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VERTICAL AND HORIZONTAL GRAZING

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ABSTRACT. Grazing solutions of non-autonomous system with variable moments of impulses are examined. Appropriate definitions for vertical and horizontal grazing in non-autonomous systems are given and interpreted geometrically. The linearization for the periodic solutions which have vertical or horizontal grazing is obtained. Examples are presented to demonstrate the practicality of our results and they are visualized by the simulations.

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1. INTRODUCTION

There can be found two different approaches in the literature for the definition of grazing. One of them is that the grazing occurs whenever the trajectory meets with zero velocity to the surface of discontinuity [1], [2], [3]. Another one is that the trajectory meets with the surface tangentially [4]–[11]. In the light of the papers [4]–[11], we focused on the analytical expression of the tangency at the grazing point to define the horizontal grazing and vertical grazing.

There are wide ranges of studies about grazing phenomenon [1]–[12]. All existing studies are conducted on autonomous systems [3, 13], the systems with discontinuous right hand side [6, 8] and non-autonomous system with autonomous surfaces of discontinuity [2]. In [11], a criterion for horizontal grazing motions in a dry friction oscillator is determined by means of the local theory of non-smooth dynamical systems on the connectible and accessible domains. In the study [2], the creation of periodic orbits associated with grazing bifurcations in the models of impacting systems and some sufficient conditions are obtained for the existence of a family of periodic solutions. In [14], two distinct types of grazing bifurcations are taken into account. One is that the stable motion disappears and system stabilized onto an already existing attracting solution and the other in which there is an immediate jump to chaos as part of an orbit grazes at a stop. In the paper [1], the stable periodic orbits and chaotic motions are determined analytically by utilizing the limit mapping. In [2], some sufficient conditions are obtained to determine the existence of a family of periodic orbits whose creation is caused by ramification from the grazing bifurcation

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point. The smallest appropriate parameter alteration for the horizontal grazing in a hybrid system is determined by applying numerical methods [7]. A general method is presented for the construction of suitable local maps near a horizontal grazing point for n-dimensional PWS systems in [8]. In our paper [9], we have taken into account the grazing properties of discontinuous dynamical systems and we prove the orbital stability theorem for them.

Horizontal and vertical grazing should be considered because they cannot be taken into account by utilizing the existing results in the literature. In a geometrical sense, the horizontal grazing occurs when the surface of discontinuity has a tangent plane at the grazing point which is parallel to the time axis and the vertical grazing occurs whenever the tangent plane at the grazing point is perpendicular to the time axis. The horizontal and vertical grazing are depicted in Figures 1a and 1b, respectively. The appropriate definitions of the horizontal and vertical grazing for non-autonomous system whose vector field and surfaces are defined by non-autonomous functions and the definition of horizontal grazing for non-autonomous system with cylindrical surface of discontinuity are given. The periodic solutions which have vertical or horizontal grazing are obtained in specific examples. The stabilities of them are examined by constructing proper linearization systems around the periodic solutions. The periodic solutions and their stabilities are observed through simulations and the results are depicted.

1.1. **Motivation.** Take into account the following differential equation

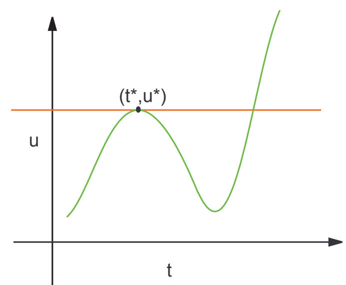
$$x'' + a(t)x' + b(t)x = f(t, x, x'), \tag{1.1}$$

where $a(t)$ is a variable damping function, $b(t)$ is a variable spring function and $f(t, x, x')$ is a force applied to the system. Assume that it is subject to impacts with a cylindrical surface $\Gamma = \{(t, x, x') \mid \Phi(x, x') = 0\}$. The type of the barrier is common for impact mechanisms. To illustrate, the surfaces $x = X_0$ and $x' = X_0'$ in (t, x, x') are cylindrical surfaces. Thus, if the grazing occurs in the non-autonomous equation (1.1), then it is mainly a horizontal one as expected.

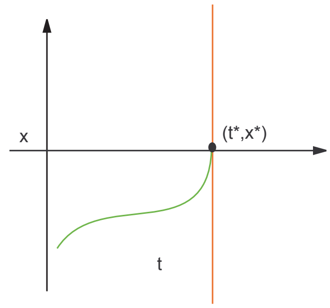
In the paper [10], the system of leaky integrate-and-fire neuron model is presented as

$$\begin{aligned} \frac{du}{dt} &= -ku + S(t), \\ u(t^+) &= 0 \quad \text{if} \quad u(t) = \Theta, \end{aligned} \tag{1.2}$$

where u is the internal state, k is the leaky parameter and $S(t)$ is the input time series which is positive. If the internal state u reaches the threshold Θ the spike occurs and the internal state immediately resets to the resting state $u = 0$. In the leaky integrate and fire neuron model, the grazing takes place whilst there exists a time T such that $\frac{du}{dt}_{t=T} = -ku + S(T) = 0$. (See Fig. (A), it is taken from the paper [10].)



(A) Horizontal grazing in neural networks.



(B) Vertical grazing.

This demonstrates that the horizontal grazing can be observed in neural networks. Moreover, it is determined that the bifurcation results in the breaking of inter spike interval attractors. In [4], it is demonstrated that the grazing bifurcation can be utilized to find the Arnol'd tongue diagram for mode-locked responses and determined that the horizontal grazing phenomenon in integrate-and-fire neuron model causes the passing to a regular firing either from a fast firing or from a doublet firing and it causes the diminish of the stability of sub-threshold oscillations.

Vertical grazing which is depicted in Fig. (B) is supposed to be useful for the analysis of singularities. The presentation of the vertical grazing is beneficial since the method of analysis can be applied for models with singularities under impacts. Such systems can be seen in the models of electrically driven robot manipulator which has slower mechanical dynamics and faster electrical dynamics. In this type of systems, we should consider the problem in two parts such as one part is slower and one part is faster dynamics [15]. For which the vertical grazing can be utilized in the analysis of faster dynamics.

