Dynamic Systems and Applications 26 (2017) 283-308

ON VARIOUS MULTIMAP CLASSES IN THE KKM THEORY AND THEIR APPLICATIONS

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ABSTRACT. Fixed point theory of convex-valued multimaps are closely related to the KKM theory from the beginning. In the last twenty-five years, we introduced the acyclic multimap class, the admissible multimap class $\mathfrak{A}_c^\kappa$, the better admissible class $\mathfrak{B}$, and the KKM admissible classes $\mathfrak{KC}$, $\mathfrak{KO}$ in the frame of the KKM theory. Our aim in this review is to collect the basic properties of our multimap classes and some mutual relations among them in general topological spaces or our abstract convex spaces. We add some new remarks and further comments to improve many of those results, and introduce some recent applications of our multimap classes.

AMS (MOS) Subject Classification. 47H04, 47H10, 47J20, 47N10, 49J53, 52A99, 54C60, 54H25, 58E35, 90C47, 91A13, 91B50.

1. INTRODUCTION

Since Kakutani obtained his celebrated fixed point theorem for convex-valued u.s.c. multimaps in 1941 and Eilenberg and Montgomerry extended it for acyclic maps in 1948, there have appeared many types of multimaps with applications in various fields in mathematics, economics, game theory, natural sciences, engineering, and others. In 1992, the author [24] obtained some coincidence theorems on acyclic maps and their applications to the newly named KKM theory originated from the celebrated intersection theorem of Knaster, Kuratowski and Mazurkiewitz in 1929. Since then a large number of applications of some results in [24] have appeared; see [43, 48, 53] and the references therein.

Moreover, in the last twenty-five years, we introduced several multimap classes in the frame of the KKM theory; namely, the acyclic multimap class, the admissible multimap class $\mathfrak{A}_c^\kappa$, the better admissible class $\mathfrak{B}$, and the KKM admissible classes $\mathfrak{KC}$, $\mathfrak{KO}$. Each of these classes contains a large number of particular multimaps.

In our previous work [43], we reviewed applications of our fixed point theorems for the multimap class of compact compositions of acyclic maps and, in [48], we collected

Received February 3, 2017

most of fixed point theorems related to the KKM theory due to the author. Moreover, applications of our versions of the Fan-Browder fixed point theorem were introduced in [49]. Furthermore, in a later work [53], we reviewed applications of our fixed point theorems and our multimap classes, appeared mainly in other authors' works. Most of them are not treated in [43, 48, 49].

Our aim in this review is to collect the basic properties of our multimap classes and some mutual relations among them in general topological spaces or our abstract convex spaces. We add some new remarks and further comments to improve some of those results, and introduce some recent applications of our multimap classes. This would be informative to peoples working in certain related fields.

This review article is organized as follows. Section 2 is a preliminary on abstract convex spaces due to ourselves. Since 2007 such spaces became the main theme of the KKM theory and many new results on them have appeared mainly by the present author.

Section 3 deals with convex-valued multimaps in the KKM theory and analytical fixed point theory, that is, one of the most important applications of the KKM theory. Usually, a Kakutani map is a convex-valued u.s.c. multimap. The upper semicontinuity related to topological vector spaces are extended to upper demicontinuity, to upper hemicontinuity, and to generalized upper hemicontinuity. In Section 4, we deal with a particular type of convex-valued multimaps called Fan-Browder maps whose fibers are open. Section 5 concerns with acyclic maps which are u.s.c. and have compact acyclic values. Recall that acyclic maps were introduced by Eilenberg and Montgomerry and studied by ourselves first in the KKM theory.

In Section 6, we deal with our admissible multimap class $\mathfrak{A}_c^\kappa$, which has been studied first by ourselves and followed by a large number of authors. Section 7 concerns with basic facts on the better admissible multimap classes $\mathfrak{B}$ and various fixed point theorems on them. In Section 8, we recall the KKM admissible multimap classes $\mathfrak{KC}$, $\mathfrak{KO}$ and their properties. Finally, in Section 9, we introduce some basic theorems related $\mathfrak{KC}$, $\mathfrak{KO}$, from which we can deduce several useful equivalent formulations in the KKM theory of abstract convex spaces.

The present review may be regarded as a continuation of our previous work [53] and an expanded version of our previous talk [59] given at a RIMS workshop, Kyoto University, in August 30, 2017.

2. ABSTRACT CONVEX SPACES

For sets X and Y, a multimap (a multifunction or simply a map) $F : X \multimap Y$ is a function $F : X \to 2^Y$ to the power set of Y.

For the concepts on our abstract convex spaces, KKM spaces and the KKM classes $\mathfrak{KC}$, $\mathfrak{KO}$, we follow [46] with some modifications and the references therein:

Definition. Let E be a topological space, D a nonempty set, $\langle D\rangle$ the set of all nonempty finite subsets of D, and $\Gamma : \langle D\rangle \multimap E$ a multimap with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$. The triple $(E, D; \Gamma)$ is called an *abstract convex space* whenever the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to some $D' \subset D$ if, for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all} \quad A \in \langle D\rangle,$$

then G is called a *KKM map with respect to* F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp., a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, Z)$ [resp., $F \in \mathfrak{KO}(E, Z)$].

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, respectively.

Example. The following are typical examples of KKM spaces. Others can be seen in [46, 51] and the references therein.

(1) A *convex space* $(X, D) = (X, D; \Gamma)$ is a triple where X is a subset of a vector space, $D \subset X$ such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D\rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde for $X = D$.

(2) An abstract convex space $(X, D; \Gamma)$ is called an H-*space* if $\Gamma = \{\Gamma_A\}$ is a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D\rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D\rangle$. If $D = X$, $(X; \Gamma)$ is called a c-space by Horvath.

(3) A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$.

(4) A *space having a family* $\{\phi_A\}_{A \in \langle D \rangle}$ or simply a ϕ_A-*space*

$$(X, D; \{\phi_A\}_{A \in \langle D \rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$.

Every ϕ_A-space $(X, D; \Gamma)$ with $\Gamma_A := \phi_A(\Delta_n)$ for $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$ is a KKM space; see [48].

Recently, Kulpa and Szymanski [13] found some partial KKM spaces which are not KKM spaces.

Note that each of the above examples has a large number of concrete examples.

Now we have the following diagram for triples $(E, D; \Gamma)$:

$$\begin{aligned}
&\text{Simplex} \Longrightarrow \text{Convex subset of a t.v.s.} \Longrightarrow \text{Convex space} \Longrightarrow \text{H-space} \\
&\Longrightarrow \text{G-convex space} \Longrightarrow \phi_A\text{-space} \Longrightarrow \text{KKM space} \\
&\Longrightarrow \text{Partial KKM space} \Longrightarrow \text{Abstract convex space.}
\end{aligned}$$

For a short history of the KKM theory, see [45].

In this paper, a t.v.s. means a topological vector space (not necessarily Hausdorff).

3. CONVEX-VALUED MAPS

In 1941, Kakutani obtained the following fixed point theorem:

Theorem 3.1. (Kakutani [11]) *If $x \to \Phi(x)$ is an upper semicontinuous point-to-set mapping of an r-dimensional closed simplex S into the family of nonempty closed convex subset of S, then there exists an $x_0 \in S$ such that $x_0 \in \Phi(x_0)$.*

Motivated by this theorem, we define the Kakutani map as follows:

Definition. Let X be a subset of a t.v.s. A multimap $T : X \multimap X$ is called a *Kakutani map* if T is u.s.c. and has nonempty compact convex values $T(x)$ for each $x \in X$.

