Dynamic Systems and Applications 26 (2017) 367-382

GRADIENT NONLINEAR ELLIPTIC SYSTEMS DRIVEN BY A (p,q)-LAPLACIAN OPERATOR

DIEGO AVERNA[a], GABRIELE BONANNO[b], AND ELISABETTA TORNATORE[c]

[a]Department of Mathematics and Computer Science, University of Palermo,
Via Archirafi, 90123 - Palermo, Italy
diego.averna@unipa.it
[b]Department of Engineering, University of Messina,
c.da Di Dio Sant'Agata, 98166 - Messina, Italy
bonanno@unime.it
[c]Department of Mathematics and Computer Science, University of Palermo,
Via Archirafi, 90123 - Palermo, Italy
elisa.tornatore@unipa.it

Dedicated with great esteem to Professor R. P. Agarwal

ABSTRACT. In this paper, using variational methods and critical point theorems, we prove the existence of multiple weak solutions for a gradient nonlinear Dirichlet elliptic system driven by a (p,q)-Laplacian operator.

AMS (MOS) Subject Classification. 35J35, 35J60, 35J92, 58E30

Keywords: Nonlinear elliptic system, critical point, variational methods, Dirichlet conditions, multiple solutions, p-Laplacian

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a non-empty bounded open set with a smooth boundary $\partial\Omega$. In this paper, we study the following gradient nonlinear elliptic system with Dirichlet conditions

$$(1.1) \qquad \begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda F_u(x,u,v) & \text{in } \Omega, \\ -\Delta_q v + b(x)|v|^{q-2}v = \lambda F_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

where $p, q > 1$, λ is a positive real parameter, by Δ_s we denote the s-Laplacian operator defined $\Delta_s u = \operatorname{div}(|\nabla u|^{s-2}\nabla u)$ for all $u \in W_0^{1,s}(\Omega)$ $(s = p, q)$. In the statement of system (1.1) the reaction term $F : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ is a C^1-function such

Received September 13, 2017

that $F(x, 0, 0) = 0$ for every $x \in \Omega$, F_u, F_v denote the partial derivatives of F with respect to u and v respectively. We suppose, moreover that $a, b \in L^\infty(\Omega)$ and

$$\operatorname*{ess\,inf}_{x\in\Omega} a(x) = a_0 \geq 0, \quad \operatorname*{ess\,inf}_{x\in\Omega} b(x) = b_0 \geq 0. \tag{1.2}$$

In recent years the existence and structure of solutions for problem driven by p-Laplacian have found many interest and different approaches have been developed. The variational methods are used to obtain weak solutions as critical points of a suitable energy function. This approach is employed to deal with systems of gradient type, i.e. the nonlinearities are the gradient of a C^1 functional. We refer the reader to [15] for a complete overview on this subject and [1], [2], [3], [4], [5], [9] [10],[12],[17], [18] and the references therein for more developments. In [5], the authors study the existence of critical points of the energy functional for which these points are the solutions of a quasilinear elliptic system involving (p, q)-Laplacian with $1 < p, q < N$. They consider subcritical growth conditions, and under suitable conditions on the nonlinearity, prove the existence of non-trivial solutions according to various cases: sublinear, superlinear and resonant case. In [10] and [17] the authors get the existence of three solutions for a class of quasilinear elliptic systems involving (p, q)-Laplacian with $p, q > N$. In [1], the authors generalize the results obtained in [17] to systems involving $(p_1, p_2, \ldots, p_n)$-Laplacian. Similar studies, but under different boundary conditions, can be found in [3], [4] (mixed boundary conditions). It is worth noticing that in [17] precise values of parameter λ are not established. In the study of nonlinear elliptic systems in order to obtain non-zero solutions, non-variational approaches have also been used under a different set of assumptions (as for instance the monotony of the nonlinearity) and we refer to [14, 16] and the references therein for further details.

The aim of paper is to determine the existence of multiple solutions as the parameter $\lambda > 0$ varies in an appropriate interval. In this work, without losing generality, we suppose that $1 < q \leq p < N$.

The paper is arranged as follows. First, we obtain the existence of one non-zero weak solution to system (1.1) without assuming any asymptotic condition neither at zero nor at infinity (see Theorem 3.1). Next we prove the existence of at least two non zero weak solutions by using the Ambrosetti-Rabinowitz condition (see Theorem 3.2). Finally, we present an existence result three solutions under an appropriate condition on the nonlinear term F (see Theorem 3.3). Moreover the case in which F is autonomuos is presented and some examples are given.

2. PRELIMINARIES

In this section, we recall definitions and theorems used in the paper.

Let $(X, \|\cdot\|)$ be a real Banach space and Φ, $\Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals and $r \in]-\infty, +\infty]$. We say that functional $I = \Phi - \Psi$

satisfies the *Palais-Smale condition cut off upper at* r (in short $(PS)^{[r]}$-condition) if any sequence $\{u_n\}$ in X such that

(α_1) $\{I(u_n)\}$ is bounded,
(α_2) $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$,
(α_3) $\Phi(u_n) < r \quad \forall n \in \mathbb{N}$,

has a convergent subsequence.

When $r = +\infty$ the previous definition coincides with the classical (PS)-condition, while if $r < \infty$ such condition is more general than the classical one. We refer to [6] for more details.

We say that the functional I satisfies the *weak Palais-Smale condition* ((WPS)-condition) if any bounded sequence $\{u_n\}$ in X such that (α_1) and (α_2) hold, admits a convergent subsequence.

Our main tool is the local minimum theorem obtained in [6]. We recall here its version presented in [7].

Theorem 2.1 ([7, Theorem 2.3])**.** *Let X be a real Banach space, and let Φ, $\Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there exist $\gamma \in \mathbb{R}$ and $\bar{u} \in X$, with $0 < \Phi(\bar{u}) < \gamma$, such that*

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,\gamma[)} \Psi(u)}{\gamma} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})} \tag{2.1}$$

and, for each $\lambda \in \Lambda := \left] \frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{\gamma}{\sup_{u\in\Phi^{-1}(]-\infty,\gamma[)} \Psi(u)} \right[$ the functional $I_\lambda = \Phi - \lambda\Psi$ satisfied $(PS)^{[r]}$-condition.