2. Grazing non-autonomous system with variable impulse moments

Let $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}$ be the sets of all real numbers, natural numbers and integers, respectively. Let $G \subseteq \mathbb{R}^n$ be an open and connected set. The non-autonomous systems with variable moments of impulses consist of two different systems. One is that the vector field as well as surfaces are defined by non-autonomous functions and the other is the vector field defined as non-autonomous function bu the surfaces defined as an autonomous functions in other words the surfaces are cylindrical.

The first type can be considered as a following system

$$
\begin{aligned}
x' &= f(t,x), \\
\Delta x|_{t=\tau_i(x)} &= J_i(x),
\end{aligned} \tag{2.1}
$$

where $(t,i,x) \in \mathbb{R} \times \mathbb{Z} \times G$, the function $f(t,x)$ is continuously differentiable in x and t on $\mathbb{R} \times G$, and T-periodic in t, i.e. $f(t+T,x) = f(t,x)$, functions $J_i(x)$ and $\tau_i(x)$, $i \in \mathbb{Z}$, are differentiable on G and $J_i(x)$ satisfies the following equality, $J_{i+p}(x) = J_i(x)$

for a natural number p and $\tau_i(x)$ has (T,p)-property, i.e. $\tau_i(x)+T=\tau_{i+p}(x)$ for all $i\in\mathbb{Z}$.

The other type of system can be defined by the following system of impulsive differential equations

$$\begin{aligned} x' &= f(t,x), \\ \Delta x|_{x\in\Gamma} &= J_i(x), \end{aligned} \tag{2.2}$$

where Γ is a cylindrical surface of discontinuity and defined as $\Gamma=\{(t,x)|\Phi(x)=0,\ t\in\mathbb{R},\ x\in G\}$. The function $f(t,x)$ is continuously differentiable in x and t on $\mathbb{R}\times G$, and T-periodic in t, i.e. $f(t+T,x)=f(t,x)$, functions $J_i(x)$ and $\Phi(x)=0$ are differentiable on G and $J_i(x)$ satisfies the following equality, $J_{i+p}(x)=J_i(x)$ for a natural number p and for all $i\in\mathbb{Z}$.

To simplify the notation, we need the following system of ordinary differential equations

$$y' = f(t,y). \tag{2.3}$$

In what follows, the conditions will be needed.

(N1) $\tau_i(x+J_i(x))\le\tau_i(x)$ for all $i\in\mathbb{Z}$ and $x\in G$;

(N2) if $\xi(t,\tau_i(c),c+J_i(c))$, $c\in G$, $i\in\mathbb{Z}$, is a solution of (2.3), then $t\neq\tau_i(\xi(t,\tau_i(c),c+J_i(c)))$ for all $t>\tau_i(c)$ and $i\in\mathbb{Z}$ and $x\in G$;

(N3) There exist positive numbers C and N such that $CN<1$

$$\max_{(t,x)\in\mathbb{R}\times D}\|f(t,x)\|\le C,\quad \max_{x\in D}\|\frac{\partial\tau_i(x)}{\partial x}\|\le N,\quad \max_{0\le\sigma\le1}\left\langle\frac{\partial\tau_i(x+\sigma J_i(x))}{\partial x},J_i(x)\right\rangle\le 0,$$

where $\langle,\rangle$ is the usual dot product and D is a compact subspace of the phase space and for all $(t,x)\in\mathbb{R}\times D$ and $i\in\mathbb{Z}$ and $x\in G$;

Assume that the conditions (N1)–(N3) are valid. Then, the solution of (2.1) intersects the surfaces of discontinuity exactly once [16]. The surface divides c^l_ϵ into two parts. Consider a point $(t_0,x_0)\in s_\epsilon$. Take $z\in c^l_\epsilon$ such that there exists $t<\zeta_l$ and $z=x(t,t_0,x_0)$. Denote the union of all of these z, for all $(t_0,x_0)\in s^l_\epsilon$ as b^l_ϵ. Moreover, denote $K_l=\bigcup_{j=1}^{m_l}\bar{K}_l^{(j)}\setminus\bigcup_{i\neq m}(K_l^{(i)}\cap K_l^{(m)})$, where $\bar{S}$ is the closure of a set S.

Consider a periodic solution $\Psi(t)$ of (2.1). Denote by θ_i, $i\in\mathbb{Z}$, the moment of meeting of a the periodic solution with the surface $t=\tau_i(x)$, $i\in\mathbb{Z}$. The intersection moments satisfy the property that $\theta_{i+p}=\theta_i+T$, $i\in\mathbb{Z}$, where p is a positive number.

Definition 2.1. There is a *horizontal grazing* of the periodic solution $\Psi(t)$ of (2.1) at a point $(\theta_l,\Psi(\theta_l))$, $l=1,2,\dots,p$, if for some $j=1,2,\dots,n$, the conditions are fulfilled:

(i) $f_j(\theta_l,\Psi(\theta_l))=0$,

(ii) a function $t = \eta(x_j) \equiv \tau_l(\Psi_1(\theta_l), \Psi_2(\theta_l), \ldots, \Psi_{j-1}(\theta_l), x_j, \Psi_{j+1}(\theta_l), \ldots, \Psi_n(\theta_l))$ is invertible near $x_j = x_j^0 = \Psi_j(\theta_l)$ for $x_j \leq x_j^0$ or $x_j \geq x_j^0$, and the one sided derivative $[\eta^{-1}(t)]'_-|_{t=\theta_l}$ or $[\eta^{-1}(t)]'_+|_{t=\theta_l}$ is equal to zero, respectively.

Definition 2.2. A *vertical grazing* of the periodic solution $\Psi(t)$ of (2.1) at a point $(\theta_l, \Psi(\theta_l))$ exits at the point $(\theta_l, \Psi(\theta_l))$ $l = 1, 2, \ldots, p$, if for some $j = 1, 2, \ldots, n$, the following conditions are fulfilled:

(i) a function $x_j = \Psi_j(t)$ is invertible near $x_j = x_j^0 = \Psi_j(\theta_l)$ for $x_j \leq x_j^0$ or/and $x_j \geq x_j^0$, and the one sided derivative $[\Psi_j^{-1}(x_j)]'_-|_{x=x_j}$ or/and $[\Psi_j^{-1}(x_j)]'_+|_{x=x_j}$ is equal to zero, respectively.