Kakutani's theorem is the beginning of the fixed point theory of multimaps having a vital connection with the minimax theory in game theory and the equilibrium theory in economics. One of the most important applications of the Kakutani theorem was made by Nash in 1951. It was followed by several hundred applications in game theory, economic theory, mathematical programming, control theory, and theory of differential equations.

There appeared also many generalizations of fixed point theorems for Kakutani maps. For the literature, see [31]. The following is well-known:

Theorem 3.2 (Himmelberg [10]) *Let X be a convex subset of a locally convex Hausdorff t.v.s. Then any compact Kakutani map $T : X \multimap X$ has a fixed point.*

Motivated by Kakutani maps, convex-valued maps are further extended as follows:

Definition. Let X be a topological space, E a t.v.s., E^* its topological dual, and $F : X \multimap E$ a map. Then

(i) F is *upper semi-continuous* (u.s.c.) if for each $x \in X$ and each open set U in E containing $F(x)$, there exists an open neighborhood N of x in X such that $F(N) \subset U$;

(ii) F is *upper demi-continuous* (u.d.c.) if for each $x \in X$ and each open half-space H in E containing $F(x)$, there exists an open neighborhood N of x in X such that $F(N) \subset H$;

(iii) F is *upper hemi-continuous* (u.h.c.) if for each $f \in E^*$ and for any real α, the set $\{x \in X \mid \sup f(F(x)) < \alpha\}$ is open in X; and

(iv) F is *generalized u.h.c.* if for each $p \in E^*$, the set $\{x \in X \mid \sup p(F(x)) \geq p(x)\}$ is closed in X.

For such class of convex-valued multimaps, the analytical fixed point theory is extensively studied. According to Lassonde, we need some preparation as follows:

Recall that a *convex space* X is a nonempty convex set with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. A nonempty subset L of a convex space X is called a *c-compact set* if for each finite set $S \subset X$ there is a compact convex set $L_S \subset X$ such that $L \cup S \subset L_S$. Let $[x, L]$ denote the closed convex hull of $\{x\} \cup L$ in X, where $x \in X$.

Let $cc(E)$ denote the set of nonempty closed convex subsets of a t.v.s. E and $kc(E)$ the set of nonempty compact convex subsets of E. Bd, Int, and $\overline{}$ denote the boundary, interior, and closure, resp., with respect to E.

Let $X \subset E$ and $x \in E$. According to Halpern, the *inward* and *outward sets* of X at x, $I_X(x)$ and $O_X(x)$, are defined as follows:

$$I_X(x) = x + \bigcup_{r>0} r(X - x), \quad O_X(x) = x + \bigcup_{r<0} r(X - x).$$

For $p \in \{\mathrm{Re}\, h : h \in E^*\}$ and $U, V \subset E$, let

$$d_p(U, V) = \inf\{|p(u - v)| \ : \ u \in U, \ v \in V\}.$$

Let X be a nonempty convex subset of a vector space E. Following Ky Fan, the *algebraic boundary* $\delta_E(X)$ of X in E is the set of all $x \in X$ for which there exists $y \in E$ such that $x + ry \notin X$ for all $r > 0$.

The following is a most general fixed point theorem on convex-valued multimaps originated from [22, 25, 32]:

Theorem 3.3. *Let X be a convex space, L a c-compact subset of X, K a nonempty compact subset of X, E a t.v.s. containing X as a subset, and F a map satisfying either*

(A) *E^* separates points of E and $F : X \to kc(E)$, or*

(B) *E is locally convex and $F : X \to cc(E)$.*

(I) *Suppose that for each $p \in E^*$,*

(0) *$p|_X$ is continuous on X;*

(1) *$X_p = \{x \in X : \inf p(F(x)) \leq p(x)\}$ is closed in X;*

(2) *$d_p(F(x), \overline{I}_X(x)) = 0$ for every $x \in K \cap \delta_E(X)$; and*

(3) *$d_p(F(x), \overline{I}_L(x)) = 0$ for every $x \in X \setminus K$.*

Then there exists an $x \in X$ such that $x \in F(x)$.

(II) *Suppose that for each $p \in E^*$,*

(0) *$p|_X$ is continuous on X;*

(1)′ *$X_p = \{x \in X : \sup p(F(x)) \geq p(x)\}$ is closed in X;*

(2)′ *$d_p(F(x), \overline{O}_X(x)) = 0$ for every $x \in K \cap \delta_E(X)$; and*

(3)′ *$d_p(F(x), \overline{O}_L(x)) = 0$ for every $x \in X \setminus K$.*

Then there exists an $x \in X$ such that $x \in F(x)$. Further, if F is u.h.c., then $F(X) \supset X$.

Recall that this theorem subsumes more than 50 previously known results.

Applications: The following are only a few examples of abstracts of articles containing some important fixed point theorems on convex-valued multimaps:

Kakutani, Duke (1941) [11] — In order to give simple proofs of von Neumann's minimax theorem in 1928 and his intersection lemma in 1937, Kakutani extended Brouwer fixed point theorem to multimaps.

Himmelberg, JMAA 38 (1972) [10] — The object of this note is to obtain two generalizations of the well-known fixed point theorem of Fan. A slight modification of Fan's proof yields one; the second is then an easy corollary, which, though interesting, seems never to be mentioned in the literature. We conclude with a generalization of the minimax theorem.

Park, JKMS 29 (1992) [22] — We apply our existence theorem to obtain new coincidence, fixed point, and surjectivity theorems, and existence theorems on critical points for a larger class of multifunctions than upper hemicontinuous ones defined on convex sets.

Park, JKMS 30 (1993) [25] — The purpose in this paper is, first, to give common generalizations of some results of Park, Park and Bae, and Idzik. This will give more adequate understanding on the nature of the results on convex-valued multifunctions in the previous article. Our second purpose is to obtain new fixed point or related results on compact composites of non-convex valued "admissible" upper semicontinuous multifunctions defined on convex subsets of topological vector spaces having sufficiently many linear functionals.

Park, VJM 27 (1999) [31] — This historical article is to survey the developments of the fields of mathematics directly related to the nearly ninety-year-old Brouwer fixed point theorem. We are mainly concerned with equivalent formulations and generalizations of the theorem. Also we deal with the KKM theory and various equilibrium problems closely related to the Brouwer theorem.

Park, AMV 27 (2002) [32] — We give new fixed point theorems on a generalized upper hemicontinuous multimap whose domain and range may have different topologies. These include known theorems appeared in almost 50 published works. See Theorem 3.3 above.

Park, ICFPTA-2007 (2008) [41] — This is to review various generalizations of the Himmelberg fixed point theorem within the category of topological vector spaces. We consider the Lassonde type, the Idzik type, and the KKM type generalizations for Kakutani maps, and other types of generalizations for acyclic maps. Finally, generalizations for various "better" admissible maps on admissible almost convex domains to Klee approximable ranges are discussed.

Park, NA 71 (2009) [42] — This is to establish fixed point theorems for multimaps in abstract convex uniform spaces. Our new results generalize corresponding ones in topological vector spaces (t.v.s.), convex spaces due to Lassonde, c-spaces due to Horvath, and G-convex spaces due to Park. We show that fixed point theorems on multimaps of the Fan-Browder type, multimaps having ranges of the Zima-Hadzic type, and multimaps whose ranges are Φ-sets or Klee approximable sets can be established in abstract convex spaces or KKM spaces.

