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INEQUALITIES AND EXPONENTIAL DECAY OF CERTAIN DIFFERENTIAL EQUATIONS OF FIRST ORDER IN TIME VARYING DELAY

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ABSTRACT. In this paper, we give sufficient conditions to guarantee exponential decay of solutions to zero of the time varying delay differential equation of first order. By using the Lyapunov-Krasovskii functional approach, we establish new results on the exponential decay of solutions, which include and improve some related results in the literature.

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1. INTRODUCTION

In this paper we consider the scalar linear differential equation with time varying delay

$$x'(t) = b(t) f(x) - \sum_{i=1}^{n} a_i(t) x(t - h_i(t)) \tag{1.1}$$

where $a_i(t)$, $b(t)$ and $h_i(t)$ are continuous with $0 < h_i(t) \le r_0$ $(i = 1, 2, \ldots, n)$ for positive constant r_0 and the function $t - h_i(t)$ is strictly increasing so that it has an inverse $r(t)$. By using the Lyapunov-Krasovskii functional, we will give sufficient conditions for instability and exponentially stable of the zero solution of (1.1).

There exist many works on the stability, boundedness, asymptotically stability, exponentially stability and unstable. In the literature some of them presented in [1]-[13]. In the present work, we have motivated from [1, 2, 6]. In [6], Cable and Raffoul obtained by using Liapunov functionals sufficient conditions that guarantee exponential decay of solutions to zero of the multi delays differential equations $x'(t) = a(t)x(t) - \sum_{i=1}^{n} b_i(t)x(t-h_i)$ where a, b are continuous with $0 < h_i \le h^*$ for $i = 1, 2, \ldots, n$ for some positive constant h^*. In [1], Adıvar and Raffoul used Liapunov functionals to obtain sufficient conditions that guarentee exponential decay of solutions to zero of the time varying delay differential equation

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$x'(t) = b(t)x(t) - a(t)x(t-h(t))$ where a, b and h are continuous with $0 < h(t) \leq r_0$ for positive constant r_0. In [2], Adıvar and Raffoul used Liapunov functionals to obtain sufficient conditions that ensure exponential stability of the nonlinear Volterra integro differential equation $x'(t) = P(t)x(t) - \int_{t-\tau}^{t} q(t,s)x(s)ds$ where the constant τ is positive, the function p does not need to obey any sign condition and the kernel q is continuous. In addition, authors gived a new criteria for instability. In [5], Burton compared two methods by applying the fixed point theory to some differential equations in which Liapunov's direct methods had ineffecient results in aspect of stability. For more detail we refer [3, 4], [7]–[13].

Let $\psi : [-r_0, 0] \to (-\infty, \infty)$ be a given continuous initial function with

$$\|\psi\| = \max_{-r_0 \leq s \leq 0} |\psi(s)|.$$

and

$$f_1(x) = \begin{cases} \frac{f(x)}{x}, & x \neq 0 \\ f'(0), & x = 0 \end{cases}$$

It should cause no confusion to denote the norm of a continuous function $\varphi : [-r, \infty) \to (-\infty, \infty)$ with

$$\|\varphi\| = \sup_{-r \leq s < \infty} |\varphi(s)|.$$

The notation x_t means that $x_t(\tau) = x(t+\tau)$, $\tau \in [-r_0, 0]$ as long as $x(t+\tau)$ is defined. Thus, x_t is a function mapping an interval $[-r_0, 0]$ into $\mathbb{R}$. We say that $x(t) \equiv x(t, t_0, \psi)$ is a solution of (1.1) if $x(t)$ satisfies (1.1) for $t \geq t_0$ and $x_{t_0}(s) = x(t_0 + s) = \psi(s)$, $s \in [-r_0, 0]$.