Then, for each $\lambda \in \Lambda := \left] \frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{\gamma}{\sup_{u\in\Phi^{-1}(]-\infty,\gamma[)} \Psi(u)} \right[$, there is $u_\lambda \in \Phi^{-1}(]0,\gamma[)$ such that $I_\lambda(u_\lambda) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(]0,\gamma[)$ and $I'_\lambda(u_\lambda) = 0$.

Now, we also recall a recent result obtained in [9] that insures the existence of at least two non-zero critical points for differentiable functionals.

Theorem 2.2 (8, Theorem 2.1])**.** *Let X be a real Banach space and let Φ, $\Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there exist $\gamma \in \mathbb{R}$ and $\bar{u} \in X$, with $0 < \Phi(\bar{u}) < \gamma$, such that*

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,\gamma])} \Psi(u)}{\gamma} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})}$$

and for each $\lambda \in \Lambda := \left] \frac{\Phi(\bar{u})}{\Psi(\bar{u})} \frac{\gamma}{\sup_{u\in\Phi^{-1}(]-\infty,\gamma])} \Psi(u)} \right[$, the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies (PS)-condition and it is unbounded from below.

Then, for each $\lambda \in \Lambda := \left] \frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{\gamma}{\sup_{u\in\Phi^{-1}(]-\infty,\gamma])} \Psi(u)} \right[$, the functional I_λ admits two non-zero critical points $u_{\lambda,1}$, $u_{\lambda,2}$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

Finally, we point out an other result which insures the existence of at least three critical points. Theorem 2.3. has been obtained in [6], it is a more precise version of Theorem 3.2 of [8] and Theorem 3.6 of [11].

Theorem 2.3 ([9, Theorem 2.1])**.** *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals with Φ bounded from below and $\Phi(0) = \Psi(0) = 0$.*

Assume that there exist $\gamma \in \mathbb{R}$ and $\bar{u} \in X$, with $0 < \gamma < \Phi(\bar{u})$, such that

(i) $\frac{\sup_{u \in \Phi^{-1}(]-\infty,\gamma])} \Psi(u)}{\gamma} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})}$

(ii) *for each $\lambda \in \Lambda := \left] \frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{\gamma}{\sup_{u \in \Phi^{-1}(]-\infty,\gamma])} \Psi(u)} \right[$ the functional $I_\lambda = \Phi - \lambda\Psi$ is bounded from below and satisfies (PS)-condition.*

Then, for each $\lambda \in \Lambda$, the functional $I_\lambda = \Phi - \lambda\Psi$ has at least three distinct critical points in X.

Throughout in the paper, we suppose that the following condition holds

(H) there exist two non negative constants a_1, a_2 and two constants $s \in [1, \frac{pN}{N-p}[$ and $r \in [1, \frac{qN}{N-q}[$ such that

$$|F_{t_1}(x, t_1, t_2)| \leq a_1 + a_2|t_1|^{s-1}$$

$$|F_{t_2}(x, t_1, t_2)| \leq a_1 + a_2|t_2|^{r-1}$$

for every $(x, t) \in \Omega \times \mathbb{R}^2$.

Clearly, from (H) follows

$$|F(x, t_1, t_2)| \leq a_1(|t_1| + |t_2|) + a_2 \left(\frac{|t_1|^s}{s} + \frac{|t_2|^r}{r} \right) \text{ for every } (x, t) \in \Omega \times \mathbb{R}^2. \tag{2.2}$$

In fact, there exists $0 < \theta < 1$ such that

$$|F(x, t)| = |F(x, t) - F(x, 0)| = |\nabla F(x, \theta t) \cdot t|$$

by using (H), we have

$$\begin{aligned}
|F(x, t)| &\leq \sum_{i=1}^{2} \int_0^1 |F_{t_i}(x, \theta t) t_i| d\theta \\
&\leq \int_0^1 [(a_1 + a_2|\theta t_1|^{s-1})|t_1| + (a_1 + |\theta t_2|^{r-1})|t_2|] d\theta \\
&\leq a_1(|t_1| + |t_2|) + a_2 \left(\frac{|t_1|^s}{s} + \frac{|t_2|^r}{r} \right).
\end{aligned}$$

We consider the Sobolev space $X = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ endowed with the norm

$$\|(u, v)\| := \|u\|_{W_0^{1,p}(\Omega)} + \|v\|_{W_0^{1,q}(\Omega)}$$

for all $(u, v) \in X$, where

$$\|u\|_{W_0^{1,p}(\Omega)} := \left(\int_\Omega (|\nabla u(x)|^p + a(x)|u(x)|^p)dx\right)^{\frac{1}{p}},$$

$$\|v\|_{W_0^{1,q}(\Omega)} := \left(\int_\Omega (|\nabla v(x)|^q + b(x)|v(x)|^q)dx\right)^{\frac{1}{q}},$$

that are, taking into account (1.2), equivalent to the usual one.

A function $(u, v) \in X$ is said a weak solution to system (1.1) if

$$\begin{aligned}\int_\Omega &[|\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla w_1(x)dx + |\nabla v(x)|^{q-2}\nabla v(x) \cdot \nabla w_2(x)]dx \\ &+ \int_\Omega [a(x)|u(x)|^{p-2}u(x)w_1(x) + b(x)|v(x)|^{q-2}v(x)w_2(x)]dx \\ &- \lambda \int_\Omega [F_u(x, u(x), v(x))w_1(x) + F_v(x, u(x), v(x))w_2(x)]dx = 0\end{aligned}$$

for every $(w_1, w_2) \in X$.

Now, consider $1 < h < N$ and put $h^* = \frac{hN}{N-h}$. Denote by Γ the Gamma function defined by

$$\Gamma(s) = \int_0^{+\infty} z^{s-1}e^z dz, \quad \forall s > 0.$$

From the Sobolev embedding theorem, for every $u \in W_0^{1,h}(\Omega)$ there exists a constant $c(N, h) \in \mathbb{R}_+$ such that

$$\|u\|_{L^{h^*}(\Omega)} \le c(N, h)\|u\|_{W_0^{1,h}(\Omega)} \tag{2.3}$$

the best constant that appears in (2.3) is

$$c(N, h) = \pi^{-\frac{1}{2}} N^{-\frac{1}{h}} \left(\frac{h-1}{N-h}\right)^{1-\frac{1}{h}} \left(\frac{\Gamma\left(1+\frac{N}{2}\right)\Gamma(N)}{\Gamma\left(\frac{N}{h}\right)\Gamma\left(1+N-\frac{N}{h}\right)}\right)^{\frac{1}{N}}$$

(see [19]).