Park, CANA 18 (2011) [47] — In this short note, we give some variants of the fixed point theorems on generalized upper hemicontinuous (g.u.h.c.) multimaps whose domains and ranges may have different topologies. Our new theorems refine our previous results and simply generalize Balaj's two map versions of Halpern's fixed point theorems.

4. FAN-BROWDER MAPS

In 1968, Browder established the following useful fixed point theorem on a particular type of convex-valued multimaps and its applications:

Theorem 4.1. (Browder [8]) *Let K be a nonempty compact convex subset of a topological vector space. Let T be a map of K into 2^K, where for each $x \in K$, $T(x)$ is a nonempty convex subset of K. Suppose further that for each y in K, $T^{-1}(y) = \{x \in K : y \in T(x)\}$ is open in K. Then there exists x_0 in K such that $x_0 \in T(x_0)$.*

The map T in Theorem 4.1 is usually called a *Browder map* or a *Fan-Browder* map. The Browder fixed point theorem [8] has a very large number of generalizations and variations; see [31, 54, 56, 57]. The following is a very useful generalizations frequently appeared in the literature:

Theorem 4.2. (Park [22]) *Let X be a convex subset of a t.v.s. (not necessarily Hausdorff), $G : X \multimap X$, and K a nonempty compact subset of X. Suppose that*

(1) *for each $x \in X$, Gx is convex;*

(2) *for each $x \in K$, Gx is nonempty;*

(3) *for each $y \in X$, $G^- y$ is open; and*

(4) *for each nonempty finite $N \subset X$, there exists a compact convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, $Gx \cap L_N \neq \emptyset$.*

Then G has a fixed point $x_0 \in X$.

For a multimap $S : D \multimap E$, consider the following related four conditions:

(a) $\bigcup_{y\in D} S(z) = E$ implies $\bigcup_{y\in D} \operatorname{Int} S(y) = E$.

(b) $\operatorname{Int} \bigcup_{y\in D} S(y) = \bigcup_{y\in D} \operatorname{Int} S(y)$ (S is *unionly open-valued* (Luc et al. [17])).

(c) $\bigcup_{y\in D} S(y) = \bigcup_{y\in D} \operatorname{Int} S(y)$ (S is *transfer open-valued*).

(d) S is open-valued.

Theorems 4.1 and 4.2 are extended to the following Fan-Browder alternatives in abstract convex spaces:

Theorem 4.3. (Park [54]) *Let $(E, D; \Gamma)$ be a partial KKM space, and $S : E \multimap D$, $T : E \multimap E$ maps. Suppose that*

(1) *for each $x \in E$, $\operatorname{co}_\Gamma S(x) \subset T(x)$;*

(2) *there exists a nonempty compact subset K of E such that either*

(a) $\bigcap_{z\in M} \overline{E \setminus S^-(z)} \subset K$ *for some $M \in \langle D\rangle$; or*

(b) *for each nonempty finite* $N \subset D$, *there exists a compact* Γ*-convex subset* L_N *of* E *relative to some* $D' \subset D$ *such that* $N \subset D'$ *and*

$$\overline{L_N} \cap \bigcap_{z \in D'} \overline{E \setminus S^-(z)} \subset K.$$

(α) *If* S^- *is transfer open-valued, then either* (i) T *has a fixed point, or* (ii) S *has a maximal element in* K.

(β) *If* S^- *is unionly open-valued, then either* (i) T *has a fixed point, or* (ii) S *has a maximal element in* E.

This subsumes a large number of particular results previously known. See Park [54].

Applications. We present only a few articles related to Fan-Browder maps:

Browder, Math. Ann. 177 (1968) [8] — Browder restated Fan's geometric lemmm in the convenient form of a fixed point theorem by means of the Brouwer theorem and the partition of the unity argument. His theorem is applied to a systematic treatment of interconnections between fixed point theorems, minimax theorems, variational inequalities, and monotone extension theorems.

Park, TopA 135 (2004) [34] — We show that the KKM principle implies two new general fixed point theorems for the Kakutani maps or the Browder maps. Consequently, we give unified transparent proofs of many of well-known results.

Park, JNAS 52(2) (2013) [54] — In this paper, from a general form of the KKM type theorems or some properties of KKM type maps on abstract convex spaces, we deduce several Fan-Browder type alternatives, coincidence or fixed point theorems, and other results. These theorems unify and generalize various particular results of the same kinds recently due to a number of authors for particular types of abstract convex spaces.

Park, NACA 2013 (2016) [56] — Corresponding to each stage of development of the KKM theory, the Fan-Browder fixed point theorem on Fan-Browder type multimaps has been generalized to hundreds of different forms or reformulated to the maximal element theorem with numerous generalizations. Recall that the theorem can be stated as an alternative form; that is, its conclusion is "the Fan-Browder map has either a fixed point or a maximal element." Our aim in this paper is to trace the evolution of the Fan-Browder type alternatives from the origin to the most recent generalization of them.

Park, JNCA 17 (2016) [57] — We begin with a modification of a characterization of (partial) KKM spaces using a general Fan-Browder type fixed point property and show that this characterization implies an alternative theorem. This theorem unifies and contains a number of historically well-known important fixed point or maximal element theorems. We list some of them chronologically and give simple proofs.

Finally, we introduce some recent works related to the generalized Fan-Browder type alternatives.

5. ACYCLIC MAPS

Convexity is directly implies the following acyclicity:

Definition. A topological space is *acyclic* if all its reduced Čech homology groups over rationals vanish. A multimap is called *acyclic* if it is u.s.c. with compact acyclic values.

A *polyhedron* is a subset of a Euclidean space $\mathbb{R}^n$ which is homeomorphic to the union of a finite number of compact convex subsets.

The following is due to Eilenberg and Montgomery in 1946 as a generalization of the Kakutani fixed point theorem:

Theorem 5.1. *Let Z be an acyclic polyhedron and $T : Z \multimap Z$ an acyclic map* (*that is, u.s.c. with acyclic values*)*. Then T has a fixed point $\hat{x} \in Z$.*

The following is given in 1992. See also [64]:

Theorem 5.2. (Park [24]) *Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space E and $F : X \multimap X$ be an acyclic map. If F is compact, then it has a fixed point.*

This is the beginning of usage of acyclic maps in the KKM theory. Theorem 5.2 reduces to Himmelberg's theorem when F is convex-valued. We obtained a large number of generalizations of Theorem 5.2; see [41, 44, 53] and the references therein. Especially, in [41], we reviewed various generalizations of the Himmelberg fixed point theorem within topological vector spaces. We considered there the Lassonde type, the Idzik type, and the KKM type generalizations for Kakutani maps, and other type of generalizations for acyclic maps. Moreover, generalizations for various 'better' admissible maps on admissible almost convex domains or maps having Klee approximable ranges were also discussed.