Equation (1.1) can be written as the following

$$\begin{aligned} x'(t) &= \left(b(t)f_1(x) - \sum_{i=1}^{n} \frac{a_i(r(t))}{1 - h_i'(r(t))} \right) x(t) \\ &\quad + \frac{d}{dt} \left[\sum_{i=1}^{n} \int_{t-h_i(t)}^{t} \frac{a_i(r(s))}{1 - h_i'(r(s))} x(s)ds \right], \\ &= \left(b(t)f_1(x) - \sum_{i=1}^{n} c_i(t) \right) x(t) \\ &\quad + \frac{d}{dt} \left[\sum_{i=1}^{n} \int_{t-h_i(t)}^{t} c_i(s)\, x(s)ds \right], \end{aligned} \tag{1.2}$$

where

$$c_i(t) = \sum_{i=1}^{n} \frac{a_i(r(t))}{1 - h_i'(r(t))}.$$

2. **Exponential Stability**

For convenience, let $Q(t,x) := b(t)f_1(x) - \sum_{i=1}^{n} c_i(t)$.

Lemma 2.1. *Let*

$$-\frac{\delta}{r_0(\delta+n)} \leq Q(t,x) \leq -r_0\delta \sum_{i=1}^{n} c_i^2(t), \tag{2.1}$$

for $\delta > 0$. If

$$V(t) = \left(x(t) - \sum_{i=1}^{n} \int_{t-h_i(t)}^{t} c_i\,(s)\,x(s)ds \right)^2 + \delta \int_{-r_0}^{0} \int_{t+s}^{t} \sum_{i=1}^{n} c_i^2(z)x^2(z)dzds, \tag{2.2}$$

then, along the solution of (1.1) *we have*

$$V'(t) \leq Q(t,x)V(t).$$

Proof. Because of condition (2.1) it is clear that $Q(t,x) < 0$ for all $t \geq 0$. Let $x(t) = x(t,t_0,\psi)$ be a solution of (1.1). Calculating the time derivative of the functional $V(t)$ along solution $x(t)$ of (1.1) we get

$$\begin{aligned}
V'(t) = {} & 2\left(x(t) - \sum_{i=1}^{n} \int_{t-h_i(t)}^{t} c_i\,(s)\,x(s)ds \right) Q(t,x)x(t) + r_0\delta \sum_{i=1}^{n} c_i^2(t)x^2(t) \\
& - \delta \int_{-r_0}^{0} \sum_{i=1}^{n} c_i^2(t+s)x^2(t+s)ds \\
\leq {} & Q(t,x)\left[x^2(t) - 2x(t) \sum_{i=1}^{n} \int_{t-h_i(t)}^{t} c_i\,(s)\,x(s)ds \right] + r_0\delta \sum_{i=1}^{n} c_i^2(t)x^2(t) \\
& - \delta \int_{-r_0}^{0} \sum_{i=1}^{n} c_i^2(t+s)x^2(t+s)ds + Q(t,x)x^2(t) \\
= {} & Q(t,x)V(t) - Q(t,x)\left(\sum_{i=1}^{n} \int_{t-h_i(t)}^{t} c_i\,(s)\,x(s)ds \right)^2 \\
& - \delta Q(t,x) \int_{-r_0}^{0} \int_{t+s}^{t} \sum_{i=1}^{n} c_i^2(z)x^2(z)dzds \\
& + \left(r_0\delta \sum_{i=1}^{n} c_i^2(t) + Q(t,x) \right) x^2(t) - \delta \int_{-r_0}^{0} \sum_{i=1}^{n} c_i^2(t+s)x^2(t+s)ds.
\end{aligned} \tag{2.3}$$

We can write expressions following to simplify (2.3). Firstly, if we let $u = t+s$, then

$$\int_{-r_0}^{0} \sum_{i=1}^{n} c_i^2(t+s)x^2(t+s)ds = \int_{t-r_0}^{t} \sum_{i=1}^{n} c_i^2(s)x^2(s)ds. \tag{2.4}$$

Secondly, from Hölder's inequality and $2\,|ab| \leq a^2 + b^2$, we obtain

$$\left(\sum_{i=1}^{n} \int_{t-hi(t)}^{t} c_i(s)x(s)ds \right)^2 \leq nr_0 \int_{t-r_0}^{t} \sum_{i=1}^{n} c_i^2(s)x^2(s)ds. \tag{2.5}$$

