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SOME RESULTS ON PREDATOR-PREY DYNAMIC SYSTEMS WITH BEDDINGTON-DEANGELIS TYPE FUNCTIONAL RESPONSE ON TIME SCALE CALCULUS

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ABSTRACT. We consider two dimensional predator-prey system with Beddington-DeAngelis type functional response on time scales. For this special case, we try to find under which conditions the system is permanent and globally attractive. This study gives beneficial results for continuous and discrete cases and also for solving open problems related to the dynamical properties of the systems which include the species that have unusual life cycle.

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1. Introduction

The subject of mathematical ecology in biomathematics is the relationship between species and outer environment. Moreover, the connections between different species describe predator-prey dynamic systems. In this type of dynamic sytems, most important issues are global attractivity and permanence. Global attractivity shows the stability of the species in their circumstances and permanence shows whether the considered species are permanent against environmental conditions. The conditions that make different predator-prey dynamic systems permanent and globally attractive were studied in [4], [9] and [18]. Another important notion in predator-prey systems is functional response, which explains the effect of predator on prey and vice versa. Therefore, various types of functional responses such as semi-ratio dependent, Holling-type functional responses have been investigated in several studies like [8], [11], [16], [17], [19].

In this paper, we consider the predator-prey systems with Beddington DeAngelis type functional response. This type of functional response was first considered in [1] and [7] by Beddington and DeAngelis respectively. At low densities, this type

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of functional response can avoid singular behavior of ratio-dependent models. Also predator feeding can be described by this functional response much better over a range of predator-prey abundances. Because of these advantages of the Beddington-DeAngelis type functional response, we have preferred to study on that system. For such kind of systems, boundedness of solution, permanence and global attractivity are important topics for the mathematical analysis which give information about the future of the population of the species.

On the other hand, when the size of the population is rarely small or has non-overlaping generation, then discrete models are more appropriate than continuous ones. Since time scale models unify discrete and continuous models, this type of models becomes more applicable to real life than the others. Many studies about global attractivity and permanence of the predator-prey dynamic systems with Beddington-DeAngelis functional response on continuous and discrete cases have been done. [6] and [14] are some of the examples for continuous case and [9] is the example for discrete case. Additionally, some examples on predator-prey dynamic systems for time scale case are [3], [10] and [15].

What we study in this paper is permanence and global attractivity of the solutions for general time scales case of the predator-prey system with Beddington-DeAngelis type functional response. We have found some conditions for permanence and global attractivity of the considered system. It enables us to make analysis about future of the species.

2. **Preliminaries**

By a time scale, denoted by $\mathbb{T}$, we mean a nonempty closed subset of $\mathbb{R}$. The theory of time scales give a way to unify continuous and discrete analysis. The followings are some important notions about the time scales calculus which are taken from [2] and [13].

The set $\mathbb{T}^\kappa$ is defined by $\mathbb{T}^\kappa := \mathbb{T}/(\rho(\sup \mathbb{T}), \sup \mathbb{T}]$ and the set $\mathbb{T}_\kappa$ is defined by $\mathbb{T}_\kappa := \mathbb{T}/[\inf \mathbb{T}, \sigma(\inf \mathbb{T}))$. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup(-\infty, t)_\mathbb{T}$, for $t \in \mathbb{T}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf(t, \infty)_\mathbb{T}$, for $t \in \mathbb{T}$. Here, it is assumed that $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{T}$, we denote the Δ-derivative of f at $t \in \mathbb{T}^\kappa$ as $f^\Delta(t)$ and it is defined as follows: For all $\epsilon > 0$, there exists a neighborhood $U \subset \mathbb{T}$ of $t \in \mathbb{T}^\kappa$ such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|,$$

for all $s \in U$.

A function $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous if it is continuous at right dense points in $\mathbb{T}$ and its left-sided limits exist at left-dense points in $\mathbb{T}$. The class of real rd-continuous functions defined on a time scale $\mathbb{T}$ is denoted by $C_{rd}(\mathbb{T},\mathbb{R})$. If $f \in C_{rd}(\mathbb{T},\mathbb{R})$, then there exists a function $F(t)$ such that $F^{\Delta}(t) = f(t)$. The delta integral is defined by $\int_a^b f(x)\Delta x = F(b) - F(a)$.