In 1992, we also obtained the following *cyclic coincidence theorem* for acyclic maps, where $\mathbb{Z}_k := \{0, 1, \dots, k-1\}$ with $(k-1)+1$ interpreted as 0:

Theorem 5.3. (Park [23]) *Let $k \geq 1$ and, for each $h \in \mathbb{Z}_k$, let Y_h be a nonempty compact convex subset of a locally convex space E_h, and $V_h \in \mathbb{V}(Y_h, Y_{h+1})$. Then there exists $(y_0, y_1, \dots, y_{k-1}) \in Y_0 \times Y_1 \times \cdots \times Y_{k-1}$ such that $y_{h+1} \in \mathbb{V}_h y_h$ for all $h \in \mathbb{Z}_k$.*

In 1994, Theorem 5.2 is extended to more general $\mathbb{V}_c$ than $\mathbb{V}$ as follows:

Theorem 5.4. (Park et al. [64]) *Let X be a nonempty convex subset of a locally convex space E and $T \in \mathbb{V}_c(X, X)$. If T is compact, then T has a fixed point $x_0 \in X$.*

From this we obtained the following *best approximation result*:

Theorem 5.5. (Park et al. [64]) *Let C be a nonempty approximatively compact, convex subset of a locally convex space E, and suppose that $\mathbb{V}_c(C,E)$ is a compact map. Then for each continuous seminorm p on E there exists an $(x_0,y_0) \in \mathrm{Gr}(F)$ such that*

$$p(x_0 - y_0) \le p(x - y_0) \ \ \textit{for all} \ \ x \in \overline{I_C(x_0)}.$$

The following is a particular case of Park and Kim ([61], Theorem 4):

Theorem 5.6. *Let X be a nonempty compact admissible subset of a hyperconvex metric space (H,d) and $F : X \multimap X$ an acyclic map. Then F has a fixed point.*

Applications. In our previous work [53], we listed 25 papers of other authors on applications of our fixed point theorems on acyclic maps or related results. We give here only a few of our articles related to acyclic maps:

Park, FPTA (K.-K. Tan, ed.) (1992)[24] — From a Lefschetz type fixed point theorem for composites of acyclic maps, we obtain a general Fan-Browder type coincidence theorem, which can be shown to be equivalent to a matching theorem and a KKM type theorem. From the main result, we deduce the Himmelberg type fixed point theorem for acyclic compact multifunctions, acyclic versions of general geometric properties of convex sets, abstract variational inequality theorems, new minimax theorems, and non-continuous versions of the Brouwer and Kakutani type fixed point theorems with very generous boundary conditions.

Park et al., PAMS 121 (1994) [64]— We obtain fixed point theorems for a new class of multifunctions containing compact composites of acyclic maps defined on a convex subset of a locally convex Hausdorif topological vector space. Our new results are applied to approximatively compact, convex sets or to Banach spaces with the Oshman property.

Park, NA-TMA 24 (1995) [27] — In this paper, we obtain fixed point theorems for acyclic maps in $\mathbb{V}(X,E)$ generalizing corresponding ones for Kakutani maps in $\mathbb{K}(X,E)$ with certain boundary conditions, where X is a compact convex subset of a Hausdorff locally convex topological vector space E. Consequently, we generalize results in many articles. We mainly follow the method of Ha and Park.

Park, WCNA'92 (1996) [28] — Sufficient conditions for the existence of fixed points of acyclic maps defined on a convex subset of a topological vector space E on which E^* separates points are obtained. Main consequences are acyclic versions of fixed point theorems due to Fan, Halpern and Bergman, Himmelberg, Reich, Granas and Liu, and many others.

Park, VJM 37 (2009) [43] — We review applications of our fixed point theorems on compact compositions of acyclic maps. Our applications are mainly on acyclic polyhedra, locally convex topological vector spaces, admissible (in the sense of Klee) convex sets, and almost convex or Klee approximable sets in topological vector spaces. Those applications are concerned with general equilibrium problems like as (collective) fixed point theorems, the von Neumann type intersection theorems, the von Neumann type minimax theorems, the Nash type equilibrium theorems, cyclic coincidence theorems, best approximation theorems, (quasi-) variational inequalities, and the Gale-Nikaido-Debreu theorem. Finally, we briefly introduce some related results mainly appeared in other author's works.

6. ADMISSIBLE MULTIMAP CLASS $\mathfrak{A}_c^\kappa$

The following 1961 KKM Lemma of Ky Fan is one of the most important milestone on the history of the KKM theory:

Lemma 5.1. (Fan [9]) *Let X be an arbitrary set in a Hausdorff topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

(i) *The convex hull of any finite subset $\{x_1, x_2, \cdots, x_n\}$ of X is contained in $\bigcup_{i=1}^n F(x_i)$.*

(ii) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x\in X} F(x) \neq \emptyset$.

The Lemma was followed by a large number of applications, generalizations, and modifications. In order to unify such generalizations, we introduced the following:

Let X and Y be topological spaces. In the following, a *polytope* is a homeomorphic image of a simplex. The following due to the author is well-known:

Definition. An *admissible class* $\mathfrak{A}_c^\kappa(X, Y)$ of maps $T : X \multimap Y$ is the one such that, for each compact subset K of X, there exists a map $S \in \mathfrak{A}_c(K, Y)$ satisfying $S(x) \subset T(x)$ for all $x \in K$; where $\mathfrak{A}_c$ is consisting of finite compositions of maps in $\mathfrak{A}$, and $\mathfrak{A}$ is a class of maps satisfying the following properties:

(1) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;

(2) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and

(3) for each polytope P, each $T \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces of compositions are suitably chosen for each $\mathfrak{A}$.

Example. Examples of the function space $\mathfrak{A}$ are the classes of continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$, the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$, the Powers maps $\mathbb{V}_c$, the O'Neil maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ (whose domains and codomains are subsets of uniform spaces), admissible maps of Górniewicz, the

Simons maps $\mathbb{K}_c$, σ-selectionable maps of Haddad and Lasry, permissible maps of Dzedzej, and others. Further, the Fan-Browder maps (codomains are convex sets), locally selectionable maps having convex values, $\mathbb{K}_c^+$ due to Lassonde, $\mathbb{V}_c^+$ due to Park et al., and approximable maps $\mathbb{A}_c^\kappa$ due to Ben-El-Mechaiekh and Idzik are examples of the function space $\mathfrak{A}_c^\kappa$.

For the literature, see Park [26, 29, 30], Park and H. Kim [60, 62, 63] and the references therein.

The following is one of the earliest generalizations and unifies so many generalizations of Ky Fan's 1961 KKM Lemma:

Theorem 5.2. (Park [26]) *Let (X, D) be a convex space, Y a Hausdorff space, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Let $G : D \to 2^Y$ be a multifunction such that*

(1) *for each $x \in D$, $G(x)$ is compactly closed in Y;*

(2) *for any $N \in \langle D\rangle$, $F(\operatorname{co} N) \subset G(N)$; and*

(3) *there exist a nonempty compact subset K of Y and, for each $N \in \langle D\rangle$, a compact D-convex subset L_N of X containing N such that $F(L_N) \cap \bigcap\{G(x) \mid x \in L_N \cap D\} \subset K$.*

Then $\overline{F(X)} \cap K \cap \bigcap\{G(x) \mid x \in D\} \neq \emptyset$.

After this one, there have appeared more elegant and general KKM type theorems on abstract convex spaces. In fact, Theorem 5.2 was extended to G-convex spaces by Park and Kim [61, 63], and to abstract convex spaces and the map class $\mathfrak{KC}$ by Park [39].

Applications. The admissible class due to Park was first applied to the KKM theory and fixed point problems. Later many authors applied the class to various problems. In fact, 19 papers on such applications were introduced in [53]. Here we give the contents of *some* of typical works applying our admissible class.