Lastly, by changing the order of integration we write

$$\int_{-r_0}^{0}\int_{t+s}^{t}\sum_{i=1}^{n}c_i^2(z)x^2(z)dzds \le r_0\int_{t-r_0}^{t}\sum_{i=1}^{n}c_i^2(s)x^2(s)ds \tag{2.6}$$

By using expressions (2.2), (2.4)–(2.6) into (2.3) obtain

$$\begin{aligned} V'(t) &\le Q(t,x)V(t) + \left(r_0\delta\sum_{i=1}^{n}c_i^2(t)+Q(t,x)\right)x^2(t) \\ &\quad + [-(\delta r_0 + nr_0)Q(t,x)-\delta]\int\limits_{t-r_0}^{t}\sum_{i=1}^{n}c_i^2(s)x^2(s)ds \\ &\le Q(t,x)V(t). \end{aligned} \tag{2.7}$$

This completes the proof. □

In the next theorem we will furnish two inequalities; one for $t \ge t_0 + \gamma r_0$ and the other for $t \in [t_0, t_0 + \gamma r_0]$, for $\gamma > 0$.

Theorem 2.2. *Assume that all the conditions of Lemma* 2.1 *holds and let* $1 < \alpha \le 2$. *If*

$$\left(\frac{\alpha-1}{\alpha}\right)r_0 \le h_i\left(t\right) \le r_0, \text{ for all } t \ge 0, \tag{2.8}$$

then any solution $x\left(t\right) = x\left(t, t_0, \psi\right)$ *of* (1.1) *satisfies the exponential inequalities*

$$\left|x\left(t\right)\right| \le \sqrt{2\frac{1+\left(\frac{\alpha-1}{n\alpha}\right)\delta}{\left(\frac{\alpha-1}{n\alpha}\right)\delta}V\left(t_0\right)e^{\frac{1}{2}\int_{t_0}^{t-\left(\frac{\alpha-1}{\alpha}\right)r_0}Q(s,x(s))ds}} \tag{2.9}$$

for $t \ge t_0 + \left(\frac{\alpha-1}{\alpha}\right)r_0$, *and*

$$\left|x\left(t\right)\right| \le \left\|\psi\right\| e^{\int_{t_0}^{t}b(s)f_1(x(s))ds}\left[1+\int_{t_0}^{t}\left|a\left(u\right)\right|e^{-\int_{t_0}^{t}b(s)f_1(x(s))ds}du\right]. \tag{2.10}$$

for $t \in \left[t_0, t_0 + \left(\frac{\alpha-1}{\alpha}\right)r_0\right]$.

Proof. By changing the order of integration we have

$$\begin{aligned} &\int_{-r_0}^{0}\int_{t+s}^{t}\sum_{i=1}^{n}c_i^2(z)x^2\left(z\right)dzds \\ &= \int_{t-r_0}^{t}\int_{-r_0}^{z-t}\sum_{i=1}^{n}c_i^2(z)x^2\left(z\right)dsdz \\ &= \int_{t-r_0}^{t}\sum_{i=1}^{n}c_i^2(z)x^2\left(z\right)\left(z-t+r_0\right)dz. \end{aligned} \tag{2.11}$$

For $1 < \alpha \leq 2$, and if $t - \frac{r_0}{\alpha} \leq z \leq t$, then $\left(\frac{\alpha-1}{\alpha}\right) r_0 \leq z - t + r_0 \leq r_0$. Expression (2.11) yields,