3. **Predator-Prey Dynamic System with Beddington DeAngelis Type Functional Response**

We investigate the following equation:

$$\begin{aligned} x^{\Delta}(t) &= a(t) - b(t)\exp(x(t)) - \frac{c(t)\exp(y(t))}{\alpha(t) + \beta(t)\exp(x(t)) + m(t)\exp(y(t))}, \\ y^{\Delta}(t) &= -d(t) + \frac{f(t)\exp(x(t))}{\alpha(t) + \beta(t)\exp(x(t)) + m(t)\exp(y(t))}. \end{aligned} \tag{3.1}$$

In this sytem;

1. $a(t) - b(t)\exp(x(t))$ is the specific growth rate of the prey in the absense of predator.
2. $d(t)$ is the death rate of predator.
3. $-\frac{c(t)\exp(y(t))}{\alpha(t)+\beta(t)\exp(x(t))+m(t)\exp(y(t))}$, is the Beddington DeAngelis type effect of predator on prey.
4. $\frac{f(t)\exp(x(t))}{\alpha(t)+\beta(t)\exp(x(t))+m(t)\exp(y(t))}$ is the Beddington DeAngelis type effect of prey on predator.

Consider the following system:

$$\begin{aligned} \tilde{x}'(t) &= a(t)\tilde{x}(t) - b(t)\tilde{x}^2(t) - \frac{c(t)\tilde{y}(t)\tilde{x}(t)}{\alpha(t) + \beta(t)\tilde{x}(t) + m(t)\tilde{y}(t)}, \\ \tilde{y}'(t) &= -d(t)\tilde{y}(t) + \frac{f(t)\tilde{x}(t)\tilde{y}(t)}{\alpha(t) + \beta(t)\tilde{x}(t) + m(t)\tilde{y}(t)}. \end{aligned} \tag{3.2}$$

The following information is taken from [14]. Let $\mathbb{T} = \mathbb{R}$, then in (3.1), by taking $\exp(x(t)) = \tilde{x}(t)$ and $\exp(y(t)) = \tilde{y}(t)$, we obtain the equality (3.2), which is the standart predator-prey system with Beddington-DeAngelis functional response governed by ordinary differential equations. Many studies have been done on this system (see [5], [6] and [12]).

Let $\mathbb{T} = \mathbb{Z}$, by using equality (3.1), we get

$$\begin{aligned} x(t+1) - x(t) &= a(t) - b(t)\exp(x(t)) - \frac{c(t)\exp(y(t))}{\alpha(t) + \beta(t)\exp(x(t)) + m(t)\exp(y(t))}, \\ y(t+1) - y(t) &= -d(t) + \frac{f(t)\exp(x(t))}{\alpha(t) + \beta(t)\exp(x(t)) + m(t)\exp(y(t))}. \end{aligned}$$

Here, again by taking $\exp(x(t)) = \tilde{x}(t)$ and $\exp(y(t)) = \tilde{y}(t)$, we obtain

$$\begin{aligned} \tilde{x}(t+1) &= \tilde{x}(t)\exp\left[a(t) - b(t)\tilde{x}(t) - \frac{c(t)\tilde{y}(t)}{\alpha(t) + \beta(t)\tilde{x}(t) + m(t)\tilde{y}(t)}\right], \\ \tilde{y}(t+1) &= \tilde{y}(t)\exp\left[-d(t) + \frac{f(t)\tilde{x}(t)}{\alpha(t) + \beta(t)\tilde{x}(t) + m(t)\tilde{y}(t)}\right], \end{aligned} \tag{3.3}$$

which is the discrete time predator-prey system with Beddington-DeAngelis type functional response and also the discrete analogue of (3.2). This system was studied in [9], [21] and [20]. Since (3.1) incorporates (3.2) and (3.3) as special cases, we call (3.1) the predator-prey dynamic system with Beddington DeAngelis functional response on time scales.

For equation (3.1), $\exp(x(t))$ and $\exp(y(t))$ denote the density of prey and the predator. Therefore $x(t)$ and $y(t)$ could be negative. By taking the exponentials of $x(t)$ and $y(t)$, we obtain the number of preys and predators that are living per unit of an area. In other words, for the general time scale case, our equation is based on the natural logarithm of the density of the predator and prey. Hence, $x(t)$ and $y(t)$ could be negative.

For equations (3.2) and (3.3), since $\exp(x(t)) = \tilde{x}(t)$ and $\exp(y(t)) = \tilde{y}(t)$, the given dynamic systems directly depend on the density of the prey and predator.