Park and H.Kim, PCNS-SNU 18 (1993) [60] — The first author introduced certain general classes of upper semicontinuous multimaps defined on convex spaces which were shown to be adequate to establish theories on fixed points, coincidence points, KKM maps, variational inequalities, best approximations, and many others. Later we found that, in certain cases, the convex spaces can be replaced by new classes of more general spaces. In this paper we collect examples of such classes of multimaps and generalized convex spaces. Some fundamental properties of such examples are also discussed.

Park and H. Kim, JMAA 197 (1996) [63] — We defined admissible classes of maps which are general enough to include composites of maps appearing in nonlinear analysis or algebraic topology, and generalized convex spaces which are generalizations of many general convexity structures. In this paper we obtain a coincidence theorem for admissible maps defined on generalized convex spaces. Our new result is applied to obtain an abstract variational inequality, a KKM type theorem, and fixed point theorems.

Agarwal and O'Regan, TMNA 21 (2003) [1] — This paper presents a continuation theory for $\mathfrak{A}_c^\kappa$ maps. The analysis is elementary and relies on properties of retractions and fixed point theory for selfmaps. Also its authors present a separate theory for a certain subclass of $\mathfrak{A}_c^\kappa$ maps, namely the PK maps.

Agarwal and O'Regan, Comm. Math. XLIV(1) (2004) [2] — New fixed point theory is presented for compact $\mathfrak{A}_c^\kappa(X,X)$ maps where X is an admissible subset of a t.v.s. The aim of this paper is to generalize results of [30, 4] and others. The authors defined extension spaces (ES), ES admissible subsets, Borsuk ES admissible subsets, Klee approximable extension spaces (KAES), Borsuk KAES admissible spaces, q-Borsuk KAES admissible subsets, etc. They show that any compact $\mathfrak{A}_c^\kappa(X,X)$ map on these spaces has a fixed point. Finally, they present a continuation theorem for particular types of admissible spaces considered in previous works of Park.

Agarwal and O'Regan, FPTA (2009) [3] — The authors present new Leray-Schauder alternatives, Krasnoselskii and Lefschetz fixed point theory for multimaps between Frèchet spaces. As an application they show that their results are directly applicable to establish the existence of integral equations over infinite intervals.

Agarwal et al., Asia-European J. Math. 4 (2011) [5] — The authors present new fixed point theorems for $\mathfrak{A}_c^\kappa$-admissible maps acting on locally convex t.v.s. They considered multimaps need not be compact, and merely assume that multimaps are weakly compact and map weakly compact sets into relatively compact sets. Their fixed point results are obtained under Schauder, Leray-Schauder and Furi-Pera type conditions.

O'Regan, AMC 219 (2012) [18] —- Several continuation principles in a variety of settings are presented which guarantee the existence of coincidence points for a general class of multimaps. Recall $\mathfrak{A}_c^\kappa$ is closed under compositions. The class $\mathfrak{A}_c^\kappa$ contains almost all the well-known maps in the literature. It is also possible to consider more general maps and in this paper the author considers a class $\mathbf{A}$ of maps.

7. BETTER ADMISSIBLE MULTIMAP CLASS $\mathfrak{B}$

The following is the concept of a slightly new multimap classes related to the KKM theory:

Definition. Let X and Y be topological spaces. We define *the better admissible multimap class* $\mathfrak{B}$ of maps from X into Y as follows:

$F \in \mathfrak{B}(X,Y) \iff F : X \multimap Y$ is a map such that, for any natural $n \in \mathbb{N}$, any continuous function $\phi : \Delta_n \to X$, and any continuous function $p : F\phi(\Delta_n) \to \Delta_n$, the composition

$$\Delta_n \xrightarrow{\phi} \phi(\Delta_n) \subset X \overset{F}{\multimap} F\phi(\Delta_n) \xrightarrow{p} \Delta_n$$

has a fixed point.

Proposition 7.1. *For any topological spaces X, Y, we have $\mathfrak{A}_c^\kappa(X,Y) \subset \mathfrak{B}(X,Y)$.*

When X is a subset of an abstract convex space, the preceding definition reduces to the following previous one in [45]:

Definition. Let $(E, D; \Gamma)$ be an abstract convex space, X a nonempty subset of E, and Y a topological space. We define *the better admissible class* $\mathfrak{B}$ of maps from X into Y as follows:

$F \in \mathfrak{B}(X,Y) \iff F : X \multimap Y$ is a map such that, for any $\Gamma_N \subset X$, where $N \in \langle D \rangle$ with the cardinality $|N| = n+1$, and for any continuous function $p : F(\Gamma_N) \to \Delta_n$, there exists a continuous function $\phi_N : \Delta_n \to \Gamma_N$ such that the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

has a fixed point. Note that Γ_N can be replaced by the compact set $\phi_N(\Delta_n) \subset X$.

This concept extends the corresponding one for G-convex spaces appeared in [38], where lots of examples were given.

The above definition also works for ϕ_A-spaces $(X, D; \Gamma)$ with $\Gamma_A := \phi_A(\Delta_n)$ for $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$.

Let X be a convex space and Y a Hausdorff space. More early in 1997 [19], we introduced a 'better' admissible class $\mathfrak{B}$ of multimaps as follows:

$F \in \mathfrak{B}(X,Y) \iff F : X \multimap Y$ such that, for any polytope P in X and any continuous map $f : F(P) \to P$, $f(F|_P)$ has a fixed point.

The following KKM theorem is due to the author [29, Theorem 3]:

Theorem 7.2. *Let X be a convex space, Y a Hausdorff space, $F \in \mathfrak{B}(X,Y)$ a compact map, and $S : X \multimap Y$ a map. Suppose that*

(1) *for each $x \in X$, $S(x)$ is closed; and*

(2) *for each $N \in \langle X \rangle$, $F(\operatorname{co} N) \subset S(N)$.*

Then $\overline{F(X)} \cap \bigcap \{S(x) \mid x \in X\} \neq \emptyset$.

Later this KKM theorem was applied to a minimax inequality related to admissible multimaps, from which we deduced generalized versions of lopsided saddle point theorems, fixed point theorems, existence of maximizable linear functionals, the Warlas excess demand theorem, and the Gale-Nikaido-Debreu theorem.

Example. For a G-convex space $(X, D; \Gamma)$ and any space Y, an admissible class $\mathfrak{A}_c^\kappa(X,Y)$ is a subclass of $\mathfrak{B}(X,Y)$. There are maps in $\mathfrak{B}$ not belonging to $\mathfrak{A}_c^\kappa$, for example, the connectivity map due to Nash and Girolo; see [30]. Some other examples; see [48].

Recall that a nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every nonempty compact subset K of X and every 0-neighborhood $V \in \mathcal{V}$, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

In 1998, we obtained the following [30, Theorem 10.1]:

Theorem 7.3. *Let E be a Hausdorff t.v.s. and X an admissible* (*in the sense of Klee*) *convex subset of E. Then any compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

In [30], it was shown that Theorem 7.3 subsumes more than sixty known or possible particular cases and generalizes them in terms of the involving spaces and multimaps as well. Later, further examples of maps in the class $\mathfrak{B}$ were known.

It is not known whether the admissibility of X can be eliminated in Theorem 7.3. However, Theorem 7.3 can be generalized by switching the admissibility of domain of the map to the Klee approximability of its ranges as follows:

Let X be a subset of a t.v.s. E. A compact subset K of X is said to be *Klee approximable into* X if for any $V \in \mathcal{V}$, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a polytope in X.