$$\begin{aligned}\int_{-\dot{r}_0}^{0}\int_{t+s}^{t}\sum_{i=1}^{n} c_i^2(z)x^2(z)\,dzds &= \int_{t-r_0}^{t}\sum_{i=1}^{n} c_i^2(z)x^2(z)(z-t+r_0)\,dz \\ &= \int_{t-r_0}^{t-\frac{r_0}{\alpha}}\sum_{i=1}^{n} c_i^2(z)x^2(z)(z-t+r_0)\,dz \\ &\quad + \int_{t-\frac{r_0}{\alpha}}^{t}\sum_{i=1}^{n} c_i^2(z)x^2(z)(z-t+r_0)\,dz \\ &\geq \int_{t-\frac{r_0}{\alpha}}^{t}\sum_{i=1}^{n} c_i^2(z)x^2(z)(z-t+r_0)\,dz \\ &\geq \left(\frac{\alpha-1}{\alpha}\right) r_0 \int_{t-\frac{r_0}{\alpha}}^{t}\sum_{i=1}^{n} c_i^2(z)x^2(z)\,dz. \qquad (2.12)\end{aligned}$$

Function of $V(t)$ can be write as

$$\begin{aligned}V(t) &\geq \delta\int_{-r_0}^{0}\int_{t+s}^{t}\sum_{i=1}^{n} c_i^2(z)x^2(z)\,dzds \\ &\geq \delta\left(\frac{\alpha-1}{\alpha}\right) r_0 \int_{t-\frac{r_0}{\alpha}}^{t}\sum_{i=1}^{n} c_i^2(z)x^2(z)\,dz.\end{aligned}$$

This implies that, for $1 < \alpha \leq 2$, we have $-r_0 + \frac{r_0}{\alpha} \geq -\frac{r_0}{\alpha}$ and hence

$$\begin{aligned}V\left(t-\left(\frac{\alpha-1}{\alpha}\right)r_0\right) &\geq \delta\left(\frac{\alpha-1}{\alpha}\right) r_0 \int_{t-r_0}^{t-r_0+\frac{r_0}{\alpha}}\sum_{i=1}^{n} c_i^2(z)x^2(z)\,dz \\ &\geq \delta\left(\frac{\alpha-1}{\alpha}\right) r_0 \int_{t-r_0}^{t-\frac{r_0}{\alpha}}\sum_{i=1}^{n} c_i^2(z)x^2(z)\,dz. \qquad (2.13)\end{aligned}$$

Note that since $V'(t) \leq 0$ we have for $t \geq t_0 + \left(\frac{\alpha-1}{\alpha}\right) r_0$ that

$$0 \leq V(t) + V\left(t-\left(\frac{\alpha-1}{\alpha}\right)r_0\right) \leq 2V\left(t-\left(\frac{\alpha-1}{\alpha}\right)r_0\right).$$

We note that

$$nr_0\int_{t-r_0}^{t}\sum_{i=1}^{n} c_i^2(z)x^2(z)\,dz \geq \left(\sum_{i=1}^{n}\int_{t-hi(t)}^{t} c_i(s)x(s)ds\right)^2. \qquad (2.14)$$

From inequalities (2.13)–(2.14) we obtain

$$\begin{aligned}V(t) + V\left(t-\left(\frac{\alpha-1}{\alpha}\right)r_0\right) &\geq \left(x(t) - \sum_{i=1}^{n}\int_{t-h_i(t)}^{t} c_i(s)\,x(s)ds\right)^2 \\ &\quad + \delta\int_{-\dot{r}_0}^{0}\int_{t+s}^{t}\sum_{i=1}^{n} c_i^2(z)x^2(z)\,dzds\end{aligned}$$