(ii) $\tau_{lx_j}(\Psi(\theta_l)) = 0$.

Consider a periodic solution $\Psi(t)$ of (2.2). Denote by θ_i, $i \in \mathbb{Z}$, the meeting moments of $\Psi(t)$ with the surface $\Phi(x) = 0$. They satisfy the property for all $i \in \mathbb{Z}$, $\theta_{i+p} = \theta_i + T$, where p is a positive number.

Definition 2.3. There is a *horizontal grazing* of the periodic solution $\Psi(t)$ of the system (2.2) at a point $(\theta_l, \Psi(\theta_l))$, where $l = 1, 2, \ldots, p$, if the equality at the point $\langle \Phi(\Psi(\theta_l)), f(\theta_l, \Psi(\theta_l)) = 0$ is valid.

Next, we will construct B-equivalent system to the system (2.1) [16], which reduces the systems with variable moments of impulses to that with fixed moments of impulses. For the system (2.2), it can be obtained similarly. Consider a point $(\theta_i, x) \in \mathbb{R} \times G$ on the periodic solution with a fixed $i \in \mathbb{Z}$. Let $\xi_i = \xi_i(x)$ be the meeting moment of the solution $x(t) = x(t, \theta_i, x)$ of (2.3). Additionally, assume that the solution $x_1(t) = x(t, \theta_i, x(\theta_i))$ of (2.3) exists on $\widehat{[\theta_i, \xi_i]}$. The B-map $W : x \to x_1(\xi)$ can be constructed as

$$W_i(x) = \int_{\theta_i}^{\xi_i} f(u, x(u))du + J_i\left(x + \int_{\theta_i}^{\xi_i} f(u, x(u))du\right) + \int_{\xi_i}^{\theta_i} f(u, x_1(u))du. \tag{2.4}$$

Let us take into account the following system of differential equations with fixed moments of impulses

$$\begin{aligned} y' &= f(t, y), \\ \Delta y|_{t=\theta_i} &= W_i(y). \end{aligned} \tag{2.5}$$

Due to the way of construction of $W_i(x)$ systems (2.1) and (2.5) are B-equivalent [16, 17] in the neighborhood of $\Psi(t)$. That is, if $x(t) : U \to G$ is a solution of (2.1), then coincides with a solution $y(t) : U \to G$ when $y(t_0) = x(t_0)$, for $t_0 \in U \backslash \cup_{i\in\mathbb{Z}} \widehat{[\theta_i, \xi_i]}$. Particularly, $x(\theta_i) = y(\theta_i+)$, $x(\xi_i) = y(\xi_i)$, if $\theta_i > \xi_i$, $x(\theta_i) = y(\theta_i)$, $x(\xi_i+) = y(\xi_i)$, if $\theta_i < \xi_i$. It is easy to see that $\Psi(t)$ is also a solution of (2.5) as well. In the remaining part of the paper, we will consider (2.5) instead of (2.1).

Assume that the periodic solution $\Psi(t)$ of (2.1) meets the surface $t = \tau_i(x)$ at the moment $t = \theta_i$, transversally. Let us start with the derivative of the equation $\theta_i(x) = \tau_i(x(\theta_i(x)))$, [16],

$$\nabla\theta_i(\Psi(\theta_i)) = \frac{\nabla\tau_i(\Psi(\theta_i))U(\theta_i)}{1 - \nabla\tau_i(\Psi(\theta_i))f(\theta_i, \Psi(\theta_i))}, \tag{2.6}$$

where $U(t)$, is a fundamental matrix of $u' = f_x(t, \Psi(t))u$ with $U(\theta_i) = I$, where I is $n \times n$ identity matrix.

By taking the derivative of the B-map defined by (2.4) with respect to x, we can determine the matrix D_i as

$$\begin{aligned} D_i = W_{ix}(\Psi(\theta_i)) &= (f(\theta_i, \Psi(\theta_i)) - f(\theta_i, \Psi(\theta_i)))\theta_i'(\Psi(\theta_i)) \\ &+ J(\Psi(\theta_i))\theta_i'(x) + J_{ix}(I + f(\theta_i, \Psi(\theta_i))\theta_i'(\Psi(\theta_i))), \end{aligned} \tag{2.7}$$

where the Jacobian matrix can be obtained as $W_{ix}(\Psi(\theta_i)) = [\frac{\partial W_i}{\partial x_1}, \frac{\partial W_i}{\partial x_2}, \ldots, \frac{\partial W_i}{\partial x_n}]$.

It is easy to see that the linearization at the point $(\theta_i, \Psi(\theta_i))$ can be obtained as

$$\Delta u|_{t=\theta_i} = D_i u, \tag{2.8}$$

with $D_{i+p} = D_i$, $i \in \mathbb{Z}$.

For Examples 2.5 and 2.6, one can utilize the formulas (2.6) and (2.7) to obtain a linearization at the point $(\theta_i, \Psi(\theta_i))$, $i \in \mathbb{Z}$. For Example 2.4, we cannot apply formulas due to the appearance of singularity in the formula (2.7) at the grazing point. For this example, we will consider another approach to obtain a linearization at the grazing point.