Example. We give some examples of Klee approximable sets:

(1) If a subset X of E is admissible (in the sense of Klee), then every compact subset K of X is Klee approximable into E.

(2) Any polytope in a subset X of a t.v.s. is Klee approximable into X.

(3) Any compact subset K of a convex subset X in a locally convex t.v.s. is Klee approximable into X.

(4) Any compact subset K of a convex and locally convex subset X of a t.v.s. is Klee approximable into X.

(5) Any compact subset K of an admissible convex subset X of a t.v.s. is Klee approximable into X.

(6) Let X be an almost convex dense subset of an admissible subset Y of a t.v.s. E. Then every compact subset K of Y is Klee approximable into X.

Note that (6)⇒(5)⇒(4)⇒(3).

In 2004 [35], Theorem 7.3 is generalized as follows:

Theorem 7.4. *Let X be a subset of a Hausdorff t.v.s. E and $F \in \mathfrak{B}(X, X)$ a compact closed multimap. If $F(X)$ is Klee approximable into X, then F has a fixed point.*

The following are obtained in 2007 [38], where it should be $\mathfrak{B}^p = \mathfrak{B}$:

Corollary 7.5. *Let X be an almost convex admissible subset of a Hausdorff t.v.s. E and $F \in \mathfrak{B}(X, X)$ a compact closed map. Then F has a fixed point.*

Corollary 7.6. *Let X be an almost convex subset of a locally convex Hausdorff t.v.s. E and $F \in \mathfrak{B}(X, X)$ a compact closed map. Then F has a fixed point.*

One of the most simple known example is that every compact continuous selfmap on an almost convex subset in a Euclidean space has a fixed point. This generalizes the Brouwer fixed point theorem.

Moreover, since the class $\mathfrak{B}(X, X)$ contains a large number of special types of function spaces, we can apply Theorem 7.4 to them. For example, since any Kakutani map belongs to $\mathfrak{B}$, Theorem 7.4 and Corollaries 7.5 and 7.6 can be applied to them.

Applications. In [53], 26 papers on applications of the better admissible maps or the KKM admissible maps were introduced. Here we give the contents of *some* of typical works on applications of our better admissible multimap classes:

Park, NA 30 (1997) [29] — Recently, in a sequence of papers, the author introduced the admissible classes $\mathfrak{A}_c^\kappa$ of multimaps, which are large enough to include most of multimaps appearing in nonlinear analysis and algebraic topology. In this paper, we define a new 'better' admissible class $\mathfrak{B}$ of multimaps.

In Section 2, we obtain a basic coincidence theorem for the class $\mathfrak{B}$. Section 3 deals with a matching theorem and KKM theorem, which are basis of the KKM theory and have many applications. In Sections 4 and 5, we deduce fixed point theorems for compact or condensing multimaps in $\mathfrak{B}$ or in some related classes of multimaps.

Park, JMAA 329 (2007) [37] — We obtain new fixed point theorems on multimaps in the class $\mathfrak{B}^p$ defined on almost convex subsets of topological vector spaces. Our main results are applied to deduce various fixed point theorems, coincidence theorems, almost fixed point theorems, intersection theorems, and minimax theorems. Consequently, our new results generalize well-known works of Kakutani, Fan, Browder, Himmelberg, Lassonde, and others.

Park, PanAm. Math. J. 18 (2008) [40] — Using recent results in analytical fixed point theory, some known basic fixed point and coincidence theorems for families of multimaps are generalized and improved by removing some redundant restrictions. Especially, the author is mainly concerned with the class of locally selectionable multimaps having convex values instead of the Fan-Browder maps, which played main role in a number of previous works.

Balaj and Lin, NA 73 (2010) [7] — Theorem 7.2 is equivalent to some existence theorems of variational inclusion problems. These are applied to existence theorems of common fixed point, generalized maximal element theorems, a general coincidence theorems and a section theorem.

O'Regan and Perán, JMAA 380 (2011) [20] — The authors set out a rigorous presentation of Park's class of admissible multimaps, within the general framework of multimaps between topological spaces, using a broad definition of convexity. In addition, they obtain a fixed point theorem for better admissible multimaps defined on a proximity space via the Samuel-Smirnov compactification.

Lu and Zhang, CMA 64 (2012) [16] — The authors introduced the concept of FWC-spaces (short form of finite weakly convex spaces) as a unified form of many known modifications of G-convex spaces and the better admissible class of multimaps on them. Note that these new concepts are inadequately defined and that no results on them can be true.

Note that their FWC-spaces are simply ϕ_A-spaces due to Park.

O'Regan and Shahzad, AFPT 2 (2012) [21] — A new Krasnoselskii fixed point result is presented for weakly sequentially upper semicontinuous maps. The proof is immediate from results of O'Regan. The authors also extend the results for a general class of maps, namely the $\mathfrak{B}^\kappa$ maps of Park.

O'Regan, AA 92 (2013) [19] — The author presents a definition of d-essential and d-L-essential maps in completely regular topological spaces and establishes a homotopy property for both d-essential and d-L-essential maps. Also using the notion of extendability, he presents new continuation theorems.

8. KKM ADMISSIBLE MULTIMAP CLASSES $\mathfrak{KC}$, $\mathfrak{KO}$

Recall that, early in 1994 [26], for a convex space (X, D) and a Hausdorff space Y, it was indicated that an acyclic map $F : X \multimap Y$ and later, more generally, a map $F \in \mathfrak{A}_c^\kappa(X, Y)$ belongs to the class $\mathfrak{KC}$. This was the origin of the study of the so-called KKM admissible class of multimaps. Later, in 1997 [63], the fact was extended to G-convex spaces $(X, D; \Gamma)$ instead of convex spaces.

Since then, in the KKM theory on abstract convex spaces, there have appeared multimap classes $\mathfrak{A}_c^\kappa$, KKM, S-KKM, s-KKM, $\mathfrak{B}$, $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$, and various modifications of them. Park [45] reviewed certain mutual relations among such spaces. In fact, the author showed that the multimap class S-KKM is included in the class $\mathfrak{KC}$, and that most of known fixed point theorems on s-KKM maps follow from the corresponding ones on $\mathfrak{B}$-maps. Consequently, the author could unify all the classes KKM, S-KKM and s-KKM to $\mathfrak{KC}$-maps. Note that compact closed maps in the classes KKM and s-KKM belong to the class $\mathfrak{B}$; see [35].

The following is known [40, Lemma 6]:

Proposition 8.1. *Let $(E, D; \Gamma)$ be a G-convex space and Z a topological space. Then*

(1) $\mathbb{C}(E,Z) \subset \mathfrak{A}_c^\kappa(E,Z) \subset \mathfrak{B}(E,Z)$;

(2) $\mathbb{C}(E,Z) \subset \mathfrak{KC}(E,Z) \cap \mathfrak{KO}(E,Z)$; *and*

(3) $\mathfrak{A}_c^\kappa(E,Z) \subset \mathfrak{KC}(E,Z) \cap \mathfrak{KO}(E,Z)$ *if* Z *is Hausdorff.*

Consider the following condition for a G-convex space $(E \supset D;\Gamma)$:

$(*)$ $\Gamma_{\{x\}} = \{x\}$ for each $x \in D$; and, for each $N \in \langle D\rangle$ with the cardinality $|N| = n+1$, there exists a continuous function $\phi_N : \Delta_n \to \Gamma_N$ such that $\phi_N(\Delta_n) = \Gamma_N$ and that $J \in \langle N\rangle$ implies $\phi_N(\Delta_J) = \Gamma_J$.