Fixing $s \in [1, h^*[$ in virtue of Sobolev embedding theorem, for every $u \in W_0^{1,h}(\Omega)$, there exists a positive constant c_{s,h^*} such that

$$\|u\|_{L^s(\Omega)} \le c_{s,h^*}\|u\|_{W_0^{1,h}(\Omega)} \tag{2.4}$$

and, in virtue of Rellich theorem the embedding is compact.

By using Hölder's inequality, we have

$$c_{s,h^*} \le \mu(\Omega)^{\frac{h^*-s}{h^*s}} c(N, h) \tag{2.5}$$

where $\mu(\Omega)$ denotes the Lebesgue measure of the set Ω. Now, we put

$$c_{1,1} = \max\{c_{1,p^*}, c_{1,q^*}\}, \quad c_{r,s} = \max\{c_{s,p^*}^s, c_{r,q^*}^r\}, \tag{2.6}$$

where the constants s and r are given by (H).

Moreover, let

$$D := \sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega). \tag{2.7}$$

Simple calculations show that there is $x_0 \in \Omega$ such that $B(x_0, D) \subseteq \Omega$.

Finally, we set

$$\kappa = \frac{\pi^{\frac{N}{2}} D^N}{\Gamma(1+\frac{N}{2})}, \tag{2.8}$$

$$\sigma = \frac{2^p}{q} \max\left\{ \frac{1}{D^p}\left(1-\frac{1}{2^N}\right) + \|a\|_\infty,\ \frac{1}{D^q}\left(1-\frac{1}{2^N}\right) + \|b\|_\infty \right\}. \tag{2.9}$$

$$\tau = \frac{2^q}{p} \min\left\{ \frac{1}{D^p}\left(1-\frac{1}{2^N}\right) + a_0,\ \frac{1}{D^q}\left(1-\frac{1}{2^N}\right) + b_0 \right\}. \tag{2.10}$$

In order to study problem (1.1), we will use the functionals $\Phi, \Psi : X \to \mathbb{R}$ defined by putting

$$\Phi(u,v) := \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)} + \frac{1}{q}\|v\|^q_{W_0^{1,q}(\Omega)}, \quad \Psi(u,v) := \int_\Omega F(x, u(x), v(x))dx \tag{2.11}$$

for every $(u,v) \in X$ and put $I_\lambda = \Phi - \lambda\Psi$ for $\lambda > 0$.

Clearly, Φ is a coercive, weakly sequentially lower semicontinuous, continuously Gâteaux differentiable and its derivative at point $(u,v) \in X$ is defined by

$$\begin{aligned}\Phi'(u,v)(w_1,w_2) = &\int_\Omega [|\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla w_1(x) + a(x)|u(x)|^{p-2}u(x)w_1(x)]dx \\ &+ \int_\Omega [|\nabla v(x)|^{q-2}\nabla v(x)\cdot\nabla w_2(x) + b(x)|v(x)|^{q-2}v(x)w_2(x))]dx\end{aligned}$$

for every $(w_1, w_2) \in X$.

Moreover, Ψ is well defined, weakly sequentially upper semicontinuous continuously Gâteaux differentiable with compact derivative and its derivative at point $(u,v) \in X$ is defined by

$$\Psi'(u,v)(w_1,w_2) = \int_\Omega [F_u(x,u(x),v(x))w_1(x) + F_v(x,u(x),v(x))w_2(x)]dx,$$

for every $(w_1, w_2) \in X$.

A critical point for the functional $I_\lambda := \Phi - \lambda\Psi$ is any $(u,v) \in X$ such that

$$\Phi'(u,v)(w_1,w_2) - \lambda\Psi'(u,v)(w_1,w_2) = 0 \quad \forall (w_1,w_2) \in X.$$

Hence, the critical points for functional $I_\lambda := \Phi - \lambda\Psi$ are exactly the weak solutions to system (1.1).

We have the following result

Lemma 2.4. *Fix $\lambda > 0$ the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the (WPS)-condition.*

Proof. Fixed $\lambda > 0$, we claim that the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the (WPS)-condition. For this end, let $\{(u_n, v_n)\}$ be a bounded sequence in X such that $I_\lambda(u_n, v_n)$ is bounded and $I'_\lambda(u_n, v_n)(\omega_1 - u_n, \omega_2 - v_n) \geq -\varepsilon_n \|(\omega_1 - u_n, \omega_2 - v_n)\|$ for all $(\omega_1, \omega_2) \in X$ and where $\varepsilon_n \to 0^+$. Hence, taking a subsequence if necessary, we have

$$
\begin{aligned}
&(u_n, v_n) \rightharpoonup (u, v) \text{ in } X,\\
&u_n \to u \text{ in } L^\alpha(\Omega) \text{ for all } \alpha \in [1, p^*[\\
&v_n \to v \text{ in } L^\beta(\Omega) \text{ for all } \beta \in [1, q^*[
\end{aligned}
$$

From the previous relation, written with $\omega_1 = u$ and $\omega_2 = v$ we infer

$$
\Phi'(u_n, v_n)(u - u_n, v - v_n) - \lambda\Psi'(u_n, v_n)(u - u_n, v - v_n) \geq -\varepsilon_n \|(u - u_n, v - v_n)\|. \tag{2.12}
$$

We observe that

$$
\begin{aligned}
\Phi'(u_n, v_n)(u - u_n, v - v_n) &= -\|u_n\|^p_{W_0^{1,p}(\Omega)} - \|v_n\|^q_{W_0^{1,q}(\Omega)}\\
&+ \int_\Omega [|\nabla u_n(x)|^{p-2} \nabla u_n(x) \cdot \nabla u(x) + a(x)|u_n(x)|^{p-2} u_n(x) u(x)] dx\\
&+ \int_\Omega [|\nabla v_n(x)|^{p-2} \nabla v_n(x) \cdot \nabla u(x) + b(x)|v_n(x)|^{p-2} v_n(x) v(x)] dx
\end{aligned}
$$

and, bearing in mind that for all $a, b \in \mathbb{R}$ and $p > 1$,

$$
|a|^{p-1}|b| \leq \frac{p-1}{p}|a|^p + \frac{1}{p}|b|^p
$$

one has

$$
\begin{aligned}
\Phi'(u_n, v_n)(u - u_n, v - v_n) \leq{}& \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)} + \frac{1}{q}\|v\|^q_{W_0^{1,q}(\Omega)}\\
&- \frac{1}{p}\|u_n\|^p_{W_0^{1,p}(\Omega)} - \frac{1}{q}\|v_n\|^q_{W_0^{1,q}(\Omega)}.
\end{aligned} \tag{2.13}
$$