Example 2.4. In this example, the motion of one degree of freedom mechanical oscillator which is subjected to impacts with a rigid wall is considered and it can be expressed as

$$\begin{aligned} x'' + 0.22x' + x &= 1 + 0.22\sin(t), \\ \Delta x'|_{(t,x,x')\in\Gamma} &= -(1 + 0.9x')x', \end{aligned} \tag{2.9}$$

where the surface of discontinuity is $\Gamma = \{(t, x, x') \mid x = 0, \ t \in \mathbb{R}\}$. System (2.9) admits 2π-periodic continuous solution of the form $\Psi(t) = 1 - \cos(t)$. Defining variables as $x = x_1$ and $x' = x_2$, we have

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -0.22x_2 - x_1 + 1 + 0.22\sin(t), \\ \Delta x_2|_{(t,x_1,x_2)\in\Gamma} &= -(1 + 0.9x_2)x_2, \end{aligned} \tag{2.10}$$

where $\Gamma = \{(t, x_1, x_2) \mid x_1 = 0, t \in \mathbb{R}\}$ and the points $(\theta_i, \Psi(\theta_i), \Psi'(\theta_i)) = (2\pi i, 0, 0)$, $i \in \mathbb{Z}$, are grazing as well. Denote by $x(\theta_i) = (x_1(\theta_i), x_2(\theta_i))$. In what follows, we will apply formula (2.7) in the basis of system (2.10).

Fix $i \in \mathbb{Z}$, and consider a near solution $x(t) = (x_1(t), x_2(t)) = x(t, \theta_i, \Psi(\theta_i) + \Delta x)$, to $\Psi(t)$ of the differential part of the system (2.10). The solution $x(t)$ impacts the barrier at a moment $t = \xi_i$ near to $t = \theta_i$ and at the point $(x_1, x_2) = (x_1(\xi_i), x_2(\xi_i))$. Let also, $\tilde{x}(t) = (\tilde{x}_1(t), \tilde{x}_2(t))$ be a solution of the equation such that $\tilde{x}(\xi_i) = x(\xi_i) + J(x(\xi_i))$. Define the following map

$$
\begin{aligned}
W_i(x) = {} & \int_{\theta_i}^{\xi_i} \begin{bmatrix} x_2(s) \\ x_1(s) - 0.22x_2(s) + 1 + 0.22\sin(s) \end{bmatrix} ds \\
& + J\left(x + \int_{\theta_i}^{\xi_i} \begin{bmatrix} x_2(s) \\ -x_1(s) - 0.22x_2(s) + 1 + 0.22\sin(s) \end{bmatrix} ds \right) \\
& + \int_{\xi_i}^{\theta_i} \begin{bmatrix} \tilde{x}_2(s) \\ -\tilde{x}_1(s) - 0.22\tilde{x}_2(s) + 1 + 0.22\sin(s) \end{bmatrix} ds.
\end{aligned} \tag{2.11}
$$

Let us start with a linearization for inside continuous solutions. The solutions, inside of the cycle, do not impact the barrier with non-zero velocity and are continuous. Thus, the linearization for these solutions is the following system [18],

$$
\begin{aligned}
u_1' &= u_2, \\
u_2' &= -u_1 - 0.22u_2.
\end{aligned} \tag{2.12}
$$

The multipliers of the system are $\rho_1^{(1)} = 0.5006 - 0.0191i$, $\rho_2^{(1)} = 0.5006 + 0.0191i$, where $i^2 = -1$. Since the multipliers are inside the unit circle, the cycle $\Psi(t)$ is asymptotically stable with respect to inside continuous solutions.

Now, we will continue with the linearization for the outside discontinuous solutions. The linearization system around the cycle $\Psi(t)$ for solutions which are outside of the cycle has the form, [16],

$$
\begin{aligned}
u_1' &= u_2, \\
u_2' &= -u_1 - 0.22u_2, \\
\Delta u|_{t=\theta_i} &= W_{ix}(x^*)u,
\end{aligned} \tag{2.13}
$$

where $\theta_i = 2\pi i$ and $u = (u_1, u_2)^T$, where T denotes transpose of a matrix. The matrices $W_{ix}(x^*)$ will be evaluated below. Assume that the solution $x(t)$ meets the barrier at the moment $t = \tau$ and denote the meeting point as $\bar{x} = x(\tau) = (x_1(\tau), x_2(\tau))$, where $x_1(\tau) = 0$, $x_2(\tau) < 0$ and $\tau \approx 2\pi$. It is easy to see that any impacting solution near to $\Psi(t)$ meets the barrier transversely. Taking derivative of (2.11) and substituting $x = \bar{x}$ to the derivative, we obtain that

$$\begin{aligned}\frac{\partial W_i(\bar{x})}{\partial x_1^0} &= \begin{bmatrix} \bar{x}_2 \\ -\bar{x}_1 - 0.22\bar{x}_2 + 1 + 0.22\sin(\tau) \end{bmatrix} \frac{\partial \xi_i(\bar{x})}{\partial x_1^0} + \begin{bmatrix} 1 & 0 \\ 0 & 1.96\bar{x}_2 \end{bmatrix} \\ &\quad \times \left(e_1 + \begin{bmatrix} \bar{x}_2 \\ -\bar{x}_1 - 0.22\bar{x}_2 + 1 + 0.22\sin(\tau) \end{bmatrix} \frac{\partial \xi_i(\bar{x})}{\partial x_1^0} \right) \\ &\quad - \begin{bmatrix} -0.98(\bar{x}_2)^2 \\ -\bar{x}_1 + 0.2156(\bar{x}_2)^2 + 1 + 0.22\sin(\tau) \end{bmatrix} \frac{\partial \xi_i(\bar{x})}{\partial x_1^0}.\end{aligned} \tag{2.14}$$

Moreover, differentiating $\Phi(x(\xi_i(x))) = 0$, we have

$$\frac{\partial \xi_i(x(\theta_i))}{\partial x_j} = -\frac{\Phi_x(x(\theta_i))\frac{\partial x(\theta_i)}{\partial x_{0j}}}{\Phi_x(x(\theta_i))f(\theta_i, x(\theta_i))}, \quad j = 1, 2, \tag{2.15}$$

for the transversal point $\bar{x} = (\bar{x}_1, \bar{x}_2)$, the first component $\frac{\partial \xi_i(\bar{x})}{\partial x_1^0}$ can be evaluated as $\frac{\partial \xi_i(\bar{x})}{\partial x_1^0} = -\frac{1}{\bar{x}_2}$. From the last equality, it is seen how the singularity appears at the grazing point $x^* = (x_1^*, x_2^*) = (0, 0)$. Finally, we obtain that