Note that every convex space satisfies the condition $(*)$. We had the following [40, Theorem 16]:

Theorem 8.2. *Let* $(E,D;\Gamma)$ *be a G-convex space and* Z *a topological space.*

(1) *If* Z *is a Hausdorff space, then every compact map* $F \in \mathfrak{B}(E,Z)$ *belongs to* $\mathfrak{KC}(E,Z)$.

(2) *If* $F : E \multimap Z$ *is a closed map such that* $F\phi_N \in \mathfrak{KC}(\Delta_n, Z)$ *for any* $N \in \langle D\rangle$ *with the cardinality* $|N| = n+1$, *then* $F \in \mathfrak{B}(E,Z)$.

(3) *In the class of closed maps defined on a G-convex space* $(E \supset D;\Gamma)$ *satisfying condition* $(*)$ *into a space* Z, *a map* $F \in \mathfrak{KC}(E,Z)$ *belongs to* $\mathfrak{B}(E,Z)$.

Remark. In (2), note that for any map $F \in \mathfrak{A}_c^\kappa(E,Z)$, we have $F\phi_N \in \mathfrak{A}_c^\kappa(\Delta_n,Z) \subset \mathfrak{KC}(\Delta_n,Z) \cap \mathfrak{KO}(\Delta_n,Z)$ when Z is Hausdorff; see [13].

The following are [40, Corollaries 16.1 and 16.2], respectively.

Corollary 8.3. *In the class of compact closed maps defined on a G-convex space* $(E \supset D;\Gamma)$ *satisfying condition* $(*)$ *into a Hausdorff space* Z, *two subclasses* $\mathfrak{KC}(E,Z)$ *and* $\mathfrak{B}(E,Z)$ *are identical.*

Corollary 8.4. *In the class of compact closed maps defined on a convex space* (X,D) *into a Hausdorff space* Z, *two subclasses* $\mathfrak{KC}(X,Z)$ *and* $\mathfrak{B}(X,Z)$ *are identical.*

Remark. 1. This is noted in [29] by a different method. In view of Corollary 8.4, the class $\mathfrak{B}$ is favorable to use for convex spaces since it has already plenty of examples and is much easier to find examples.

2. Proposition 8.1, Theorem 8.2, and Corollary 8.3 hold also for ϕ_A-spaces $(X,D;\Gamma)$ with $\Gamma_A := \phi_A(\Delta_n)$ for $A \in \langle D\rangle$ with $|A| = n+1$.

Corollary 8.5. *Let* X *be a subset of a Hausdorff t.v.s.,* I *a nonempty set,* $s : I \to X$ *a map such that* $\operatorname{co} s(I) \subset X$, *and* $T \in s$*-KKM*(I,X,X). *If* T *is closed and compact, then* $T \in \mathfrak{B}(X,X)$.

Proof. Note that $(X,s(I))$ is a convex space and the class s-KKM(I,X,X) is $\mathfrak{KC}(X,X)$. The conclusion follows from Corollary 8.4. $\square$

In 2004, the author [35] showed that a compact closed s-KKM map from a convex subset of a t.v.s. into itself belongs to $\mathfrak{B}$ whenever $s : I \to X$ is a surjection.

Corollary 8.6. *Let X be a subset of a Hausdorff t.v.s., I a nonempty set, $s : I \to X$ a map such that $\operatorname{co} s(I) \subset X$, and Y a Hausdorff space. Then, in the class of closed compact maps, four classes $\mathfrak{KC}(X,Y)$, $\mathrm{KKM}(X,Y)$, s-$\mathrm{KKM}(I,X,Y)$, and $\mathfrak{B}(X,Y)$ coincide.*

Proof. For the convex space $(X, s(I))$, we have $\mathfrak{KC}(X,Y) = \mathrm{KKM}(X,Y) = \mathfrak{B}(X,Y)$ by Theorem 8.2(1) and (3). Note that $\mathrm{KKM}(X,Y) = s{-}\mathrm{KKM}(I,X,Y)$ by following the proof of [12, Proposition 2.3]. □

In view of Corollary 8.6, all fixed point theorems on s-KKM maps on a Hausdorff t.v.s. are consequences of corresponding ones on $\mathfrak{B}$-maps.

Moreover, if $F : X \to Y$ is a continuous single-valued map or if $F : X \multimap Y$ has a continuous selection, then it is easy to check that $F \in \mathfrak{KC}(X,Y) \cap \mathfrak{KO}(X,Y)$. Note that there are many known selection theorems due to Michael and others; see [50].

For convex subsets of a t.v.s., from the KKM principle, we had the following almost fixed point theorems for the class $\mathfrak{KC}$ and $\mathfrak{KO}$ [33]:

Theorem 8.7. *Let X be a convex subset of a t.v.s. E and $F \in \mathfrak{KC}(X,X)$ such that $F(X)$ is totally bounded. Then for any convex neighborhood V of 0 in E, there exists an $x_* \in X$ such that $F(x_*) \cap (x_* + V) \neq \emptyset$.*

Theorem 8.8. *Let X be a totally bounded convex subset of a t.v.s. E and $F \in \mathfrak{KO}(X,X)$. Then for each closed convex neighborhood V of 0 in E, there exists an $x_* \in X$ such that $F(x_*) \cap (x_* + V) \neq \emptyset$.*

Note that E is not necessarily Hausdorff in Theorems 8.7 and 8.8. From Theorem 8.7, we immediately have the following with a routine proof:

Corollary 8.9 *Let X be a convex subset of a locally convex Hausdorff t.v.s. E. Then any compact closed map $F \in \mathfrak{KC}(X,X)$ has a fixed point.*

Applications. Some of relatively recent works related the classes $\mathfrak{KC}$ and $\mathfrak{KO}$ are as follows:

Shahzad, NA 56 (2004) [64] — This paper discusses new fixed point and approximation theorems for multimaps in the class S-KKM.

Park, NAF 11 (2006) [36] — We introduce basic results in the KKM theory on abstract convex spaces and the map classes $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$, and $\mathfrak{B}$. We study the nature of Kakutani type maps, $\mathfrak{B}$-maps, and $\mathfrak{KC}$-maps in G-convex spaces; and show that generalizations of the key results in four papers are consequences of the G-convex space theory and the new abstract convex space theory.

Amini et al., NA 66 (2007) [6] — An abstract convex space (X, C) consists of a nonempty set X and a family C of subsets of X such that X, $\emptyset \in C$ and C_i is closed under arbitrary intersections. In this paper the authors introduce the class S-KKM mappings for their abstract convex spaces (X, C). They obtain some fixed point theorems for multimaps with the S-KKM property on Φ-spaces.

Yang et al., FPTA (2011) [67] — The authors first prove that the product of a family of LΓ-spaces in the sense of Park is also an LΓ-space. Then, by using a Himmelberg type fixed point theorem in LΓ-spaces due to Park, they establish existence theorems of solutions for systems of generalized quasivariational inclusion problems, systems of variational equations, and systems of generalized quasiequilibrium problems in LΓ-spaces. Applications of the existence theorem of solutions for systems of generalized quasiequilibrium problems to optimization problems are given in LΓ-spaces.

