Dynamic Systems and Applications 26 (2017) 121-130

STABILITY AND BOUNDEDNESS IN VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH DELAY

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ABSTRACT. In this paper, a class of non-linear vector Volterra integro-differential equations of first order with constant delay is considered. The stability and boundedness of solutions are investigated. The technique of proofs involves defining appropriate Lyapunov functionals. The obtained results include and improve the results obtained in literature.

AMS (MOS) Subject Classification. 34D05, 34K20, 45J05.

1. INTRODUCTION

During the last years, many good results have been obtained on the qualitative behaviors in Volterra integro-differential equations without delay. In particular, for some works on the stability and boundedness in certain Volterra integro-differential equations without delay, we referee the interested reader to the papers of Becker ([1], [2], [3]), Burton ([4], [5], [6]), Burton et al. [7], Burton and Mahfoud [8], Corduneanu [9], Furumochi and Matsuoka [11], Gripenberg et al. [13], Hara et al.[14], Miller [16], Staffans [19], Tunç [20], Tunç and Ayhan [22], Vanualailai and Nakagiri [24] and theirs references.

Besides, concerning stability and boundedness in Volterra integro-differential equations with delay, we can find a few interesting results in the papers by Graef and Tunç [12], Raffoul [17], Raffoul and Ünal [18], Tunç [21] and in theirs references.

In 1982, Burton [5] considered the following non-linear homogeneous scalar Volterra integro-differential equation without delay

$$x'(t) = A(t)f(x(t)) + \int_0^t B(t,s)g(x(s))ds, \tag{1.1}$$

where $t \geq 0$, $x \in \Re$, $A(t) : \Re \to [0,\infty)$ and $f, g : \Re \to \Re$ are continuous functions with $f(0) = g(0) = 0$, and $B(t,s)$ is a continuous function for all $0 \leq s \leq t < \infty$. The author studied the stability, boundedness, convergence of bounded solutions of equation (1.1) by using the Lyapunov functionals.

Received April 11, 2016

Next, in the same paper, Burton [5] considered the following non-linear homogeneous vector Volterra integro-differential equation without delay of the form

$$x'(t) = Ax(t) + \int_0^t B(t,s)E(x(s))x(s)ds, \tag{1.2}$$

where $t \geq 0$, x is an n-vector, $n \geq 1$, A is an $n \times n$-constant matrix, $B(t,s)$ is an $n \times n$-continuous matrix function for $0 \leq s \leq t < \infty$, $E(x)$ is $n \times n-$matrix valued continuous function for $x \in \Re^n$. Burton [5] discussed the stability, boundedness, convergence of bounded solutions and square integrability of solutions of equation (1.2) by defining a Lyapunov functional.

Later, in 1999, Furumochi and Matsuoka [11] discussed the stability and boundedness of solutions of vector Volterra integro-differential equations without delay of the form

$$x'(t) = a(x(t)) + \int_0^t C(t,s)f(x(s))ds + g(t,x(t)), \tag{1.3}$$

when $g(t,x(t)) \equiv 0$ and $g(t,x(t)) \neq 0$, respectively. In the proofs, the Lyapunov functionals are applied by Furumochi and Matsuoka [11].

Recently, the author in [21] considered the non-linear scalar Volterra integro-differential equation with delay

$$x'(t) = -a(t)f(x(t)) + \int_{t-\tau}^t B(t,s)g(x(s))ds, \tag{1.4}$$

where $t \geq 0$, τ is a positive constant, fixed delay, $x \in \Re$, $a(t) : [0,\infty) \to (0,\infty)$, $f, g : \Re \to \Re$ are continuous functions with $f(0) = g(0) = 0$, $B(t,s)$ is a continuous function for $0 \leq s \leq t < \infty$. The author discussed the stability, boundedness and convergence of bounded solutions of equation (1.4) when $t \to \infty$ by defining suitable Lyapunov functionals.