4. **Permenance**

Taking $\tilde{x}(t) = \exp(x(t))$ and $\tilde{y}(t) = \exp(y(t))$ in (3.1), then we get

$$\begin{aligned} (\ln(\tilde{x}(t)))^{\Delta} &= a(t) - b(t)\tilde{x}(t) - \frac{c(t)\tilde{y}(t)}{\alpha(t) + \beta(t)\tilde{x}(t) + m(t)\tilde{y}(t)}, \\ (\ln(\tilde{y}(t)))^{\Delta} &= -d(t) + \frac{f(t)\tilde{x}(t)}{\alpha(t) + \beta(t)\tilde{x}(t) + m(t)\tilde{y}(t)}. \end{aligned} \tag{4.1}$$

Assume $a(t)$, $b(t)$, $c(t)$, $d(t)$, $f(t)$, $\beta(t)$, $m(t) > 0$ and $\alpha(t) \geq 0$. Also suppose that these functions are bounded from above. Each of them is from $C_{rd}(\mathbb{T}, \mathbb{R})$ and $x(t), y(t) \in C_{rd}(\mathbb{T}, \mathbb{R})$. Additionally, $\sup_{t\in\mathbb{T}} a(t) = a^u$, $\inf_{t\in\mathbb{T}} a(t) = a^l$. Similar representations are used for supremum and infimum of the other coefficient functions of system (4.1).

Definition 4.1. System (4.1) is called permanent if there exists positive constants r_1, r_2, R_1, and R_2 such that solution $(x(t), y(t))$ of system (4.1) satisfies

$$r_1 \leq \liminf_{t\to\infty} \tilde{x}(t) \leq \limsup_{t\to\infty} \tilde{x}(t) \leq R_1,$$
$$r_2 \leq \liminf_{t\to\infty} \tilde{y}(t) \leq \limsup_{t\to\infty} \tilde{y}(t) \leq R_2.$$

Lemma 4.2. *If* $-\int_0^w d(t)\Delta t + \int_0^w \frac{f(t)}{\beta(t)}\Delta t < 0$, *then for all positive solutions of system* (4.1), $\exp(y(t))$ *tends to* 0 *as* t *tends to infinity in system* (3.1).

Proof. Integrate

$$y^{\Delta}(t) = -d(t) + \frac{f(t)\exp(x(t))}{\alpha(t) + \beta(t)\exp(x(t)) + m(t)\exp(y(t))} \leq -d(t) + \frac{f(t)}{\beta(t)}$$

from 0 to t. Then, we get

$$y(t) \leq y(0) + \int_0^t -d(s) + \frac{f(s)}{\beta(s)}\Delta s.$$

Thus,

$$\exp(y(t)) \leq \exp(y(0)) \exp\left(\int_0^t -d(s) + \frac{f(s)}{\beta(s)}\Delta s\right).$$

From the assumption, we have $\lim_{t\to\infty} \exp\left(\int_0^t -d(s)\frac{f(s)}{\beta(s)}\Delta s\right) = 0$. Hence, $\lim_{t\to\infty} \exp(y(t)) = 0$. $\square$

Remark 4.3. If (4.1) satisfies conditions of Lemma 4.2, then the system can not be permanent by Definition 4.1.

Lemma 4.4. *If conditions for the coefficient functions of system* (4.1) *are satisfied, then* $\tilde{x}(t) \leq \frac{a^u}{b^l}\exp(\mu a^u) := G_1$. *In addition to conditions on the coefficient functions of system* (4.1) *if* $-d^l + \frac{f^u}{\beta^l} \geq 0$ *is satisfied, we have the following*

$$\tilde{y}(t) \leq \frac{f^u G_1}{d^l m^l} \exp\left(\mu\Big(-d^l + \frac{f^u}{\beta^l}\Big)\right) := G_2,$$

where $\mu = \max_{t\in\mathbb{T}} \mu(t)$.

Proof. Let us start with the first equation of (4.1),

$$\begin{aligned}(\ln(\tilde{x}(t)))^\Delta &= a(t) - b(t)\tilde{x}(t) - \frac{c(t)\tilde{y}(t)}{\alpha(t)+\beta(t)\tilde{x}(t)+m(t)\tilde{y}(t)} \\ &\leq a(t) - b(t)\tilde{x}(t) \leq a^u - b^l\tilde{x}(t).\end{aligned} \tag{4.2}$$

Set $M_1 := \frac{a^u}{b^l}(k+1)$, where $0 < k < \exp\{\mu a^u\} - 1$. If $\tilde{x}(t)$ is not oscillatory about M_1, there exists $T_1 > 0$ such that $\tilde{x}(t) > M_1$ for $t > T_1$ or $\tilde{x}(t) < M_1$ for $t > T_1$.

If $\tilde{x}(t) < M_1$ for $t > T_1$, then $\tilde{x}(t) \leq \frac{a^u}{b^l}\exp\{\mu a^u\}$. If $\tilde{x}(t) > M_1$ for $t > T_1$, then $(\ln(\tilde{x}(t)))^\Delta \leq -ka^u$. Hence, there exists $T_2 = T_1 + \tau$ such that for $t > T_2$, $\tilde{x}(t) < M_1$, which is a contradiction.