Moreover, by using (H) we have

$$
\begin{aligned}
|\Psi'(u_n, v_n)(u - u_n, v - v_n)| &\leq a_1 \left(\|u_n - u\|_{L^1(\Omega)} + \|v_n - v\|_{L^1(\Omega)}\right)\\
&+ a_2 \left(\|u_n\|^{s-1}_{L^{p^*}(\Omega)} \|u_n - u\|_{L^\alpha(\Omega)} + \|v_n\|^{r-1}_{L^{q^*}(\Omega)} \|v_n - v\|_{L^\beta(\Omega)}\right)
\end{aligned}
$$

where $\alpha = \frac{p^*}{p^* - s + 1}$ and $\beta = \frac{q^*}{q^* - r + 1}$, hence observing that $\alpha < p^*$ and $\beta < q^*$, we obtain

$$
\lim_{n \to +\infty} \Psi'(u_n, v_n)(u - u_n, v - v_n) = 0. \tag{2.14}
$$

From (2.12) and (2.13) we obtain

$$
\begin{aligned}
-\varepsilon_n \|(u - u_n, v - v_n)\| &+ \frac{1}{p}\|u_n\|^p_{W_0^{1,p}(\Omega)} + \frac{1}{q}\|v_n\|^q_{W_0^{1,q}(\Omega)}\\
&\leq \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)} + \frac{1}{q}\|v\|^q_{W_0^{1,q}(\Omega)} - \lambda\Psi'(u_n, v_n)(u - u_n, v - v_n),
\end{aligned}
$$

from this, taking into account (2.14) we have

$$
\limsup_{n \to +\infty} \left(\frac{1}{p}\|u_n\|^p_{W_0^{1,p}(\Omega)} + \frac{1}{q}\|v_n\|^q_{W_0^{1,q}(\Omega)}\right) \leq \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)} + \frac{1}{q}\|v\|^q_{W_0^{1,q}(\Omega)}
$$

thus, since X is uniformly convex, Proposition III.30 of [13] ensures that $\{(u_n, v_n)\}$ converges to (u, v) in X. Hence our claim is proved. $\square$

3. MAIN RESULTS

By using the notation of Section 2 we have our main results

Theorem 3.1. *We suppose that (H) holds and assume that*

(i_1) $F(x,t) \geq 0$ *for every* $(x,t) \in \Omega \times \mathbb{R}^2_+$ *where* $\mathbb{R}^2_+ = \{t = (t_1, t_2) \in \mathbb{R}^2 \, : \, t_i \geq 0 \quad i = 1, 2\}$;

(i_2) *there exist two positive constants* γ *and* δ *with*

$$\delta^p + \delta^q < \frac{q\gamma}{\kappa\sigma},$$

such that

$$\frac{\inf_{x\in\Omega} F(x,\delta,\delta)}{\delta^p + \delta^q} > \frac{2^N\sigma}{q}\left[a_1 c_{1,1}\left(p^{\frac{1}{p}}\gamma^{\frac{1}{p}-1} + q^{\frac{1}{q}}\gamma^{\frac{1}{q}-1}\right) + a_2 c_{r,s}\left(\frac{p^{\frac{s}{p}}}{s}\gamma^{\frac{s}{p}-1} + \frac{q^{\frac{r}{q}}}{r}\gamma^{\frac{r}{q}-1}\right)\right]$$

where a_1, a_2, s *and* r *are given by (H) and* κ, σ *are given by* (2.8) *and* (2.9).

Then, for each $\lambda \in \left]\frac{2^N\sigma(\delta^p+\delta^q)}{q\inf_{x\in\Omega} F(x,\delta,\delta)}, \frac{1}{a_1 c_{1,1}\left(p^{\frac{1}{p}}\gamma^{\frac{1}{p}-1} + q^{\frac{1}{q}}\gamma^{\frac{1}{q}-1}\right) + a_2 c_{r,s}\left(\frac{p^{\frac{s}{p}}}{s}\gamma^{\frac{s}{p}-1} + \frac{q^{\frac{r}{q}}}{r}\gamma^{\frac{r}{q}-1}\right)}\right[$*, the system* (1.1) *has at least one non-zero weak solution.*

Proof. Our goal is to apply Theorem 2.1. Consider the Sobolev space X and the operators defined in (2.11).

Taking into account (2.2), it follows that

$$\begin{aligned} \Psi(u,v) &= \int_\Omega F(x, u(x), v(x))dx \\ &\leq a_1(\|u\|_{L^1(\Omega)} + \|v\|_{L^1(\Omega)}) + a_2\left(\frac{\|u\|^s_{L^s(\Omega)}}{s} + \frac{\|v\|^r_{L^r(\Omega)}}{r}\right). \end{aligned} \tag{3.1}$$

Let $\gamma \in]0, +\infty[$, then for every $(u, v) \in X$ such that $\Phi(u, v) < \gamma$, by using (2.4) and (2.6) we get

$$\Psi(u,v) \leq a_1 c_{1,1}((p\gamma)^{\frac{1}{p}} + (q\gamma)^{\frac{1}{q}}) + a_2 c_{r,s}\left(\frac{(p\gamma)^{\frac{s}{p}}}{s} + \frac{(q\gamma)^{\frac{r}{q}}}{r}\right). \tag{3.2}$$

Hence, from (3.2), we have

$$\frac{\sup_{(u,v)\in\Phi^{-1}(]-\infty,\gamma[)} \Psi(u,v)}{\gamma} \leq a_1 c_{1,1}\left(p^{\frac{1}{p}}\gamma^{\frac{1}{p}-1} + q^{\frac{1}{q}}\gamma^{\frac{1}{q}-1}\right) + a_2 c_{r,s}\left(\frac{p^{\frac{s}{p}}}{s}\gamma^{\frac{s}{p}-1} + \frac{q^{\frac{r}{q}}}{r}\gamma^{\frac{r}{q}-1}\right), \tag{3.3}$$

for every $\gamma > 0$.