In this paper, we consider the vector Volterra integro-differential equations with delay of the form:

$$x' = -Ax + H(x) + \int_{t-\tau}^t C(t,s)G(s,x(s))ds + E(t,x), \tag{1.5}$$

where $t \geq 0$, τ is a positive constant fixed delay, x is an n-vector, $n \geq 1$, A is an $n\times n$-symmetric matrix, $H : \Re^n \to \Re^n$ is a continuous function with $H(0) = 0$, $C(t,s)$ is an $n\times n$-continuous symmetric matrix function for $0 \leq s \leq t < \infty$, $G, E : \Re^+ \times \Re^n \to \Re^n$ are continuous functions with $G(t,0) = 0$, and $\Re^+ = [0,\infty)$.

The objective of this paper is to investigate sufficient conditions for the stability of zero solution and boundedness of solutions of equation (1.5) by employing Lyapunov functionals, when $E(t,x) \equiv 0$ and $E(t,x) \neq 0$, respectively. It is clear that equation (1.1)–(1.3) and equation (1.4) are special cases of equation (1.1) when $\tau = 0$ and $\tau \neq 0$, respectively.

It should be noted any investigation of the stability and boundedness in a Volterra integro-differential equation, using the Lyapunov functional method, first requires the definition or construction of a suitable Laypunov functional. In fact, this case can be an arduous task. The situation becomes more difficult when we replace the ordinary differential equation with a functional integro-differential equation. However, once a viable Lyapunov functional has been defined or constructed, researchers may end up with working with it for a long time, deriving more more information about stability. To arrive at the objective of this paper, we define two new suitable Lyapunov functionals.

In view of the mentioned information, it follows that the Volterra integro-differential equations discussed by Burton [5] and Furumochi and Matsuoka [11] are without delay. However, in this paper, the Volterra integro-differential equations to be studied are with delay. This is a novelty and improvement for the case without delay to the case with delay. That is from the ordinary case to the functional case. Besides, our equation, equation (1.5) includes and extends the equations discussed by Burton [4], and Furumochi and Matsuoka [11], when $\tau = 0$. In addition, our equation includes and improves equation (1.4) in [21] from the scalar case to the system form.

Our results will also be different from that obtained in the literature (see, Becker ([1], [2], [3]), Burton ([4], [5], [6]), Burton et al. [7], Burton and Mahfoud [8], Corduneanu [9], Furumochi and Matsuoka [11], Gripenberg et al. [13], Hara et al. [14], Miller [16], Rafffoul [17], Raffoul and Ünal [18], Staffans [19], Tunç ([20], [21]), Vanualailai and Nakagiri [24] and the references thereof). By this way, we mean that the Volterra integro-differential equations considered and the assumptions to be established here are different from that in the mentioned papers above. This paper has also a contribution to the subject in the literature, and it may be useful for researchers working on the qualitative behaviors of solutions to Volterra integro-differential equations. In view of all the mentioned information, it can be checked the novelty and originality of the present paper.

We give some basic information related equation (1.5).

We use the following notation throughout this paper.

For any $t_0 \geq 0$ and initial function $\varphi \in [t_0 - \tau, t_0]$, let $x(t) = x(t, t_0, \varphi)$ denote the solution of equation (1.5) on $[t_0 - \tau, \infty)$ such that $x(t) = \varphi(t)$ on $\varphi \in [t_0 - \tau, t_0]$.

Let $C[t_0, t_1]$ and $C[t_0, \infty)$ denote the set of all continuous real-valued functions on $[t_0, t_1]$, $[t_0, \infty)$, respectively.

For $\varphi \in C[0, t_0]$, $|\varphi|_{t_0} := \sup\{|\varphi(t)| : 0 \leq t \leq t_0\}$.

Definition 1.1. The zero solution of equation (1.5) with $E(t, x) \equiv 0$ is stable if for each $\varepsilon > 0$ and each $t_0 \geq 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $\varphi \in C[0, t_0]$ with $|\varphi(t)|_{t_0} < \delta$ and $t \geq t_0$ imply $|x(t, t_0, \varphi)| < \varepsilon$.