If $\tilde{x}(t)$ is oscillatory about M_1 for $t > T_1$ and $\sigma(\tilde{t})$ be an arbitrary local maximum of $\ln(\tilde{x}(t))$, then

$$0 \leq (\ln(\tilde{x}(\tilde{t})))^\Delta = a(\tilde{t}) - b(\tilde{t})\tilde{x}(\tilde{t}) - \frac{c(\tilde{t})\tilde{y}(\tilde{t})}{\alpha(\tilde{t})+\beta(\tilde{t})\tilde{x}(\tilde{t})+m(\tilde{t})\tilde{y}(\tilde{t})} \leq a(\tilde{t}) - b(t)\tilde{x}(\tilde{t}) \leq a^u.$$

Therefore $\tilde{x}(\tilde{t}) \leq \frac{a(\tilde{t})}{b(\tilde{t})}$. If $\tilde{t}$ is right dense, then $\tilde{x}(\sigma(\tilde{t})) \leq \frac{a(\tilde{t})}{b(\tilde{t})}$. If $\tilde{t}$ is right scattered, by integrating first equation of (4.1) from $\tilde{t}$ to $\sigma(\tilde{t})$ and using (4.2), we obtain

$$\begin{aligned}\int_{\tilde{t}}^{\sigma(\tilde{t})} (\ln(\tilde{x}(s)))^\Delta \Delta s &= \int_{\tilde{t}}^{\sigma(\tilde{t})} a(s) - b(s)\tilde{x}(s) - \frac{c(s)\tilde{y}(s)}{\alpha(s)+\beta(s)\tilde{x}(s)+m(s)\tilde{y}(s)}\Delta s \\ &\leq \mu a^u\end{aligned}$$

and

$$\tilde{x}(\sigma(\tilde{t}))) \leq \frac{a^u}{b^l}\exp(\mu a^u) = G_1. \tag{4.3}$$

Since $\sigma(\tilde{t})$ be an arbitrary local maximum of $\ln(\tilde{x}(t))$, then $\limsup_{t\to\infty}\tilde{x}(t) \leq G_1$. Hence, $\limsup_{t\to\infty} x(t) \leq R_1$.

Consider the second equation of (4.1), we get

$$(\ln(\tilde{y}(t)))^{\Delta} = -d(t) + \frac{f(t)\tilde{x}(t)}{\alpha(t) + \beta(t)\tilde{x}(t) + m(t)\tilde{y}(t)} \leq -d(t) + \frac{f(t)}{\beta(t)} \leq -d^l + \frac{f^u}{\beta^l}. \tag{4.4}$$

Define $M_2 := \frac{f^u G_1}{d^l m^l}(k+1)$, where $0 < k < \exp\left(\mu(-d^l + \frac{f^u}{\beta^l})\right) - 1$. If $\tilde{y}(t)$ is not oscillatory about M_2, there exists $T_3 > 0$ such that $\tilde{y}(t) > M_2$ for $t > T_3$ or $\tilde{y}(t) < M_2$ for $t > T_3$. If $\tilde{y}(t) < M_2$ for $t > T_3$, then $\tilde{y}(t) < \frac{f^u G_1}{d^l m^l}(k+1)$. If $\tilde{y}(t) > M_2$ for $t > T_3$, then $(\ln(\tilde{y}(t)))^{\Delta} \leq -k\frac{f^u G_1}{\alpha^u + \beta^u G_1 + m^u M_2}$. Hence, there exists $T_4 = T_3 + \tau$ such that for $t > T_4$, $\tilde{y}(t) < M_2$, which is a contradiction.

If $\tilde{y}(t)$ is oscillatory about M_2 for $t > T_3$, let $\sigma(\hat{t})$ be an arbitrary local maximum of $\ln(\tilde{y}(t))$, then by using second equation of (4.1), we can conclude that

$$\begin{aligned} 0 \leq (\ln(\tilde{y}(\hat{t})))^{\Delta} &= -d(\hat{t}) + \frac{f(\hat{t})\tilde{x}(\hat{t})}{\alpha(\hat{t}) + \beta(\hat{t})\tilde{x}(\hat{t}) + m(\hat{t})\tilde{y}(\hat{t})} \\ &= \frac{-d(\hat{t})\alpha(\hat{t}) - d(\hat{t})\beta(\hat{t})\tilde{x}(\hat{t}) - d(\hat{t})m(\hat{t})\tilde{y}(\hat{t}) + f(\hat{t})\tilde{x}(\hat{t})}{\alpha(\hat{t}) + \beta(\hat{t})\tilde{x}(\hat{t}) + m(\hat{t})\tilde{y}(\hat{t})} \\ &\leq \frac{-d(\hat{t})m(\hat{t})\tilde{y}(\hat{t}) + f(\hat{t})G_1}{\beta(\hat{t})\tilde{x}(\hat{t})}. \end{aligned}$$

Therefore, $\tilde{y}(\hat{t}) \leq \frac{f(\hat{t})G_1}{d(\hat{t})m(\hat{t})}$. If $\hat{t}$ is right dense, then $\tilde{y}(\sigma(\hat{t})) \leq \frac{f(\hat{t})G_1}{d(\hat{t})m(\hat{t})}$.