Now, we choose the function $(\bar{u}, \bar{u})$ defined by putting

$$\bar{u}(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, D) \\ \frac{2\delta}{D}\left(D - \sqrt{\sum_{j=1}^{N}(x_j - x_{j0})^2}\right) & \text{if } x \in B(x_0, D) \setminus B(x_0, \frac{D}{2}) \\ \delta & \text{if } x \in B(x_0, \frac{D}{2}) \end{cases} \tag{3.4}$$

Clearly $(\bar{u}, \bar{u}) \in X$ and by using (2.8), (2.9) and (2.10) we have

$$\frac{\delta^p + \delta^q}{p}\kappa\tau < \Phi(\bar{u}, \bar{u}) < \frac{\delta^p + \delta^q}{q}\kappa\sigma. \tag{3.5}$$

In virtue of (3.5) and bearing in mind that $\delta^p + \delta^q < \frac{q\gamma}{\kappa\sigma}$, we obtain

$$0 < \Phi(\bar{u}, \bar{u}) < \gamma$$

and by using (i_1) we have

$$\Psi(\bar{u}, \bar{u}) = \int_\Omega F(x, \bar{u}(x), \bar{u}(x))dx \geq \int_{B(x_0, \frac{D}{2})} F(x, \delta, \delta)dx \geq \frac{k}{2^N} \inf_{x\in\Omega} F(x, \delta, \delta). \tag{3.6}$$

Hence, by (3.5) and (3.6), one has

$$\frac{\Psi(\bar{u}, \bar{u})}{\Phi(\bar{u}, \bar{u})} \geq \frac{q}{2^N\sigma} \frac{\inf_{x\in\Omega} F(x, \delta, \delta)}{\delta^p + \delta^q}. \tag{3.7}$$

By using (3.3), (3.7) and taking into account (i_2), we get

$$\begin{aligned} \frac{\sup_{(u,v)\in\Phi^{-1}(]-\infty,\gamma[)} \Psi(u, v)}{\gamma} &\leq a_1 c_{1,1}\left(p^{\frac{1}{p}}\gamma^{\frac{1}{p}-1} + q^{\frac{1}{q}}\gamma^{\frac{1}{q}-1}\right) + a_2 c_{r,s}\left(\frac{p^{\frac{s}{p}}}{s}\gamma^{\frac{s}{p}-1} + \frac{q^{\frac{r}{q}}}{r}\gamma^{\frac{r}{q}-1}\right) \\ &< \frac{q}{2^N\sigma(\delta^p + \delta^q)} \inf_{x\in\Omega} F(x, \delta, \delta) \\ &\leq \frac{\Psi(\bar{u}, \bar{u})}{\Phi(\bar{u}, \bar{u})}. \end{aligned}$$

Moreover, let be $r_2 > 0$ and $\{(u_n, v_n)\}$ a sequence in X such that (α_3) holds, since Φ is coercive we have that $\{(u_n, v_n)\}$ is bounded. Then by using Lemma 2.1. we obtain that (WPS)-condition implies $(PS)^{[r_2]}$-condition.

Therefore, all the assumptions of Theorem 2.1 are satisfied. So, for each

$$\begin{aligned} \lambda \in &\left] \frac{2^N\sigma(\delta^p + \delta^q)}{q \inf_{x\in\Omega} F(x, \delta, \delta)}, \frac{1}{a_1 c_{1,1}\left(p^{\frac{1}{p}}\gamma^{\frac{1}{p}-1} + q^{\frac{1}{q}}\gamma^{\frac{1}{q}-1}\right) + a_2 c_{r,s}\left(\frac{p^{\frac{s}{p}}}{s}\gamma^{\frac{s}{p}-1} + \frac{q^{\frac{r}{q}}}{r}\gamma^{\frac{r}{q}-1}\right)} \right[\\ \subseteq &\left] \frac{\Phi(\bar{u}, \bar{u})}{\Psi(\bar{u}, \bar{u})}, \frac{\gamma}{\sup_{(u,v)\in\Phi^{-1}(]-\infty,\gamma[)} \Psi(u, v)} \right[\end{aligned}$$

the functional I_λ has at least one non-zero critical point that is weak solution of system (1.1). □

The following result, in which Ambrosetti-Rabinowitz condition is also used, ensures the existence at least two non-zero weak solutions.

Theorem 3.2. *We suppose that (H) holds. Assume that*

(j_1) $F(x,t) \geq 0$ *for every* $(x,t) \in \Omega \times \mathbb{R}^2_+$ *where* $\mathbb{R}^2_+ = \{t = (t_1,t_2) \in \mathbb{R}^2 : t_i \geq 0 \quad i = 1,2\}$*;*

(j_2) *there are two positive constants* γ *and* δ *with*

$$\delta^p + \delta^q < \frac{q\gamma}{\kappa\sigma},$$

such that

$$\frac{\inf_{x\in\Omega} F(x,\delta,\delta)}{\delta^p + \delta^q} > \frac{2^N\sigma}{q}\left[a_1c_{1,1}\left(p^{\frac{1}{p}}\gamma^{\frac{1}{p}-1} + q^{\frac{1}{q}}\gamma^{\frac{1}{q}-1}\right) + a_2c_{r,s}\left(\frac{p^{\frac{s}{p}}}{s}\gamma^{\frac{s}{p}-1} + \frac{q^{\frac{r}{q}}}{r}\gamma^{\frac{r}{q}-1}\right)\right]$$

where a_1*,* a_2*,* s *and* r *are given by (H) and* κ*,* σ *are given by* (2.8) *and* (2.9)*, and that there are two positive constants* $\mu > p$ *and* R *such that*

$$0 < \mu F(x,t) \leq t \cdot \nabla_t F(x,t) \tag{AR}$$

for all $x \in \Omega$ *and* $|t| > R$*.*

Then, for each $\lambda \in \left]\frac{2^N\sigma(\delta^p+\delta^q)}{q\inf_{x\in\Omega} F(x,\delta,\delta)}, \frac{1}{a_1c_{1,1}\left(p^{\frac{1}{p}}\gamma^{\frac{1}{p}-1}+q^{\frac{1}{q}}\gamma^{\frac{1}{q}-1}\right)+a_2c_{r,s}\left(\frac{p^{\frac{s}{p}}}{s}\gamma^{\frac{s}{p}-1}+\frac{q^{\frac{r}{q}}}{r}\gamma^{\frac{r}{q}-1}\right)}\right[$*, the system* (1.1) *has at least two non-zero weak solutions.*