Yang and Huang, BKMS 49 (2012) [66] — A coincidence theorem for a compact $\mathfrak{KC}$-map is proved in an abstract convex space in the sense of Park. Several more general coincidence theorems for noncompact $\mathfrak{KC}$-maps are derived in abstract convex spaces. Some examples are given to illustrate the coincidence theorems. As applications, an alternative theorem concerning the existence of maximal elements, an alternative theorem concerning equilibrium problems and a minimax inequality for three functions are proved in abstract convex spaces.

Lu and Hu, JFSA (2013) [15] — The authors establish a new collectively fixed point theorem in noncompact abstract convex spaces in the sense of Park. As applications of this theorem, they obtain some new existence theorems of equilibria for generalized abstract economies in noncompact abstract convex spaces.

Park, NACA (2016) [58] — In the last three decades, we introduced several fixed point theorems for multimap classes on various types of abstract convex spaces. Such are the classes of acyclic maps, the Fan-Browder type maps, admissible maps $\mathfrak{A}_c^\kappa$, better admissible maps $\mathfrak{B}$, and the KKM maps $\mathfrak{KC}$ and $\mathfrak{KO}$. In our previous reviews, several hundred papers related to applications of such fixed point theorems were introduced. In the present review, we introduce some recent results in analytical fixed point theory based on our previous works. Most of them are not appeared in our previous reviews.

9. BASIC THEOREMS IN THE KKM THEORY

In our KKM theory on abstract convex spaces given in [40, 49], there exist some basic theorems from which we can deduce several equivalent formulations and useful applications. In this section, we introduce some of such basic theorems in [40].

We begin with the following prototype of KKM type theorems on the finite intersection property:

Theorem 9.1. *Let $(E, D; \Gamma)$ be an abstract convex space, Y a topological space, and $F \in \mathfrak{KO}(E, Y)$ [resp., $F \in \mathfrak{KC}(E, Y)$]. Let $G : D \multimap Y$ be a map such that*

(1) *for any $N \in \langle D \rangle$, $F(\Gamma_N) \subset G(N)$; and*

(2) *G is open-valued [resp., closed-valued].*

Then for each $N \in \langle D \rangle$, $F(E) \cap \bigcap\{G(y) : y \in N\} \neq \emptyset$.

Remark. 1. If $E = Y$ and if the identity map $1_E = F \in \mathfrak{K}(E, E)$, then Condition (1) says that G is a KKM map.

2. If $E = Y = \Delta_n$ is an n-simplex, D is the set of its vertices, and $\Gamma = \mathrm{co}$ is the convex hull operation, then the celebrated KKM theorem says that $1_E \in \mathfrak{KC}(E, E)$.

3. If D is a nonempty subset of a t.v.s. $E = Y$ (not necessarily Hausdorff), Fan's KKM lemma says that $1_E \in \mathfrak{KC}(E, E)$.

4. For another forms of the KKM theorem for convex spaces, H-spaces, or G-convex spaces and their applications, there exists a large number of works.

From Theorem 9.1, we have another finite intersection property as follows:

Theorem 9.2. *Let $(E, D; \Gamma)$ be an abstract convex space, Y a topological space, and $F \in \mathfrak{KO}(E, Y)$ [resp., $F \in \mathfrak{KC}(E, Y)$]. Let $G : D \multimap Y$ and $H : E \multimap Y$ be maps satisfying*

(1) *G is open-valued [resp., closed-valued];*

(2) *for each $x \in E$, $F(x) \subset H(x)$; and*

(3) *for each $y \in F(E)$, $M \in \langle D \setminus G^-(z) \rangle$ implies $\Gamma_M \subset E \setminus H^-(z)$.*

Then $F(E) \cap \bigcap\{G(z) : z \in N\} \neq \emptyset$ for all $N \in \langle D \rangle$.

The following coincidence theorem follows from Theorem 9.2:

Theorem 9.3. *Let $(E, D; \Gamma)$ be an abstract convex space, Y a topological space, $S : D \multimap Y$, $T : E \multimap Y$ maps, and $F \in \mathfrak{KO}(E, Y)$ [resp., $F \in \mathfrak{KC}(E, Y)$]. Suppose that*

(1) *S is open-valued [resp., closed-valued];*

(2) *for each $y \in F(E)$, $\mathrm{co}_\Gamma S^-(y) \subset T^-(y)$; and*

(3) *$F(E) \subset S(N)$ for some $N \in \langle D \rangle$.*

Then there exists an $\bar{x} \in E$ such that $F(\bar{x}) \cap T(\bar{x}) \neq \emptyset$.

From Theorem 9.3, we obtain the following Ky Fan type matching theorem:

Theorem 9.4. *Let $(E, D; \Gamma)$ be an abstract convex space, Y a topological space, $S : D \multimap Y$, and $F \in \mathfrak{KO}(E, Y)$ [resp., $F \in \mathfrak{KC}(E, Y)$] satisfying (1) and (3) of Theorem 9.3. Then there exists an $M \in \langle D \rangle$ such that $F(\Gamma_M) \cap \bigcap\{S(x) : x \in M\} \neq \emptyset$.*

Theorem 9.4 can be stated in its contrapositive form and in terms of the complement $G(z)$ of $S(z)$ in Y. Then we obtain Theorem 9.1. Therefore, Theorems 9.1–9.4 are mutually equivalent and can be applied to various results in the KKM theory on our abstract convex spaces.

If we add certain compactness or coercivity condition to Theorem 9.1, then we obtain certain KKM type whole intersection theorems as follows in Park [52, 55]:

Theorem 9.5. *Let $(X, D; \Gamma)$ be an abstract convex space, Z a topological space, $F \in \mathfrak{KC}(X, Z)$, and $G : D \multimap Z$ a map such that*

(1) *$\overline{G}$ is a KKM map w.r.t. F; and*

(2) *there exists a nonempty compact subset K of Z such that either*

(i) *$K \supset \bigcap\{\overline{G(y)} : y \in M\}$ for some $M \in \langle D\rangle$; or*

(ii) *for each $N \in \langle D\rangle$, there exists a Γ-convex subset L_N of X relative to some $D' \subset D$ such that $N \subset D'$, $F(L_N)$ is compact, and*

$$K \supset \overline{F(L_N)} \cap \bigcap\{\overline{G(z)} : z \in D'\}.$$

Then we have

$$\overline{F(X)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,

(α) *if G is transfer closed-valued, then $\overline{F(X)} \cap K \cap \bigcap\{G(z) : z \in D\} \neq \emptyset$; and*

(β) *if G is intersectionally closed-valued, then $\bigcap\{G(z) : z \in D\} \neq \emptyset$.*

Here, intersectionally [resp., transfer] closed sets are complements of intersectionally [resp., transfer] open sets.

Note that Theorem 9.5 is the basis of hundreds of all statements in the KKM theory.

Remark. Since we introduced the multimap classes $\mathfrak{A}_c^\kappa$, $\mathfrak{B}$, $\mathfrak{KC}$, and $\mathfrak{KO}$, many authors or printers mistook $\mathfrak{A}$ for $\mathcal{U}$ or $\mathbb{U}$, $\mathfrak{B}$ for $\mathcal{B}$ or $\mathbb{B}$, and $\mathfrak{KO}$ for $\mathcal{KD}$ or $\mathfrak{RO}$. *The author cordially asks his followers to keep the original notations.*

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