$$
\begin{aligned}
&+\delta\left(\frac{\alpha-1}{\alpha}\right)r_0\int_{t-r_0}^{t-\frac{r_0}{\alpha}}\sum_{i=1}^{n}c_i^2(z)x^2\left(z\right)dz\\
&\geq\left(x\left(t\right)-\sum_{i=1}^{n}\int_{t-h_i(t)}^{t}c_i\left(s\right)x(s)ds\right)^2\\
&+\delta\left(\frac{\alpha-1}{\alpha}\right)r_0\int_{t-\frac{r_0}{\alpha}}^{t}\sum_{i=1}^{n}c_i^2(z)x^2\left(z\right)dz\\
&+\delta\left(\frac{\alpha-1}{\alpha}\right)r_0\int_{t-r_0}^{t-\frac{r_0}{\alpha}}\sum_{i=1}^{n}c_i^2(z)x^2\left(z\right)dz\\
&=\left(x\left(t\right)-\sum_{i=1}^{n}\int_{t-h_i(t)}^{t}c_i\left(s\right)x(s)ds\right)^2\\
&+\delta\left(\frac{\alpha-1}{\alpha}\right)r_0\int_{t-r_0}^{t}\sum_{i=1}^{n}c_i^2(z)x^2\left(z\right)dz\\
&\geq\left(x\left(t\right)-\sum_{i=1}^{n}\int_{t-h_i(t)}^{t}c_i\left(s\right)x(s)ds\right)^2\\
&+\delta\left(\frac{\alpha-1}{n\alpha}\right)\left(\sum_{i=1}^{n}\int_{t-hi(t)}^{t}c_i(s)x(s)ds\right)^2\\
&=\frac{\left(\frac{\alpha-1}{n\alpha}\right)\delta}{1+\left(\frac{\alpha-1}{n\alpha}\right)\delta}x^2\left(t\right)+\left[\frac{1}{\sqrt{1+\delta\left(\frac{\alpha-1}{n\alpha}\right)}}x\left(t\right)\right.\\
&\left.-\sqrt{1+\delta\left(\frac{\alpha-1}{n\alpha}\right)}\left(\sum_{i=1}^{n}\int_{t-hi(t)}^{t}c_i(s)x(s)ds\right)\right]^2\\
&\geq\frac{\left(\frac{\alpha-1}{n\alpha}\right)\delta}{1+\left(\frac{\alpha-1}{n\alpha}\right)\delta}x^2\left(t\right). \qquad (2.15)
\end{aligned}
$$

In this way, (2.15) shows that

$$
\begin{aligned}
\frac{\left(\frac{\alpha-1}{n\alpha}\right)\delta}{1+\left(\frac{\alpha-1}{n\alpha}\right)\delta}x^2\left(t\right)&\leq V\left(t\right)+V\left(t-\left(\frac{\alpha-1}{\alpha}\right)r_0\right)\\
&\leq 2V\left(t-\left(\frac{\alpha-1}{\alpha}\right)r_0\right). \qquad (2.16)
\end{aligned}
$$

Integrating inequality (2.7) from t_0 to t, we have

$$V\left(t\right)\leq V\left(t_0\right)e^{\int_{t_0}^{t}Q(s,x(s))ds}.$$

As a result of (2.16),

$$V\left(t-\left(\frac{\alpha-1}{\alpha}\right)r_0\right)\leq V\left(t_0\right)e^{\int_{t_0}^{t-\left(\frac{\alpha-1}{\alpha}\right)r_0}Q(s,x(s))ds},$$

and

$$|x(t)| \le \sqrt{2\frac{1+\left(\frac{\alpha-1}{n\alpha}\right)\delta}{\left(\frac{\alpha-1}{n\alpha}\right)\delta}V(t_0)}e^{\frac{1}{2}\int_{t_0}^{t-\left(\frac{\alpha-1}{\alpha}\right)r_0}Q(s,x(s))ds}$$

for $t \ge t_0 + \left(\frac{\alpha-1}{\alpha}\right) r_0$. For $t \in \left[t_0, t_0 + \left(\frac{\alpha-1}{\alpha}\right) r_0\right]$, we observe form (2.8) that

$$\left(\frac{\alpha-1}{\alpha}\right) r_0 - h_i(t)\left(t_0 - \left(\frac{\alpha-1}{\alpha}\right) r_0\right) \le 0. \tag{2.17}$$

Thus for $t \in \left[t_0, t_0 + \left(\frac{\alpha-1}{\alpha}\right) r_0\right]$ and by (2.17), we have $x'(t) = b(t) f(x) - \sum_{i=1}^n a_i(t) \times x(t-h_i(t)) = b(t)f(x) - \sum_{i=1}^n a_i(t)\psi(t)$. Since $\psi(t)$ is the known initial function, we can easily solve for $x(t)$ using the variations of the parameters formula. That is

$$x(t) = e^{\int_{t_0}^t b(s)f_1(x)ds}\left[\psi(t_0) - \int_{t_0}^t \sum_{i=1}^n a_i(u)\psi(u)e^{-\int_{t_0}^t b(s)f_1(x)ds}du\right].$$