Proof. Our goal is to apply Theorem 2.2. Consider the Sobolev space X and the operators defined in (2.11) taking into account that the regolarity assumptions on Φ and Ψ are satisfied. Arguing as in the proof of Theorem 3.1, put $(\bar{u},\bar{u})$ as in (3.4), by using (i_1), (j_2), (3.5) and bearing in mind that $\delta^p + \delta^q > \frac{q\gamma}{\kappa\sigma}$, we obtain

$$0 < \Phi(\bar{u},\bar{u}) < \gamma$$

and

$$\frac{\sup_{(u,v)\in\Phi^{-1}(]-\infty,\gamma])}\Psi(u,v)}{\gamma} < \frac{\Psi(\bar{u},\bar{u})}{\Phi(\bar{u},\bar{u})}.$$

Fix $\lambda \in \left]\frac{2^N\sigma(\delta^p+\delta^q)}{q\inf_{x\in\Omega} F(x,\delta,\delta)}, \frac{1}{a_1c_{1,1}\left(p^{\frac{1}{p}}\gamma^{\frac{1}{p}-1}+q^{\frac{1}{q}}\gamma^{\frac{1}{q}-1}\right)+a_2c_{r,s}\left(\frac{p^{\frac{s}{p}}}{s}\gamma^{\frac{s}{p}-1}+\frac{q^{\frac{r}{q}}}{r}\gamma^{\frac{r}{q}-1}\right)}\right[$, from (AR), by standard computations, there is a positive constant C such that

$$F(x,t) \geq C|t|^\mu \tag{3.8}$$

$\forall x \in \Omega$, $|t| > R$.

From (3.8) it follows that I_λ is unbounded from below.

Now, by using Lemma 2.4 to verify (PS)-condition it is enough to prove that any sequence of Palais-Smale is bounded. To this end, taking into account (AR) one has

$$
\begin{aligned}
(3.9)\quad & \mu I_\lambda(u_n,v_n) - \|I'_\lambda(u_n,v_n)\|_{X'}\|(u_n,v_n)\| \geq \mu I_\lambda(u_n,v_n) - I'_\lambda(u_n,v_n)(u_n,v_n) \\
&= \mu\Phi(u_n,v_n) - \lambda\mu\Psi(u_n,v_n) - \Phi'(u_n,v_n)(u_n,v_n) + \lambda\Psi'(u_n,v_n)(u_n,v_n) \\
&= \left(\frac{\mu}{p}-1\right)\|u_n\|^p_{W_0^{1,p}(\Omega)} + \left(\frac{\mu}{q}-1\right)\|v_n\|^q_{W_0^{1,q}(\Omega)} - \lambda\int_\Omega (\mu F(x,u_n(x),v_n(x)) \\
&\quad - (F_u(x,u_n(x),v_n(x))u_n(x) + F_v(x,u_n(x),v_n(x))v_n(x)) \\
&\geq \left(\frac{\mu}{p}-1\right)\|u_n\|^p_{W_0^{1,p}(\Omega)} + \left(\frac{\mu}{q}-1\right)\|v_n\|^q_{W_0^{1,q}(\Omega)} + C.
\end{aligned}
$$

where C is a constant.

If $\{(u_n,v_n)\}$ is not bounded from (3.9) we have a contradiction.

Therefore, all conditions of Theorem 2.2 are satisfied, then the system (1.1), for each $\lambda \in \left]\frac{2^N\sigma(\delta^p+\delta^q)}{q\inf_{x\in\Omega}F(x,\delta,\delta)}, \frac{1}{a_1c_{1,1}\left(p^{\frac{1}{p}}\gamma^{\frac{1}{p}-1}+q^{\frac{1}{q}}\gamma^{\frac{1}{q}-1}\right)+a_2c_{r,s}\left(\frac{p^{\frac{s}{p}}}{s}\gamma^{\frac{s}{p}-1}+\frac{q^{\frac{r}{q}}}{r}\gamma^{\frac{r}{q}-1}\right)}\right[$, admits at least two non-zero weak solutions. □

Now, we point out the following result on the existence of at least three weak solutions.

Theorem 3.3. *We suppose that (H) holds and assume that*

(j_1) $F(x,t)\geq 0$ *for every* $(x,t)\in\Omega\times\mathbb{R}^2_+$ *where* $\mathbb{R}^2_+ = \{t=(t_1,t_2)\in\mathbb{R}^2 : t_i\geq 0 \quad i=1,2\}$;

(h_2) *there exist three positive constants* α, β *and* b *with* $\alpha<p$ *and* $\beta<q$ *such that*

$$F(x,t_1,t_2)\leq b(1+|t_1|^\alpha+|t_2|^\beta)$$

for almost every $x\in\Omega$ *and for every* $(t_1,t_2)\in\mathbb{R}^2_+$;

(h_3) *there exist two positive constants* γ *and* δ *with*

$$\delta^p+\delta^q > \frac{p\gamma}{\kappa\tau},$$

such that

$$\frac{\inf_{x\in\Omega}F(x,\delta,\delta)}{\delta^p+\delta^q} > \frac{2^N\sigma}{q}\left[a_1c_{1,1}\left(p^{\frac{1}{p}}\gamma^{\frac{1}{p}-1}+q^{\frac{1}{q}}\gamma^{\frac{1}{q}-1}\right)+a_2c_{r,s}\left(\frac{p^{\frac{s}{p}}}{s}\gamma^{\frac{s}{p}-1}+\frac{q^{\frac{r}{q}}}{r}\gamma^{\frac{r}{q}-1}\right)\right]$$

where a_1, a_2, s *and* r *are given by (H) and* κ, σ *are given by* (2.8) *and* (2.9).