Definition 1.2. The solutions of equation (1.5) are bounded with $E(t,x) \neq 0$ if for each $T > 0$, there exists $D > 0$ such that $t_0 \geq 0$, $\varphi \in C[0,t_0]$, $|\varphi(t)|_{t_0} < T$ and $t \geq t_0$ imply $|x(t)| < D$.

The following lemma plays a key role in proving our main results.

Lemma 1.3 (Horn and Johnson [15]). *Let A be a real symmetric $n \times n$-matrix. Then, for any $X \in \Re^n$,*

$$a_1 \|X\|^2 \leq \langle AX, X\rangle \geq a_0 \|X\|^2 ,$$

where a_0 and a_1 are the least and greatest eigenvalues, respectively, of A.

The stability result in this paper is based in the following theorem.

Theorem 1.4 (Driver [10]). *If there exists a functional $V(t,\phi(\cdot))$, defined whenever $t \geq t_0 \geq 0$ and $\phi \in C([0,t],\Re^n)$, such that*

(i) *$V(t,0) \equiv 0$, V is continuous in t and locally Lipschitz in ϕ,*
(ii) *$V(t,\phi(.)) \geq W(|\phi(t)|)$, $W : [0,\infty) \to [0,\infty)$ is a continuous function with $W(0) = 0$, $W(r) > 0$ if $r > 0$, and W strictly increasing (positive definiteness), and*
(iii) *$V'(t,\phi(.)) \leq 0$,*

then the zero solution of equation (1.5) is stable, and

$$V(t,\phi(.)) = V(t,\phi(s) : 0 \leq s \leq t)$$

is called a Lyapunov functional for system (1.5).

2. **STABILITY**

In this section we use a Lyapunov functional and establish sufficient conditions to obtain a stability result on zero solution of equation(1.5).

Let

$$E(t,x) \equiv 0$$

and

$$\beta(t) = 2\gamma \|B\| + \|B\| \int_0^t \|C(t,s)\|\, ds + \delta^2 \|B\| \int_{t-\tau}^{\infty} \|C(u+\tau,t)\|\, du.$$

A. Assumptions. We assume the following holds:

(A1) There exist a symmetric matrix B and positive constants δ and γ such that

$$A^T B + BA = I, \quad x^T Bx > 0 \quad \text{for } x \in \Re^n,\ x \neq 0,$$

(A2) $G(t,0) = 0$, $\|G(t,x)\| \leq \delta\|x\|$ for $t \geq 0$ and $x \in \Re^n$,
$H(0) = 0$, $\|H(x)\| \leq \gamma\|x\|$ for $x \in \Re^n$.

Theorem 2.1. *Let assumptions (A1) and (A2) hold, K is a positive constant. If $\beta(t) \le K < 1$ holds for $t \ge t_0 - \tau \ge 0$, then the zero solution of equation (1.5) is stable.*

Proof. We define a functional $W_0 = W_0(t) = W_0(t, x(t))$ defined by

$$W_0 = x^T(t)Bx(t) + \|B\| \int_0^t \int_{t-\tau}^{\infty} \||C(u+\tau, s)|\| \, du \, \|G(s, x(s))\|^2 \, ds.$$

If the assumptions of Theorem 2.1 hold, then it is clear that the functional W_0 is positive definite.

