left(rf^{\Delta^3}\right)^{\Delta}(t)\left(f^{\Delta\Delta}\right)^{\sigma}(t)\Delta t\\
&= -\int_a^b \left(rf^{\Delta^3}\right)^{\Delta}(t)\left(f^{\Delta\Delta}\right)^{\sigma}(t)\Delta t\\
&= \int_a^b \left((qf^{\sigma})(t) - (pf^{\Delta})^{\Delta}(t)\right)\left(f^{\Delta\Delta}\right)^{\sigma}(t)\Delta t\\
&= -\int_a^b Q^{\Delta}(t)\left(ff^{\Delta\Delta}\right)^{\sigma}(t)\Delta t - \int_a^b \left(pf^{\Delta}\right)^{\Delta}(t)\left(f^{\Delta\Delta}\right)^{\sigma}(t)\Delta t\\
&= Q(a)\left(ff^{\Delta\Delta}\right)(a) - Q(b)\left(ff^{\Delta\Delta}\right)(b)\\
&\quad + \int_a^b Q(t)\left(ff^{\Delta\Delta}\right)^{\Delta}(t)\Delta t + \left(pf^{\Delta}\right)(a)f^{\Delta\Delta}(a)\\
&\quad - \left(pf^{\Delta}\right)(b)f^{\Delta\Delta}(b) + \int_a^b \left(pf^{\Delta}\right)(t)f^{\Delta^3}(t)\Delta t\\
&= \int_a^b Q(t)\left(f^{\Delta}f^{\Delta\Delta} + f^{\sigma}f^{\Delta^3}\right)(t)\Delta t\\
&\quad + \int_a^b p(t)f^{\Delta}(t)f^{\Delta^3}(t)\Delta t\\
&\le \int_a^b |Q(t)|\,|f(\sigma(t))|\left|f^{\Delta^3}(t)\right|\Delta t + \int_a^b |p(t)|\left|f^{\Delta}(t)\right|\left|f^{\Delta^3}(t)\right|\Delta t\\
&\quad + \int_a^b |Q(t)|\left|f^{\Delta}(t)\right|\left|f^{\Delta\Delta}(t)\right|\Delta t\\
&\le \left(K_1 + K_2 + 4L^2K_3\right)\int_a^b r(t)\left(f^{\Delta^3}(t)\right)^2\Delta t,
\end{aligned}$$

and so the claim follows after cancelling the integral. □

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