$$\begin{aligned}\frac{\partial W_i(\bar{x})}{\partial x_1^0} &= \begin{bmatrix} \bar{x}_2 \\ -\bar{x}_1 - 0.22\bar{x}_2 + 1 + 0.22\sin(\tau) \end{bmatrix} \left(\frac{-1}{\bar{x}_2} \right) \\ &\quad + \begin{bmatrix} 1 & 0 \\ 0 & 1.96\bar{x}_2 \end{bmatrix} \left(e_1 - \begin{bmatrix} \bar{x}_2 \\ \bar{x}_1 + 0.22(\bar{x}_2 - \sin(\tau)) - 1 \end{bmatrix} \left(\frac{-1}{\bar{x}_2} \right) \right) \\ &\quad + \begin{bmatrix} 0.98(\bar{x}_2)^2 \\ \bar{x}_1 + 0.2156(\bar{x}_2)^2 - 1 - 0.22\sin(\tau) \end{bmatrix} \left(\frac{-1}{\bar{x}_2} \right) \\ &= \begin{bmatrix} -1 + 0.98\bar{x}_2 \\ 0.22 + 0.2156\bar{x}_2 + 1.96(\bar{x}_1 - 0.22\sin(\tau) + 0.22\bar{x}_2 - 1) \end{bmatrix},\end{aligned} \tag{2.16}$$

where $e_1 = (1, 0)^T$.

The last expression demonstrates that the derivative is a continuous function of its arguments in a neighborhood of the grazing point. Since $\bar{x}$ is a transversal point, one can evaluate the limit as

$$\lim_{\bar{x} \to x^*} \frac{\partial W_i(\bar{x})}{\partial x_1^0} = B, \tag{2.17}$$

where $B = \begin{bmatrix} -1 \\ -1.74 \end{bmatrix}$.

To linearize system at the grazing point x^*, we should verify that the function $W_i(x)$ is differentiable at x^*. The differentiability requests that the partial derivatives $\frac{\partial W_i(x)}{\partial x_j^0}$, $j = 1, 2$, exist in a neighborhood of the grazing point and they are continuous at the point [19]. To compute the derivative $\frac{\partial W_i(x)}{\partial x_1^0}$ at x^*, the following expression will be taken into account

$$\begin{aligned}\frac{\partial W_i(x_1^*, x_2^*)}{\partial x_1^0} &= \lim_{x_1 \to x_1^*} \frac{W_i(x_1, x_2^*) - W_i(x_1^*, x_2^*)}{x_1 - x_1^*} \\ &= \lim_{x_1 \to x_1^*} \frac{W_i(x_1, x_2^*) - W_i(x_1^*, x_2^*)}{x_1 - x_1^*} - B + B. \end{aligned} \tag{2.18}$$

Applying the Mean Value Theorem [19], we obtain that

$$\lim_{x_1 \to x_1^*} \frac{\frac{\partial W_i(\zeta, x_2^*)}{\partial x_1^0}(x_1 - x_1^*) - B(x_1 - x_1^*)}{x_1 - x_1^*} + B, \tag{2.19}$$

where ζ lies between x_1 and x_1^*.

From (2.19), considering (2.17), we have that

$$\frac{\partial W_i(x_1^*, x_2^*)}{\partial x_1^0} = B. \tag{2.20}$$

So, the derivative exists and is continuous at x^*.

Now, let us check the existence and continuity of the derivative $\frac{\partial W_i(x)}{\partial x_2^0}$ at x^*. To accomplish these, we should continue with differentiating (2.11) again and substituting $\bar{x} = (\bar{x}_1, \bar{x}_2)$. Then, we obtain

$$\begin{aligned}\frac{\partial W_i(\bar{x})}{\partial x_2^0} &= \begin{bmatrix} \bar{x}_2 \\ -\bar{x}_1 - 0.22\bar{x}_2 + 1 + 0.22\sin(\tau) \end{bmatrix} \frac{\partial \xi_i(\bar{x})}{\partial x_2^0} \\ &\quad + \begin{bmatrix} 1 & 0 \\ 0 & 1.96\bar{x}_2 \end{bmatrix} i \left(e_2 + \begin{bmatrix} \bar{x}_2 \\ -\bar{x}_1 - 0.22(\bar{x}_2 - \sin(\tau)) + 1 \end{bmatrix} \frac{\partial \xi_i(\bar{x})}{\partial x_2^0} \right) \\ &\quad + \begin{bmatrix} 0.98(\bar{x}_2)^2 \\ \bar{x}_1 - 0.2156(\bar{x}_2)^2 - 1 - 0.22\sin(\tau) \end{bmatrix} \frac{\partial \xi_i(\bar{x})}{\partial x_2^0} \\ &= \begin{bmatrix} \bar{x}_2 - 0.98(\bar{x}_2)^2 \\ -\bar{x}_1 - 0.22(\bar{x}_2 - 0.98(\bar{x}_2)^2) \end{bmatrix} \frac{\partial \xi_i(\bar{x})}{\partial x_2^0} \\ &\quad + \begin{bmatrix} 1 & 0 \\ 0 & 1.96\bar{x}_2 \end{bmatrix} \left(e_2 + \begin{bmatrix} \bar{x}_2 \\ -\bar{x}_1 - 0.22(\bar{x}_2 - \sin(\tau)) + 1 \end{bmatrix} \frac{\partial \xi_i(\bar{x})}{\partial x_2^0} \right) \end{aligned} \tag{2.21}$$

where $e_2 = (0, 1)^T$. To evaluate the derivative $\frac{\partial \xi_i(\bar{x})}{\partial x_2^0}$ in (2.21), we apply formula (2.15) for the transversal point $\bar{x} = (\bar{x}_1, \bar{x}_2)$ and it is equal to $\frac{\partial \xi_i(\bar{x})}{\partial x_2^0} = 0$. This and formula (2.21) imply

$$\lim_{\bar{x} \to x^*} \frac{\partial W_i(\bar{x})}{\partial x_2^0} = C, \tag{2.22}$$

where $C = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Similar to above discussion for the first derivative, one can obtain that $\frac{\partial W_i(x^*)}{\partial x_2^0} = C$ and the derivative is continuous at x^*. Thus, both derivatives $\frac{\partial W_i(x)}{\partial x_1^0}$ and $\frac{\partial W_i(x)}{\partial x_2^0}$ exist in a neighborhood of x^* and they are continuous at x^*. That is, $W_i(x)$ is differentiable at x^*. Since of the periodicity, the linearization can be obtained for all grazing moment θ_i, $i \in \mathbb{Z}$.