If $\hat{t}$ is right scattered, integrate (4.4) from $\hat{t}$ to $\sigma(\hat{t})$ for the same w above and we obtain

$$\begin{aligned} \int_{\hat{t}}^{\sigma(\hat{t})} (\ln(\tilde{y}(s)))^{\Delta}\Delta s &= \int_{\hat{t}}^{\sigma(\hat{t})} -d(s) + \frac{f(s)\tilde{x}(s)}{\alpha(s) + \beta(s)\tilde{x}(s) + m(s)\tilde{y}(s)}\Delta s \\ &\leq \mu(-d^l + \frac{f^u}{\beta^l}). \end{aligned}$$

$$\tilde{y}(\sigma(\hat{t}))) \leq \frac{f^u G_1}{d^l m^l}\exp\left(\mu\left(-d^l + \frac{f^u}{\beta^l}\right)\right) = G_2. \tag{4.5}$$

Since $\sigma(\hat{t})$ be an arbitrary local maximum of $\ln(\tilde{y}(t))$, then $\limsup_{t\to\infty}\tilde{y}(t) \leq G_2$. Hence $\limsup_{t\to\infty} y(t) \leq R_2$. □

Remark 4.5. For all solutions of system (3.1), if $\exp(y(t))$ does not tend to 0 as t tends to infinity, Lemma 4.4 follows from Lemma 4.2.

Lemma 4.6. *For (4.1) when $\tilde{x}(t) \leq G_1$, $a^l - b^u G_1 - \frac{c^u}{m^l} \leq 0$ and $a^l - \frac{c^u}{m^l} \geq 0$ are satisfied, then*

$$\tilde{x}(t) \geq \frac{1}{b^u}\left(a^l - \frac{c^u}{m^l}\right)\exp\left(\mu\left(a^l - b^u G_1 - \frac{c^u}{m^l}\right)\right) := \tilde{g}_1$$

and when $\tilde{y}(t) \le G_2$, $-d^u\alpha^u - (d^u\beta^u - f^l)\tilde{g}_1 \ge 0$ *and* $-d^u + \frac{f^l\tilde{g}_1}{\alpha^u+\beta^u\tilde{g}_1+m^uG_2} \le 0$ *are satisfied, then*

$$\tilde{y}(t) \ge \frac{1}{d^u m^u}(-d^u\alpha^u - (d^u\beta^u - f^l)\tilde{g}_1)\exp\left(\mu\left(-d^u + \frac{f^l\tilde{g}_1}{\alpha^u+\beta^u\tilde{g}_1+m^uG_2}\right)\right),$$

where $\mu = \max_{t\in\mathbb{T}}\mu(t)$*.*

Proof. Consider the first equation of (4.1) and we obtain

$$\begin{aligned}(\ln(\tilde{x}(t)))^\Delta &= a(t) - b(t)\tilde{x}(t) - \frac{c(t)\tilde{y}(t)}{\alpha(t)+\beta(t)\tilde{x}(t)+m(t)\tilde{y}(t)}\\ &\ge a(t) - b(t)\tilde{x}(t) - \frac{c(t)}{m(t)}\\ &\ge a^l - b^u\tilde{x}(t) - \tfrac{c^u}{m^l}\\ &\ge a^l - b^uG_1 - \tfrac{c^u}{m^l}.\end{aligned} \tag{4.6}$$

If $a^l - b^uG_1 - \frac{c^u}{m^l} > 0$, then there exists $\tilde{T}$ such that $t > \tilde{T}$, $\tilde{x}(t) > G_1$, for $t > \tilde{T}$. So, there is a contradiction. Therefore $a^l - b^uG_1 - \frac{c^u}{m^l} \le 0$.