Thus for $t \in \left[t_0, t_0 + \left(\frac{\alpha-1}{\alpha}\right) r_0\right]$, the above expression implies

$$|x(t)| \le \|\psi\| e^{\int_{t_0}^t b(s)f_1(x)ds}\left[1 + \int_{t_0}^t \sum_{i=1}^n |a_i(u)| e^{-\int_{t_0}^t b(s)f_1(x)ds}du\right].$$

This completes the proof. □

Remark 2.3. Since the delay $h(t)$ is time varying, condition (2.17) is price we paid to obtain two different inequalities on two different intervals. In the case $h_i(t) = r_0$, where r_0 is constant, then condition (2.17) is automatically satisfied.

Remark 2.4. It follows form (2.1) and inequality (2.9) that

$$\begin{aligned}|x(t)| &\le \sqrt{2\frac{1+\left(\frac{\alpha-1}{n\alpha}\right)\delta}{\left(\frac{\alpha-1}{n\alpha}\right)\delta}V(t_0)}e^{\frac{1}{2}\int_{t_0}^{t-\left(\frac{\alpha-1}{\alpha}\right)r_0}Q(s,x(s))ds} \\ &\le \sqrt{2\frac{1+\left(\frac{\alpha-1}{n\alpha}\right)\delta}{\left(\frac{\alpha-1}{n\alpha}\right)\delta}V(t_0)}e^{-r_0\delta\frac{1}{2}\int_{t_0}^{t-\left(\frac{\alpha-1}{\alpha}\right)r_0}\sum_{i=1}^n c_i^2(s)ds}.\end{aligned}$$

Thus, if $\int_{t_0}^\infty \sum_{i=1}^n c_i^2(s)ds = \infty$, then the zero solution of (1.1) is exponentially stable.

3. A criterion for instability

In this section, we usea non-negative definite Lyapunov functional and obtain a criterion that can be easily applied to test for instability of the zero solution of (1.1).

Lemma 3.1. *Suppose there exists a positive constant $D > nr_0$ such that*

$$Q(t,x) - D\sum_{i=1}^n c_i^2(t) \ge 0. \tag{3.1}$$

If

$$V(t) = \left(x(t) - \sum_{i=1}^n \int_{t-h_i(t)}^t c_i(s)x(s)ds\right)^2 - D\int_{t-r_0}^t \sum_{i=1}^n c_i^2(z)x^2(z)dz, \tag{3.2}$$

then, along the solutions of (1.1) *we have*

$$V'(t) \geq Q(t,x)\,V(t). \tag{3.3}$$

Proof. Because of condition (3.1), it is clear that $Q(t,x) > 0$ for all $t \geq 0$. Let $x(t) = x(t, t_0, \psi)$ be a solution of (1.1). Using inequality (2.5) and calculating the time derivative of the functional $V(t)$ defined by (3.2) along solution $x(t)$ of (1.1) we have

$$\begin{aligned}
V'_{(1)}(t) &= 2\left(x(t) - \sum_{i=1}^{n}\int_{t-h_i(t)}^{t} c_i(s)\,x(s)ds\right)Q(t,x)\,x(t) \\
&\quad - D\sum_{i=1}^{n} c_i^2(t)x^2(t) + D\sum_{i=1}^{n} c_i^2(t-r_0)x^2(t-r_0) \\
&\geq 2\left(x(t) - \sum_{i=1}^{n}\int_{t-h_i(t)}^{t} c_i(s)\,x(s)ds\right)Q(t,x)\,x(t) - D\sum_{i=1}^{n} c_i^2(t)x^2(t) \\
&= Q(t,x)\,V(t) - Q(t,x)\left(\sum_{i=1}^{n}\int_{t-h_i(t)}^{t} c_i(s)\,x(s)ds\right)^2 \\
&\quad + DQ(t,x)\int_{t-r_0}^{t}\sum_{i=1}^{n} c_i^2(z)x^2(z)\,dz + \left(Q(t,x) - D\sum_{i=1}^{n} c_i^2(t)\right)x^2(t) \\
&\geq Q(t,x)\,V(t) + \left(Q(t,x) - D\sum_{i=1}^{n} c_i^2(t)\right)x^2(t) \\
&\quad + Q(t,x)\,(-nr_0 + D)\int_{t-r_0}^{t}\sum_{i=1}^{n} c_i^2(z)x^2(z)\,dz \\
&\geq Q(t,x)\,V(t).
\end{aligned}$$