Then, for each $\lambda \in \left]\frac{2^N\sigma(\delta^p+\delta^q)}{q\inf_{x\in\Omega}F(x,\delta,\delta)}, \frac{1}{a_1c_{1,1}\left(p^{\frac{1}{p}}\gamma^{\frac{1}{p}-1}+q^{\frac{1}{q}}\gamma^{\frac{1}{q}-1}\right)+a_2c_{r,s}\left(\frac{p^{\frac{s}{p}}}{s}\gamma^{\frac{s}{p}-1}+\frac{q^{\frac{r}{q}}}{r}\gamma^{\frac{r}{q}-1}\right)}\right[$, *the system* (1.1) *has at least three weak solutions.*

Proof. Our goal is to apply Theorem 2.3. Consider the Sobolev space X and the operators defined in (2.11) taking into account that the regolarity assumptions on Φ and Ψ are satisfied, our aim is to verify (i) and (ii). Arguing as in the proof of Theorem 3.1, put $(\bar{u}, \bar{u})$ as in (3.4), by using (3.5) and bearing in mind that $\delta^p + \delta^q > \frac{p\gamma}{\kappa\tau}$, we obtain

$$\Phi(\bar{u}, \bar{u}) > \gamma > 0.$$

Therefore, the assumption (i) of Theorem 2.3 is satisfied.

We prove that the functional $I_\lambda = \Phi - \lambda\Psi$ is coercive for all positive parameter, in fact by using condition (h_2) we have

$$\begin{aligned} I_\lambda(u,v) &= \Phi(u,v) - \lambda\Psi(u,v) \\ &= \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)} + \frac{1}{q}\|v\|^q_{W_0^{1,q}(\Omega)} - \lambda\int_\Omega F(x,u(x),v(x))dx \\ &\geq \left(\frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)} - \lambda b c_{r,s}^{\frac{\alpha}{s}} \mu(\Omega)^{\frac{p-\alpha}{p}} \|u\|^\alpha_{W_0^{1,p}(\Omega)}\right) \\ &\quad + \left(\frac{1}{q}\|v\|^q_{W_0^{1,q}(\Omega)} - \lambda b c_{r,s}^{\frac{\beta}{r}} \mu(\Omega)^{\frac{q-\beta}{q}} \|v\|^\beta_{W_0^{1,q}(\Omega)}\right) - \lambda b\mu(\Omega). \end{aligned}$$

We observe that the functional $I_\lambda = \Phi - \lambda\Psi$ is bounded from below because it is coercive and weakly sequentially lower semicontinuous.

Now, by using Lemma 2.4 to verify (PS)-condition it is enough to observe that since the functional $I_\lambda = \Phi - \lambda\Psi$ is coercive any sequence of Palais-Smale is bounded. Then also condition (ii) holds. Hence all the assumptions of Theorem 2.3 are satisfied. So, for each $\lambda \in \left]\frac{2^N\sigma(\delta^p+\delta^q)}{q\inf_{x\in\Omega} F(x,\delta,\delta)}, \frac{1}{a_1c_{1,1}\left(p^{\frac{1}{p}}\gamma^{\frac{1}{p}-1}+q^{\frac{1}{q}}\gamma^{\frac{1}{q}-1}\right)+a_2c_{r,s}\left(\frac{p^{\frac{s}{p}}}{s}\gamma^{\frac{s}{p}-1}+\frac{q^{\frac{r}{q}}}{r}\gamma^{\frac{r}{q}-1}\right)}\right[$, the functional I_λ has at least three distinct critical points that are weak solutions of system (1.1). □

Now, we point out the case when F does not depend on $x \in \Omega$, we consider the following system

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda F_u(u,v) & \text{in } \Omega, \\ -\Delta_q v + b(x)|v|^{q-2}v = \lambda F_v(u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.10}$$

we have the following result.

Corollary 3.4. *Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a nonnegative and C^1-function satisfying (H) and assume that*

$$\lim_{t\to 0^+} \frac{F(t,t)}{t^q} = +\infty.$$

Then, there is $\lambda^* > 0$ such that, for each $\lambda \in]0, \lambda^*[$, the problem (3.10) *admits at least one non-zero weak solution.*

Proof. Fix

$$\lambda^* = \frac{1}{a_1 c_{1,1}\left(p^{\frac{1}{p}} + q^{\frac{1}{q}}\right) + a_2 c_{r,s}\left(\frac{p^{\frac{s}{p}}}{s} + \frac{q^{\frac{r}{q}}}{r}\right)}.$$

where the constants a_1, a_2, $c_{1,1}$ and $c_{r,s}$ are given by condition (H) and (2.6).

By using (2.8) and (2.9) and taking into account that

$$\lim_{t\to 0^+} \frac{F(t,t)}{t^q} = +\infty$$

we obtain that for each $\lambda \in]0, \lambda^*[$ there exists $\bar{h} > 0$ such that $\frac{F(t,t)}{t^q} > \frac{2^N\sigma}{q\lambda}$ for each $|t| < \bar{h}$.

Now, consider $0 < \delta < \min\{\bar{h}, (\frac{q}{2\kappa\sigma})^{\frac{1}{q}}\}$ we have

$$\frac{F(\delta,\delta)}{\delta^p + \delta^q} > \frac{2^N\sigma}{q\lambda} > \frac{2^N\sigma}{q\lambda^*}$$

$$\delta^p + \delta^q < \frac{q}{\kappa\sigma}.$$

Then, by choosing $\gamma = 1$ all assumptions of Theorem 3.1 are satisfied and the proof is complete. □

Corollary 3.5. *Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a nonnegative and C^1-function satisfying (H),* (AR) *and assume that*

$$\lim_{t\to 0^+} \frac{F(t,t)}{t^q} = +\infty.$$

Then, there is $\lambda^* > 0$ such that, for each $\lambda \in]0, \lambda^*[$, the problem (3.10) *admits at least two non-zero weak solutions.*

Proof. Fix

$$\lambda^* = \frac{1}{a_1 c_{1,1}\left(p^{\frac{1}{p}} + q^{\frac{1}{q}}\right) + a_2 c_{r,s}\left(\frac{p^{\frac{s}{p}}}{s} + \frac{q^{\frac{r}{q}}}{r}\right)}$$

where the constants a_1, a_2, $c_{1,1}$ and $c_{r,s}$ are given by condition (H) and (2.6).

The conclusion follows arguing as in the proof of Corollary 3.4 taking into account Theorem 3.2. □

Now, we present some examples that illustrate our results.

Example 3.6. Let Ω be an open ball of radius one in $\mathbb{R}^6$.

Consider the function $F : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$F(t_1, t_2) = \log(1 + t_1^2 + t_2^2).$$

We observe that

$$F_{t_1}(t_1, t_2) = \frac{2t_1}{1 + t_1^2 + t_2^2}$$

$$F_{t_2}(t_1, t_2) = \frac{2t_2}{1 + t_1^2 + t_2^2}$$

then, choosing $q = 3$, $p = 4$, $s = r = 2$ $a_1 = 0$ and $a_2 = 2$, the condition (H) holds.