Differentiating the functional W_0 with respect to t, we obtain

$$\begin{aligned}
W_0' &= [-x^T A + H^T(x)]Bx + \left[\int_{t-\tau}^{t} G^T(s, x(s))C^T(t,s)ds\right] Bx \\
&\quad + x^T B \left[-Ax + H(x) + \int_{t-\tau}^{t} C(t,s)G(s,x(s))ds\right] \\
&\quad + \|B\| \int_{t-\tau}^{\infty} \|C(u+\tau,t)\| \, du \, \|G(t,x)\|^2 - \|B\| \int_0^t \|C(t,s)\| \, \|G(s,x(s))\|^2 \, ds \\
&= -x^T ABx - x^T BAx + 2H^T(x)Bx \\
&\quad + 2x^T B \int_{t-\tau}^{t} C(t,s)G(s,x(s))ds \\
&\quad + \|B\| \int_{t-\tau}^{\infty} \|C(u+\tau,t)\| \, du \, \|G(t,x)\|^2 \\
&\quad - \|B\| \int_0^t \|C(t,s)\| \, \|G(s,x(s))\|^2 \, ds \\
&= -x^T(AB+BA)x + 2H^T(x)Bx \\
&\quad + 2x^T B \int_{t-\tau}^{t} C(t,s)G(s,x(s))ds \\
&\quad + \|B\| \int_{t-\tau}^{\infty} \|C(u+\tau,t)\| \, du \, \|G(t,x)\|^2 \\
&\quad - \|B\| \int_0^t \|C(t,s)\| \, \|G(s,x(s))\|^2 \, ds \\
&\le -\|x\|^2 + 2\|x\| \, \|B\| \, \|H(x)\| \\
&\quad + 2\|x\| \, \|B\| \int_{t-\tau}^{t} \|C(t,s)\| \, \|G(s,x(s))\| \, ds \\
&\quad + \|B\| \int_{t-\tau}^{\infty} \|C(u+\tau,t)\| \, du \, \|G(t,x)\|^2 \\
&\quad - \|B\| \int_0^t \|C(t,s)\| \, \|G(s,x(s))\|^2 \, ds \\
&\le -\|x\|^2 + 2\gamma \|B\| \, \|x\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\|B\|\int_{t-\tau}^{t}\|C(t,s)\|\,\{\|x\|^2+\|G(s,x(s))\|^2\}ds\\
&+\delta^2\|B\|\int_{t-\tau}^{\infty}\|C(u+\tau,t)\|\,du\,\|x\|^2\\
&-\|B\|\int_{0}^{t}\|C(t,s)\|\,\|G(s,x(s))\|^2\,ds\\
=&-\|x\|^2+2\gamma\|B\|\,\|x\|^2+\|B\|\int_{t-\tau}^{t}\|C(t,s)\|\,\|x\|^2\,ds\\
&+\|B\|\int_{t-\tau}^{t}\|C(t,s)\|\,\|G(s,x(s))\|^2\,ds+\delta^2\|B\|\int_{t-\tau}^{\infty}\|C(u+\tau,t)\|\,du\,\|x\|^2\\
&-\|B\|\int_{0}^{t}\|C(t,s)\|\,\|G(s,x(s))\|^2\,ds\\
=&-\|x\|^2+2\gamma\|B\|\,\|x\|^2+\|B\|\int_{t-\tau}^{t}\|C(t,s)\|\,\|x\|^2\,ds\\
&+\delta^2\|B\|\int_{t-\tau}^{\infty}\|C(u+\tau,t)\|\,du\,\|x\|^2\\
=&-\left\{1-2\gamma\|B\|-\|B\|\int_{t-\tau}^{t}\|C(t,s)\|\,ds-\delta^2\|B\|\int_{t-\tau}^{\infty}\|C(u+\tau,t)\|\,du\right\}\|x\|^2\\
=&-\left\{1-2\gamma\|B\|-\|B\|\int_{0}^{t}\|C(t,s)\|\,ds-\delta^2\|B\|\int_{t-\tau}^{\infty}\|C(u+\tau,t)\|\,du\right\}\|x\|^2\\
&-\|B\|\int_{0}^{t-\tau}\|C(t,s)\|\,ds\\
\le&-\left\{1-2\gamma\|B\|-\|B\|\int_{0}^{t}\|C(t,s)\|\,ds-\delta^2\|B\|\int_{t-\tau}^{\infty}\|C(u+\tau,t)\|\,du\right\}\|x\|^2\\
=&-\{1-\beta(t)\}\,\|x\|^2\\
\le&-(1-K)\,\|x\|^2\le 0.
\end{aligned}
$$