Joining (2.20) and (2.22), it is obtained that

$$W_{ix}(x^*) = \begin{bmatrix} -1 & 0 \\ -1.74 & 0 \end{bmatrix}. \tag{2.23}$$

The multipliers of (2.13) are $\rho_1 = 0$ and $\rho_2 = 0.5339$. Due to the fact that the multipliers are less than unity in norm, we can conclude that the periodic solution $\Psi(t)$ is asymptotically stable.

Despite the grazing, the singularity is not obtained in the derivative (2.6) in Examples 2.5 and 2.6. So, the method presented in [16] can be utilized in the following examples to find a linearization at the grazing point.

Example 2.5. We will consider the system

$$\begin{aligned} x' &= 0, \\ \Delta x|_{t=\tau_i(x)} &= -0.2x, \end{aligned} \tag{2.24}$$

where $\tau_i(x) = 10\arccos(x+0.1) + i\pi$. One can easily determine that the system has zero solution $x(t) = 0$. We will consider it as a π-periodic solution of (2.24).

By means of the fact that $\tau_i(x)$ is an increasing function, it is easy to see that condition $(N1)$ is valid. For constants $C = 1/11$, and $N = \frac{10}{\sqrt{0.99}}$, such that $CN < 1$ and the following inequalities are valid $\max_{(t,x)\in\mathbb{R}\times D} \|f(t,x)\| = 0 \leq C$, $\max_{x\in D} \|\frac{\partial\tau_i(x)}{\partial x}\| = \max_{x\in D} \|\frac{10}{\sqrt{1-(x+0.1)^2}}\| \leq N$, and $\max_{0\leq\sigma\leq 1} \langle \frac{\partial\tau_i(x+\sigma J_i(x))}{\partial x}, J_i(x)\rangle = \max_{0\leq\sigma\leq 1} \frac{-2}{\sqrt{1-((1-0.2\sigma)x)^2}} \leq 0$. So, $(N2)$ is valid. The condition $(N3)$ is also true for this example.

The integral line $x(t) \equiv 0$ is tangent to the surface $t = \tau_0(x)$ at the point $(\theta_1, \Psi(\theta_1)) = (0,0)$. Indeed, since the right hand side $f(t,x)$ is constantly zero, the condition (i) is valid. Take into account the function $t = \eta(x) \equiv 10\arccos(x+0.1)$, which is invertible near $x = \Psi(\theta_1) = 0$, for $x \leq 0$ and the one sided derivative $[\eta^{-1}(t)]'_-|_{t=\theta_1} = 0.1\sin(\theta_1) = 0$. So, it validates the condition (ii). Therefore, the zero solution has horizontal grazing at the point $(\theta_1, \Psi(\theta_1)) = (0,0)$. Moreover, one can validate easily that $(\pi i, 0)$, $i \in \mathbb{Z}$ are also horizontal grazing points. Let us obtain a linearization system around zero solution. For a solution, $x(t) = x(t, 0, \bar{x})$, with $\bar{x} > 0$, there exists no intersection with the surfaces of discontinuity. This is why, the linearization system has the form

$$u' = 0. \tag{2.25}$$

Next, consider another solution $x(t) = x(t, 0, \bar{x})$, with $\bar{x} < 0$ of (2.24). One can easily find that the solution meets each of the surfaces of discontinuity. Due to the periodicity of the system, linearization near the zero solution at all points $(i\pi, 0)$ is the same, if exists. So, it is sufficient to consider the linearization around the grazing point $(0,0)$ for those points where $\bar{x} < 0$. Let us start with the function $\theta(x) = 10\arccos(x(\theta(x)) + 0.1)$. By taking derivative of it, we get

$$\theta'(x) = -10\frac{f(\theta(x), x(\theta(x)))\theta'(x) + 1}{\sqrt{1 - (x(\theta(x) + 0.1)^2)}}, \tag{2.26}$$

and substituting the grazing point into the equation (2.26), one can obtain that $\theta'(0) = -10/\sqrt{0.99}$.

The coefficients in the impulsive part of the linearization system have to be evaluated by formula

$$\begin{aligned} D_i = (&f(\theta(0), x(\theta(0))) - f(\theta(0), x(\theta(0)) + J(x(\theta(0)))))\theta'(0) \\ &+ J_x(1 + f(\theta(0), x(\theta(0))))\theta'(0) = -0.2, \end{aligned}$$

for all $i \in \mathbb{Z}$.

So, linearization system for the intersecting solutions with initial value $\bar{x} < 0$ can be determined as

$$\begin{aligned} &u' = 0, \\ &\Delta u|_{t=\pi i} = -0.2u. \end{aligned} \tag{2.27}$$

Consider solutions with $\bar{x} < 0$. The linearization for them is the system (2.27) and its multiplier can be evaluated as $\rho = 0.8 < 1$, and consequently solutions with negative initial values are attracted by the zero solution. Nevertheless, the solutions with positive initial values are constant. That is, one can say that the zero solution is stable for neighbors from above. On the basis of the discussion, one can conclude that zero solution is stable. It is pictured in Figure 1. The solutions $\Psi(t) = 0$, and $x(t, 0, \bar{x})$ with initial values $\bar{x} > 0$ and $\bar{x} < 0$ are depicted in black, red and magenta, respectively in Fig. 1 and the stability of the zero solution is apparently seen by virtue of simulation.

Through the last examples, it is seen that the tangent at the grazing point is parallel to the time axis. This approves why we call the phenomenon as *horizontal grazing*.