Take $N_1 = \frac{1}{b^u}(a^l - \frac{c^u}{m^l})(1-\tilde{q})$, where $\tilde{q} = 1 - \exp\left(w\left(a^l - \frac{c^u}{m^l} - b^uG_1\right)\right)$. Suppose that $\tilde{x}(t)$ is not oscillatory around N_1. Then there exists T_5, such that $\tilde{x}(t) > N_1$ for $t > T_5$ or $\tilde{x}(t) < N_1$ for $t > T_5$. If $\tilde{x}(t) > N_1$ for $t > T_5$, then $\tilde{x}(t)$ satisfies the desired result. Since $\tilde{x} \ge 0$, then the condition $a^l - \frac{c^u}{m^l} \ge 0$ must be satisfied. If $\tilde{x}(t) < N_1$ for $t > T_5$, then $(\ln(\tilde{x}(t)))^\Delta \ge (a^l - \frac{c^u}{m^l})\tilde{q}$. Since $(a^l - \frac{c^u}{m^l})\tilde{q} > 0$, there exists T_6 such that for $t > T_6$, we have $\tilde{x}(t) > N_1$ which is a contradiction. Suppose that $\tilde{x}(t)$ is oscillatory around N_1 and $\sigma(t_1)$ be an arbitrary local minimum of $\ln(\tilde{x}(t))$, thus

$$\begin{aligned}0 \ge (\ln(\tilde{x}(t_1)))^\Delta &= a(t_1) - b(t_1)\tilde{x}(t_1) - \frac{c(t_1)\tilde{y}(t_1)}{\alpha(t_1)+\beta(t_1)\tilde{x}(t_1)+m(t_1)\tilde{y}(t_1)}\\ &\ge a(t_1) - b(t_1)\tilde{x}(t_1) - \frac{c(t_1)}{m(t_1)}.\end{aligned}$$

Thus, we have,

$$\frac{1}{b^u}\left(a^l - \frac{c^u}{m^l}\right) \le \frac{1}{b(t_1)}\left(a(t_1) - \frac{c(t_1)}{m(t_1)}\right) \le \tilde{x}(t_1).$$

If t_1 is right dense, then $\tilde{x}(\sigma(t_1)) \ge \frac{1}{b^u}(a^l - \frac{c^u}{m^l})$. If t_1 is right scattered, the following is obtained by integrating (4.6) from t_1 to $\sigma(t_1)$

$$\begin{aligned}\int_{t_1}^{\sigma(t_1)}(\ln(\tilde{x}(t)))^\Delta\Delta t &= \int_{t_1}^{\sigma(t_1)} a(t) - b(t)\tilde{x}(t) - \frac{c(t)\tilde{y}(t)}{\alpha(t)+\beta(t)\tilde{x}(t)+m(t)\tilde{y}(t)}\Delta t\\ &\quad + \ln\prod_{i=1}^{q}(1+g_i)\\ &\ge \mu\left(a(t_1) - b(t_1)G_1 - \frac{c(t_1)}{m(t_1)}\Delta t\right),\end{aligned}$$

$$\tilde{x}(\sigma(t_1)) \geq \frac{1}{b^u}(a^l - \frac{c^u}{m^l}) \exp\left(\mu\left(a^l - b^u G_1 - \frac{c^u}{m^l}\right)\right) = \tilde{g}_1.$$

Since $\tilde{x}(\sigma(t_1))$ is the arbitrary local minimum, then $\liminf_{t\to\infty} \tilde{x}(t) \leq \tilde{g}_1$, i.e. $\limsup_{t\to\infty} x(t) \leq r_1$.

Considering the second equation of (4.1), we have

$$(4.7)\quad (\ln(\tilde{y}(t)))^{\Delta} = -d(t) + \frac{f(t)\tilde{x}(t)}{\alpha(t)+\beta(t)\tilde{x}(t)+m(t)\tilde{y}(t)} \geq -d^u + \frac{f^l\tilde{g}_1}{\alpha^u+\beta^u\tilde{g}_1+m^uG_2}.$$

If $-d^u + \frac{f^l\tilde{g}_1}{\alpha^u+\beta^u\tilde{g}_1+m^uG_2} > 0$, then there exits T_7 such that for $t > T_7$, one can see $\tilde{y}(t) > G_2$ which is a contradiction. Thus $-d^u + \frac{f^l\tilde{g}_1}{\alpha^u+\beta^u\tilde{g}_1+m^uG_2} \leq 0$.

Let us take N_2 such that $N_2 = \frac{1}{d^u m^u}(-d^u\alpha^u - d^u\beta^u\tilde{g}_1 + f^l\tilde{g}_1)(1-r)$ where $r = 1 - \exp\left(\mu\left(-d^u + \frac{f^l\tilde{g}_1}{\alpha^u+\beta^u\tilde{g}_1+m^uG_2}\right)\right)$. Assume $\tilde{y}(t)$ is not oscillatory around N_2. Then, there exists T_8, such that for $t > T_8$ $\tilde{y}(t) > N_2$ or $\tilde{y}(t) < N_2$.

For the first case,

$$\tilde{y}(t) > \frac{1}{d^u m^u}(-d^u\alpha^u - d^u\beta^u\tilde{g}_1 + f^l\tilde{g}_1)\exp\left(\mu\left(-d^u + \frac{f^l\tilde{g}_1}{\alpha^u+\beta^u\tilde{g}_1+m^uG_2}\right)\right).$$

Since $\tilde{y} \geq 0$, then $-d^u\alpha^u - d^u\beta^u\tilde{g}_1 + f^l\tilde{g}_1 \geq 0$.