Thus, in view of the discussion made and Theorem 1.4, we can arrive at that the zero solution of equation (1.5) is stable. This completes the proof of Theorem 2.1. □

3. BOUNDEDNESS

Let $E(t,x)\neq 0$ and

$$\beta_1(t)=-K\|x\|^2+2\theta(t)\|B\|\,\|x\|^2+2\theta(t)\|B\|\,\|x\|-L\theta(t).$$

B. Assumptions.

(B1) $\|E(t,x)\|\le\theta(t)(\|x\|+1)$, $\theta:\Re^+\to\Re^+$, $\Re^+=[0,\infty)$, θ is a continuous function such that

$$\int_0^\infty\theta(s)ds<\infty\quad\text{and}\quad\theta(t)\to 0\quad\text{as } t\to\infty.$$

(B2) There exists a positive constant K_1 such that $\beta_1(t) \le -K_1 \|x\|^2$.

Theorem 3.1. *We suppose that assumptions* $(A1)$, $(A2)$, $(B1)$ *and* $(B2)$ *hold. Then all solutions of equation* (1.5) *are bounded.*

Proof. We define a functional $W_1 = W_1(t) = W_1(t, x(t))$ by

$$W_1 = \left[1 + x^T(t)Bx(t) + \|B\| \int_0^t \int_{t-\tau}^{\infty} \||C(u+\tau, s)|\| \, du \, \|G(s, x(s))\|^2 \, ds\right] \times \exp\left(-L \int_0^t \theta(s) ds\right),$$

where L is a positive constant.

In the light of the assumptions of Theorem 3.1, it follows that the functional W_1 is positive definite.

Differentiating the functional W_1 with respect to t, using the assumptions of Theorem 3.1 and the inequality $|mn| \le 2^{-1}(m^2 + n^2)$ we have

$$\begin{aligned}
W_1' \le\ & -L\theta(t)W_1 \\
& + \exp\left(-L \int_0^t \theta(s) ds\right) \times [-x^T A + H^T(x)]Bx \\
& + \exp\left(-L \int_0^t \theta(s) ds\right) \times \left[\int_{t-\tau}^t G^T(s, x(s))C^T(t, s)ds + E^T(t, x)\right] Bx \\
& + \exp\left(-L \int_0^t \theta(s) ds\right) \times x^T B \left[-Ax + H(x) + \int_{t-\tau}^t C(t, s)G(s, x(s))ds + E(t, x)\right] \\
& + \exp\left(-L \int_0^t \theta(s) ds\right) \times \|B\| \int_{t-\tau}^{\infty} \|C(u+\tau, t)\| \, du \, \|G(t, x)\|^2 \\
& - \exp\left(-L \int_0^t \theta(s) ds\right) \times \|B\| \int_0^t \|C(t, s)\| \, \|G(s, x(s))\|^2 \, ds \\
\le\ & -L\theta(t)W_1 - \exp\left(-L \int_0^t \theta(s) ds\right) \\
& \times \left\{1 - 2\gamma \|B\| - \|B\| \int_{t-\tau}^t \|C(t, s)\| \, du - \delta^2 \|B\| \int_{t-\tau}^{\infty} \|C(u+\tau, t)\| \, du\right\} \|x\|^2 \\
& + 2\exp\left(-L \int_0^t \theta(s) ds\right) \|B\| \, \|x\| \, \|E(t, x)\| \\
\le\ & -L\theta(t)W_1 - K \exp\left(-L \int_0^t \theta(s) ds\right) \|x\|^2 \\
& + 2\exp\left(-L \int_0^t \theta(s) ds\right) \|B\| \, \|x\| \, \theta(t)(\|x\| + 1) \\
=\ & -L\theta(t)W_1 - \alpha_1 \exp\left(-L \int_0^t \theta(s) ds\right) \|x\|^2 \\
& + 2\theta(t) \exp\left(-L \int_0^t \theta(s) ds\right) \|B\| \, \|x\|^2 + 2\theta(t) \exp\left(-L \int_0^t \theta(s) ds\right) \|B\| \, \|x\|
\end{aligned}$$