Example 2.6. In order to demonstrate vertical grazing, we take into account the following system

$$\begin{aligned} &x' = \frac{1}{\sqrt{i - t}},\ t \in [i-1, i), \\ &\Delta x|_{t=\tau_i(x)} = -1, \end{aligned} \tag{2.28}$$

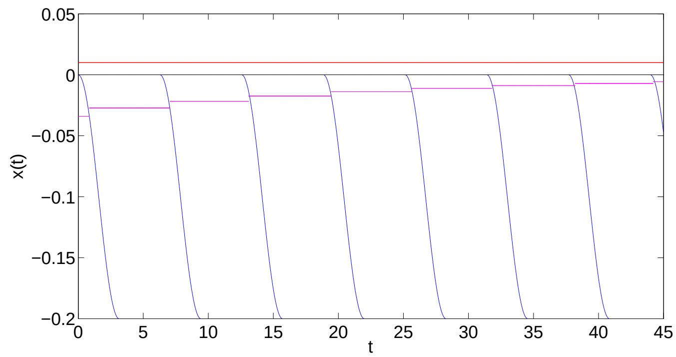


FIGURE 1. The blue curves are the discontinuity surfaces $t = \tau_i(x)$, $i = 0, 1, 2, \ldots, 7$. The solutions $\Psi(t) = 0$, and $x(t, 0, \bar{x})$ with initial values $\bar{x} = 0.01$ and $\bar{x} = -0.03$ are depicted in black, red and magenta, respectively.

where $f(t, x) = \frac{1}{\sqrt{i-t}}$ and $\tau_i(x) = \sqrt{16 - x^2} - 3 + i$, $i \in \mathbb{Z}$. The domain is equal to $G = (-0.6, 0.6)$. One can easily determine that the system has a 1-periodic solution

$$\Psi(t) = \begin{cases} -2 & \text{if } t = 0, \\ -2\sqrt{1-t} & \text{if } t \in (0, 1]. \end{cases}$$

The integral curve of the solution is tangent to the curve of discontinuity $\tau_0(x) = \sqrt{1 - x^2} + 1$ at the point $(\theta_1, \Psi(\theta_1))$, and the tangent is vertical. That is, one can find that the function $x = \Psi(t)$ is invertible near $x = \Psi(\theta_1) = 0$ and for $x < 0$, and the left hand derivative is equal to $[\Psi_j^{-1}(x_j)]'_-|_{x=x_j} = -2\frac{1}{2}x(\theta_1)^2 = 0$ and let $\tau_0(x) = \tau(x)$, and $\tau_x(x) = -\frac{x(\theta_1)}{\sqrt{16-x^2}} = 0$. Conditions (i) and (ii) are verified and the periodic solution $\Psi(t)$ has vertical grazing at the point $(\theta_1, \Psi(\eta_1))$. Similarly, the points are $(i, 0)$, $i \in \mathbb{Z}$ are vertical grazing ones. The periodic solution, $\Psi(t)$, is exhibited through simulation in Fig. 2.

Now, we will validate the conditions from $(N1)$ to $(N4)$. Every solution which meets a discontinuity surface does not intersect the same one again, which validates $(N1)$ and instead of the equation $x' = \frac{1}{\sqrt{i-t}}$, $t \in [i-1, i)$, we will take into account the differential equation $\frac{dt}{dx} = \frac{1}{\sqrt{i-t}}$, $t \in [i-1, i)$. For $C = 1$ and $N = 0.7$, such that $CN < 1$ and the following inequalities are valid $\max_{(t,x)\in\mathbb{R}\times D} \|f(t, x)\| = 1$, $\max_{x\in D} \|\frac{\partial\tau_i(x)}{\partial x}\| = \max_{x\in D} \|\frac{10}{\sqrt{1-(x+0.1)^2}}\| \le N$, and $\max_{0\le\sigma\le1}\langle\frac{\partial\tau_i(x+\sigma J_i(x))}{\partial x}, J_i(x)\rangle = \max_{0\le\sigma\le1} \frac{-2}{\sqrt{1-((1-0.2\sigma)x)^2}} \le 0$. So, $(N2)$ is verified. Conditions $(N3)$ and $(N4)$ can be validated easily.

Consider a near solution $x(t) = x(t, 0, \bar{x})$ of (2.28) to $\Psi(t)$ with $\bar{x} \neq 0$. It is easy to determine that all near solutions intersects the surface of discontinuity $\tau(x) = \tau_0(x)$. We could not evaluate the derivative $\theta'(x)$ at the grazing point, by considering the original system. For this reason, let us interchange the dependent and independent variables in the equation. Consider the system

$$\frac{dt}{dx} = \sqrt{1-t}. \tag{2.29}$$

Since the function $\Psi(t)$ is invertible on the interval $[0,1]$, its inverse satisfies the equation (2.29), as well as the surface $\tau_0(x)$ can be written as $X(t) = -\sqrt{16-(t-3)^2}$, for negative values of x. It is easy to check that the solution

$$\Psi^{-1}(x) = \begin{cases} 0 & \text{if } x = -2, \\ 1 - \frac{x^2}{4} & \text{if } x \in (-2, 0], \end{cases} \tag{2.30}$$

of the equation (2.29) has a horizontal grazing point, $(\Psi^{-1}(\theta_1), \theta_1) = (0, 1)$.

Introduce the function $\phi(t)$ as an analogue of $\theta(x)$ for the last equation. It is easy to find that

$$\theta'(0) = \frac{1}{\phi'(1)}, \tag{2.31}$$

since the functions are mutually inverse. Let us evaluate $\phi'(1)$. Issuing from the equation $\phi(t) = -\sqrt{16-(t(\phi(t))-3)^2}$, we obtain $\phi'(1) = -\frac{-2(t(\phi(1))-3)(\sqrt{1-t(\phi(1))}\phi'(1)+1)}{2\sqrt{16-(t(\phi(1))-3)^2}}$ and $\phi'(1) = -\frac{1}{\sqrt{3}}$, i.e. $\theta'(0) = -\sqrt{3}$. Taking into account the periodicity of system (2.28) as well as $\Psi(t)$, one can conclude that $\theta_i'(0)$, is equal to $-\sqrt{3}$, for all $i \in \mathbb{Z}$. By utilizing this discussion and equation (2.7), one can obtain that $D_i \equiv D = -\sqrt{3}$.