We observe

$$\lim_{t\to 0^+} \frac{F(t,t)}{t^3} = +\infty.$$

Then by using Corollary 3.4, put

$$\lambda^* = 0,46$$

$\forall \lambda \in]0, \lambda^*[$ the following system

$$\begin{cases} -\Delta_4 u + |u|^3 u = \lambda F_u(u,v) & \text{in } \Omega, \\ -\Delta_3 v + |v|^2 v = \lambda F_v(u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

admits at least one non-zero weak solution in $X = W_0^{1,4}(\Omega) \times W_0^{1,3}(\Omega)$.

Example 3.7. Let Ω be an open ball of radius one in $\mathbb{R}^6$.

Consider the function $F : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$F(t_1,t_2) = \begin{cases} \frac{1}{18}(t_1+t_2) + (t_1^4+t_2^4)e^{-\frac{1}{t_1^2+t_2^2}} & (t_1,t_2) \neq (0,0) \\ 0 & (t_1,t_2) = (0,0). \end{cases}$$

We observe that

$$F_{t_i}(t_1,t_2) = \begin{cases} \frac{1}{18} + 2\left(2t_i^2 + \frac{(t_1^4+t_2^4)}{(t_1^2+t_2^2)^2}\right) t_i e^{-\frac{1}{t_1^2+t_2^2}} & (t_1,t_2) \neq (0,0) \\ \frac{1}{18} & (t_1,t_2) = (0,0) \end{cases}$$

then, choosing $p = q = 3$, $r = s = 4$, $a_1 = 3$ and $a_2 = 6$, the condition (H) holds. Moreover, choose $\mu = 4$ and $R = 1$ we have

$$0 < 4F(t_1,t_2) \leq t_1 F_{t_1}(t_1,t_2) + t_2 F_{t_2}(t_1,t_2)$$

for every $(t_1,t_2) \in \mathbb{R}^2$ with $|(t_1,t_2)| > 1$. We observe

$$\lim_{t\to 0^+} \frac{F(t,t)}{t^3} = +\infty.$$

Then by using Corollary 3.5, put $\lambda^* = 0.061$, $\forall \lambda \in]0, \lambda^*[$ the following system

$$\begin{cases} -\Delta_3 u + |u|^2 u = \lambda F_u(u,v) & \text{in } \Omega, \\ -\Delta_3 v + |v|^2 v = \lambda F_v(u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

admits at least two non-zero weak solutions in $X = W_0^{1,3}(\Omega) \times W_0^{1,3}(\Omega)$.

Acknowledgments. The authors are members of the "Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA)" of the "Istituto Nazionale di Alta Matematica (INdAM)".

REFERENCES

[1] G. A. Afrouzi and S. Heidarkhani, Existence of three solutions for a class of Dirichlet quasilinear equations involving $(p_1, \ldots, p_m)$-Laplacian, *Nonlinear Anal.*, 70:135–143, 2009.

[2] R. P. Agarwal, D. O'Regan and K. Perera, Multiple positive solutions of singular problems by variational methods, *Proc. Amer. Math. Soc.*, 134:817–824, 2006.

[3] D. Averna and E. Tornatore, Infinitely many weak solutions for a mixed boundary value system with $(p_1, \ldots, p_m)$-Laplacian, *Electron. J. Qual. Theory Differ. Equ.* 57:1–8, 2014.

[4] D. Averna and E. Tornatore, Ordinary $(p_1, \ldots, p_m)$-Laplacian systems with mixed boundary value conditions, *Nonlinear Anal.: Real World Appl.*, 28:20–31, 2016.

[5] L. Boccardo and D. G. Defigueiredo, Some remarks on a system of quasilinear elliptic equations, *NoDEA Nonlinear Differential Equations Appl.*, 9:309–323, 2002.

[6] G. Bonanno, A critical point theorem via the Ekeland variational principle, *Nonlinear Anal.*, 75:2992–3007, 2012.

[7] G. Bonanno, Relations between the mountain pass theorem and local minima, *Adv. Nonlinear Anal.*, 1:205–220, 2012.

[8] G. Bonanno and P. Candito, Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities, *J. Differential Equations* 244:3031–3059, 2008.

[9] G. Bonanno and G. D'Aguì, Two non-zero solutions for elliptic Dirichlet problems, *Z. Anal. Anwend.* 35:449–464, 2016.

[10] G. Bonanno, S. Heidarkhani and D. O'Regan, Multiple soltions for a class of Dirichlet quasilinear elliptic systems driven by a (p, q)-laplacian operator, *Dynam. Systems Appl.*, 20:89–99, 2011.

[11] G. Bonanno and S. A. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, *Appl. Anal.*, 89:1–10, 2010.

[12] G. Bonanno and E. Tornatore, Existence and multiplicity of solutions for nonlinear elliptic Dirichlet systems, *Electron. J. Differential Equations*, 183:1–11, 2012.

[13] H. Brézis, *Analyse fonctionelle - théorie et applications*, Masson, Paris, 1983.

[14] J. A. Cid and G. Infante, A non-variational approach to the existence of nonzero positive solutions for elliptic systems, preprint.

[15] D. G. Defigueiredo, Nonlinear Elliptic Systems, *An. Acad. Brasil. Ciênc.*, 72:453–469, 2000.

[16] K. Q. Lan and Z. Zhang, Nonzero positive weak solutions of systems of p-Laplace equations, *J. Math. Anal. Appl.,*, 394:581–691, 2012.

[17] C. Li and C.-L. Tang, Three solutions for a class of quasilinear elliptic systems involving the $(p, q)-$Laplacian, *Nonlinear Anal.*, 69:3322–3329, 2008.

[18] K. Perera, R. P. Agarwal and D. O'Regan, Nontrivial solutions of p-superlinear anisotropic p-Laplacian systems via Morse theory, *Topol. Methods Nonlinear Anal.*, 35:367–378, 2010.

[19] G. Talenti, Best constant in Sobolev inequality, *Ann. Mat. Pura Appl.* 110:353–372, 1976.

[20] E. Zeidler, *Nonlinear functional analysis and its applications*, vol. III, Springer, Berlin, 1990.