$$\leq \exp\left(-L\int_0^t \theta(s)ds\right)\{-K\|x\|^2+2\theta(t)\|B\|\|x\|^2+2\theta(t)\|B\|\|x\|-L\theta(t)\}$$
$$\leq \exp\left(-L\int_0^t \theta(s)ds\right)\beta_1(t)\|x\|^2$$
$$\leq -K_1\|x\|^2 \leq 0.$$

Integrating the estimate $W_1'(t) \leq 0$ from zero t_0 to t, we get

$$\left[1+x^T(t)Bx(t)+\|B\|\int_0^t\int_{t-\tau}^{\infty}\||C(u+\tau,s)|\|\,du\,\|G(s,x(s))\|^2\,ds\right]$$
$$=W_1(t)\leq W_0(t_0)=M>0$$

Then, the boundedness of solutions can be readily followed. $\square$

Example 3.2. For the case $n=1$, as a special case of equation (1.5), we consider the nonlinear Volterra integro-differential equation with delay, $\tau \geq 0$,

$$x' = -100x+\sin x - 2\int_{t-\tau}^{t} C(t,s)\,(x(s)+\sin x(s))\,ds + \frac{x}{1+t^2},$$

for $t-\tau \geq 0$, $x \in \Re$.

When we compare this equation with equation (1.5) and consider the assumption of Theorem 2.1 and Theorem 3.1, it follows that

$$A=100, \quad B=1/200,$$
$$A^TB+BA=1,$$
$$H(x)=\sin x,$$
$$|\sin x| \leq |x|, \quad \gamma=1,$$
$$G(t,x)=x+\sin x,$$
$$|G(t,x)| \leq |x+\sin x| \leq |x|+|\sin x| \leq 2|x|, \quad \delta=2,$$
$$E(t,x)=\frac{x}{1+t^2},$$
$$|E(t,x)|=\frac{|x|}{1+t^2}\leq\frac{1}{1+t^2}(|x|+1),$$
$$\theta(t)=\frac{1}{1+t^2}, \qquad \text{and} \qquad \theta(t)\to 0 \ \text{ as } t\to\infty,$$
$$\int_0^{\infty}\theta(s)ds=\int_0^{\infty}\frac{1}{1+s^2}ds=\frac{\pi}{2}<\infty,$$

$$\beta(t)=2\gamma|B|+|B|\int_0^t|C(t,s)|\,ds+\delta^2|B|\int_{t-\tau}^{\infty}|C(u+\tau,t)|\,du$$
$$=\frac{1}{100}+\frac{1}{200}\int_0^t|C(t,s)|\,ds+\frac{1}{50}\int_{t-\tau}^{\infty}|C(u+\tau,t)|\,du,$$

$$\beta_1(t) = -K\,|x|^2 + 2\theta(t)\,|B|\,|x|^2 + 2\theta(t)\,|B|\,|x| - L\theta(t)$$
$$= -K\,|x|^2 + \frac{1}{100(t^2+1)}\,|x|^2 + \frac{1}{100(t^2+1)}\,|x| - \frac{L}{t^2+1}.$$

Hence, all the assumptions of Theorem 2.1 and Theorem 3.1 hold if $\beta(t) \leq K < 1$ and $\beta_1(t) \leq -K_1 x^2$.

4. CONCLUSION

A class of vector non-linear Volterra integro-differential equations of first order is considered. The stability and boundedness of solutions are studied by means of the Lyapunov's functional approach. The obtained results improve some results in the literature.

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