Thus, the variational system for all solutions near $\Psi(t)$ has the form

$$\begin{aligned} u' &= 0, \\ \Delta u &= -\sqrt{3}u. \end{aligned} \tag{2.32}$$

The multiplier for (2.32) can be found as $\rho = -\sqrt{3}+1$, and it is less than one in absolute value. So, the periodic solution $\Psi(t)$ is asymptotically stable. One can observe through simulations results exhibited in Fig. 2 that near solutions approach to th orbit of the cycle $\Psi(t)$ as time increases.

3. **Discussion**

This paper includes information about a non-autonomous system with non-fixed moments of impulses whose solutions have vertical and grazing points. For the horizontal grazing, a system with a non-autonomous vector field and a cylindrical surface of d iscontinuity is considered as an example and for the vertical grazing the systems with non-autonomous vector field and the surfaces of discontinuity is exemplified. By applying a novel technique, we construct a linearization system around the grazing periodic solution. Concrete models are demonstrated and some simulations are presented to visualize theoretical results. Grazing solutions are widely investigated in mechanical systems, but there is a few studies can be found in neural networks

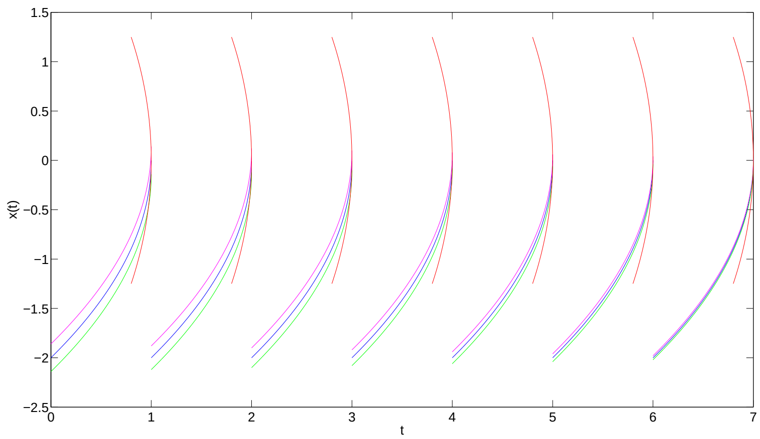


FIGURE 2. The blue curve is for the cycle $\Psi(t)$, the magenta and green curves are the solutions which start with an initial condition -1.9 and -2.1, respectively. The red curves are the surfaces of discontinuity $t = \tau_i(x)$, $i = 0, 1, \ldots, 6$.

which includes grazing. Further, we will apply our methods to investigate the stability of neural network models which have grazing points in other words which meet the threshold tangentially.

REFERENCES

[1] A. B. Nordmark, Universal limit mapping in grazing bifurcations, *Physical Review E*, 55:266–270, 1997.

[2] A. B. Nordmark. Existence of periodic orbits in grazing bifurcations of impacting mechanical oscillators. *Nonlinearity*, 14:1517–1542, 2001.

[3] P. T. Piiroinen, L. N. Virgin and A. R. Champneys, Chaos and Period-Adding: Experimental and Numerical Verification of the Grazing Bifurcation, *Journal of Nonlinear Science*, 14:383–404, 2004.

[4] S. Coombes, R. Thul and K.C.A. Wedgwood, Nonsmooth dynamics in spiking neuron models, *Physica D: Nonlinear Phenomena*, 241:2042-2057, 2012.

[5] M. di Bernardo and S. J. Hogan, Discontinuity-induced bifurcations of piecewise smooth dynamical systems, *Philosophical Transactions of The royal society A*, 368:4915–4935, 2010.

[6] M. di Bernardo, C. J. Budd, A. R. Champneys and P. Kowalczyk, *Piecewise-smooth dynamical systems theory and applications*, Springer-Verlag, London, 2008.

[7] V. Donde and I. A. Hiskens, Shooting methods for locating grazing phenomena in hybrid systems, *International Journal of Bifurcation and Chaos*, 16:671–692, 2006.

[8] M. di Bernardo, C. J. Budd and A. R. Champneys, Grazing bifurcations in n-dimensional piecewise-smooth dynamical systems, *Physica D*, 160:222–254, 2001.

[9] M. Akhmet and A. Kıvılcım, Discontinuous dynamics with grazing points, *Communications in Nonlinear Science and Numerical Simulation*,(accepted).

[10] Y. Ono, H. Suzuki and K. Aihara, Grazing bifurcation and mode-locking in reconstructing chaotic dynamics with a leaky integrate-and-fire model, *Artificial Life and Robotics*, 7:55–62, 2003.

[11] A. C. J. Luo, and B. C. Gegg, Grazing phenomena in a periodically forced, friction-induced, linear oscillator, *Communications in Nonlinear Science and Numerical Simulation*, 11:777–802, 2006.

[12] A. C. J. Luo, A theory for non-smooth dynamical systems on connectable domains, *Communications in Nonlinear Science and Numerical Simulations*, 10:1–55, 2005.

[13] C. Hosa and A. R. Champneys, Grazing bifurcations and chatter in a pressure relief valve model, *Physica D*, 241:2068–2076, 2012.

[14] S. Foale and S. R. Bishop, Dynamical Complexities of Forced Impacting Systems, *Philosophical Transactions of The royal society A*, 338:547–556, 1992.

[15] F. Verhulst, *Methods and Applications of Singular Perturbations: Boundary Layers and Multiple Timescale Dynamics*, Springer, 2005.

[16] M. U. Akhmet, *Principles of Discontinuous Dynamical Systems*, Springer, New York, 2010.

[17] M. U. Akhmet, *Nonlinear hybrid continuous/discrete-time models*, Atlantis Press, Paris, Amsterdam, 2011.

[18] P. Hartman, *Ordinary Differential Equations*, SIAM, 2002.

[19] T. M. Apostol, *Calculus, Vol. 2: Multi-Variable Calculus and Linear Algebra with Applications to Differential Equations and Probability*, Wiley, 1969.