For the second case

$$(\ln(\tilde{y}(t)))^{\Delta} > \left(\frac{1}{d^u m^u(\alpha^u+\beta^u\tilde{g}_1+m^uG_2)}(-d^u\alpha^u - d^u\beta^u\tilde{g}_1 + f^l\tilde{g}_1)\right) r.$$

Since $-d^u\alpha^u - d^u\beta^u\tilde{g}_1 + f^l\tilde{g}_1 > 0$, there exists T_9, such that $\tilde{y}(t) > N_2$ for $t > T_9$, which is a contradiction.

Assume $\tilde{y}(t)$ is oscillatory around N_2 and $\sigma(t_2)$ be an arbitrary local minumum of $\ln(\tilde{y}(t))$, then by (4.7) we have

$$\begin{aligned}0 \geq (\ln(\tilde{y}(t_2))^{\Delta} &= \frac{-d(t_2)\alpha(t_2) - d(t_2)\beta(t_2)\tilde{x}(t_2) - d(t_2)m(t_2)\tilde{y}(t_2) + f(t_2)\tilde{x}(t_2)}{\alpha(t_2)+\beta(t_2)\tilde{x}(t_2)+m(t_2)\tilde{y}(t_2)}\\ &\geq \frac{-d(t_2)\alpha(t_2) - d(t_2)\beta(t_2)\tilde{g}_1 - d(t_2)m(t_2)\tilde{y}(t_2) + f(t_2)\tilde{g}_1}{\alpha(t_2)+\beta(t_2)\tilde{g}_1+m(t_2)\tilde{y}(t_2)}.\end{aligned}$$

So, we get $\tilde{y}(t_2) \geq \frac{1}{d(t_2)m(t_2)}(-d(t_2)\alpha(t_2) - d(t_2)\beta(t_2)\tilde{g}_1 + f(t_2)\tilde{g}_1)$. Thus, $\tilde{y}(t_2) \geq \frac{1}{d^u m^u}(-d^u\alpha^u - d^u\beta^u\tilde{g}_1 + f^l\tilde{g}_1)$.

If t_2 is right dense, we have $\tilde{y}(\sigma(t_2)) \geq \frac{1}{d^u m^u}(-d^u\alpha^u - d^u\beta^u\tilde{g}_1 + f^l\tilde{g}_1)$. If t_2 is right scattered, by integrating (4.7) from t_2 to $\sigma(t_2)$ we obtain

$$\begin{aligned}\int_{t_2}^{\sigma(t_2)} (\ln(\tilde{y}(t)))^{\Delta}\Delta t &= \int_{t_2}^{\sigma(t_2)} -d(t) + \frac{f(t)\tilde{x}(t)}{\alpha(t)+\beta(t)\tilde{x}(t)+m(t)\tilde{y}(t)}\Delta t\\ &\geq \mu\left(-d^u + \frac{f^l\tilde{g}_1}{\alpha^u+\beta^u G_1+m^uG_2}\right).\end{aligned}$$

By the above inequality, we have

$$\tilde{y}(\sigma(t_2)) \geq \frac{1}{d^u m^u}(-d^u\alpha^u - d^u\beta^u\tilde{g}_1 + f^l\tilde{g}_1)\exp\left(\mu\left(-d^u + \frac{f^l\tilde{g}_1}{\alpha^u + \beta^u\tilde{g}_1 + m^u G_2}\right)\right).$$

Since $\tilde{y}(\sigma(t_2))$ is the arbitrary local minimum, then $\liminf_{t\to\infty}\tilde{y}(t) \leq \tilde{g}_2$, i.e. $\limsup_{t\to\infty} y(t) \leq r_2$. □

If (3.1) satisfies all the conditions of Lemma 4.4 and Lemma 4.6, then solution is permanent.

Example 4.7. $\mathbb{T} = [2k, 2k+1]$, $k \in \mathbb{N}$ k start with 0.

$$x^{\Delta}(t) = (2 - \frac{1}{2t+2}) - \exp(x) - \frac{0.5\exp(y)}{\exp(x)+\exp(y)},$$

$$y^{\Delta}(t) = -1 + \frac{(3+\frac{1}{t+1})\exp(x)}{\exp(x)+\exp(y)}.$$

Example 4.7 satisfies all conditions of Lemma 4.4 and Lemma 4.6, therefore, the solution is permanent.