This completes the proof. ∎

Theorem 3.2. *Assume that all the conditions of Lemma* 3.1 *holds. Then the zero solution of* (1.1) *is unstable, provided that*

$$\int_{t_0}^{\infty}\sum_{i=1}^{n} c_i^2(s)ds = \infty.$$

Proof. An integration of (3.3) from t_0 to t yields

$$V(t) \geq V(t_0)\,e^{\int_{t_0}^{t} Q(s,x(s))ds}. \tag{3.4}$$

Function of $V(t)$ given by (3.2) we can write as

$$\begin{aligned}
V(t) &= x^2(t) - 2x(t)\sum_{i=1}^{n}\int_{t-h_i(t)}^{t} c_i(s)\,x(s)ds \\
&\quad + \left[\sum_{i=1}^{n}\int_{t-h_i(t)}^{t} c_i(s)\,x(s)ds\right]^2 - D\int_{t-r_0}^{t}\sum_{i=1}^{n} c_i^2(z)x^2(z)\,dz.
\end{aligned} \tag{3.5}$$

Let $\beta = D - nr_0$. Then form

$$\left(\frac{\sqrt{nr_0}}{\sqrt{\beta}}a - \frac{\sqrt{\beta}}{\sqrt{nr_0}}b\right)^2 \geq 0,$$

we have

$$2ab \leq \frac{nr_0}{\beta}a^2 + \frac{\beta}{nr_0}b^2.$$

With this in mind we arrive at,

$$\begin{aligned}
-2x(t)\sum_{i=1}^{n}\int_{t-h_i(t)}^{t} c_i(s)\,x(s)ds &\leq 2|x(t)|\left|\sum_{i=1}^{n}\int_{t-h_i(t)}^{t} c_i(s)\,x(s)ds\right| \\
&\leq \frac{nr_0}{\beta}x^2(t) + \frac{\beta}{nr_0}\left[\sum_{i=1}^{n}\int_{t-h_i(t)}^{t} c_i(s)\,x(s)ds\right]^2 \\
&\leq \frac{nr_0}{\beta}x^2(t) + \frac{\beta}{nr_0}nr_0\int_{t-r_0}^{t}\sum_{i=1}^{n} c_i^2(t)x^2(s)\,ds.
\end{aligned}$$

A substitution of the above inequality into (3.5) yields,

$$\begin{aligned}
V(t) &\leq x^2(t) + \frac{nr_0}{\beta}x^2(t) + (\beta + nr_0 - D)\int_{t-r_0}^{t}\sum_{i=1}^{n} c_i^2(t)x^2(s)\,ds \\
&= \frac{\beta + nr_0}{\beta}x^2(t) \\
&= \frac{D}{D - nr_0}x^2(t).
\end{aligned}$$

Using inequalities (3.1) and (3.4), we get

$$\begin{aligned}
|x(t)| &\geq \sqrt{\frac{D - nr_0}{D}}V^{1/2}(t) \\
&= \sqrt{\frac{D - nr_0}{D}}V^{1/2}(t_0)\,e^{\frac{1}{2}\int_{t_0}^{t} Q(s,x(s))ds} \\
&\geq \sqrt{\frac{D - nr_0}{D}}V^{1/2}(t_0)\,e^{\frac{D}{2}\int_{t_0}^{t}\sum_{i=1}^{n} c_i^2(s)ds}.
\end{aligned}$$

This completes the proof. □

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