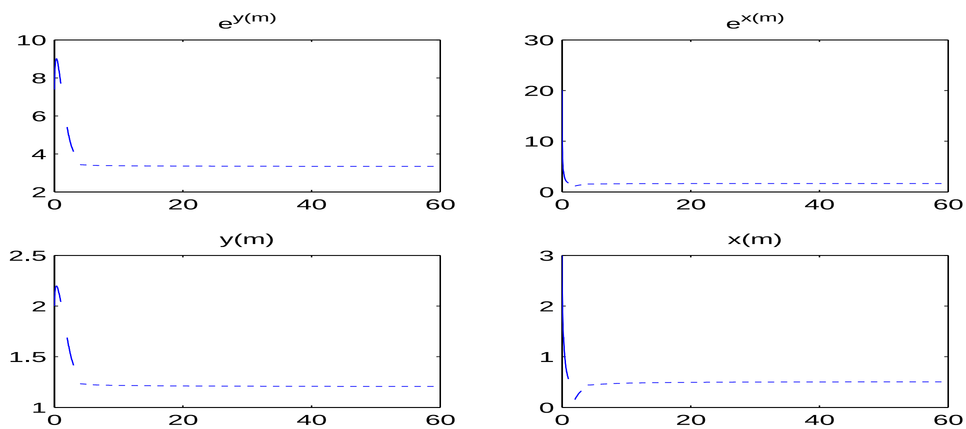

FIGURE 1. Numeric solutions of Example 4.7 show the permanence.

5. Global Attractivity

Definition 5.1. A positive solution $(x*(t), y*(t))$ of (4.1) is said to be globally attractive if any other positive solution $(x(t), y(t))$ of (4.1) satifies $\lim_{t\to\infty}|x^*(t) - x(t)| = 0$, $\lim_{t\to\infty}|y^*(t) - y(t)| = 0$.

Theorem 5.2. *In addition to conditions of Lemma* 4.4 *and Lemma* 4.6 *if* $a_1, a_2 \in (0,1)$, $\delta > 0$ *and*

$$\Bigg(a_1\min\left\{b^l, \frac{2}{G_1\mu^u} - b^u\right\} - \Bigg[a_1\frac{c^u(\beta^u)^{1/3}G_2^{1/3}}{9(\alpha^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{1/3}} + a_2\frac{f^u(\alpha^u)^{1/3}}{9(\beta^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{2/3}}$$

$$+a_2\frac{f^u(m^u)^{1/3}G_2^{2/3}}{9(\alpha^l)^{2/3}(\beta^l)^{2/3}g_1^{2/3}g_2^{1/3}}\Bigg], a_2\min\left\{\frac{f^lg_1m^l}{A^uB^u},\frac{2}{G_2\mu^u}-\frac{f^uG_1m^u}{A^lB^l}\right\}$$
$$-\left[a_1\frac{c^u(\alpha^u)^{1/3}}{9(\beta^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{2/3}}+a_1\frac{c^u(\beta^u)^{1/3}(t)G_1^{2/3}}{9(\alpha^l)^{2/3}(m^l)^{2/3}g_1^{1/3}g_2^{2/3}}\right]\Bigg)>\delta,$$

then system (4.1) *is globally attractive.*

Proof. For any positive solutions $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ of system (3.1), it follows from Lemma 4.4 and Lemma 4.6 that $\liminf_{t\to\infty} x_i(t) < g_1$, $\limsup_{t\to\infty} x_i(t) < G_1$, $\liminf_{t\to\infty} y_i(t) < g_2$ and $\limsup_{t\to\infty} x_i(t) < G_2$ for $i = 1, 2$.

Let $V_1(t) = |\ln x_1(t) - \ln x_2(t)|$, $A = \alpha + \beta x_1(t) + m(t)y_1(t)$, $B = \alpha + \beta x_2(t) + m(t)y_2(t)$. If t is right dense,

$$\begin{aligned}
V_1^{\Delta}(t) &= \frac{V_1(\sigma(t))-V_1(t)}{\mu(t)} = \frac{|\ln x_1(\sigma(t))-\ln x_2(\sigma(t))|-|\ln x_1(t)-\ln x_2(t)|}{\mu(t)}\\
&\le \frac{1}{\mu(t)}|\ln x_1(t)-\ln x_2(t)-\mu(t)b(t)[x_1(t)-x_2(t)]|\\
&\quad -\frac{1}{\mu(t)}|\ln x_1(t)-\ln x_2(t)| + c(t)\left|\frac{\beta(t)y_1(t)[x_1(t)-x_2(t)]}{AB}\right|\\
&\quad + c(t)\left|\frac{\alpha(t)[y_1(t)-y_2(t)]}{AB}\right| + c(t)\left|\frac{\beta(t)x_1(t)[y_1(t)-y_2(t)]}{AB}\right|.
\end{aligned}$$

By using mean value theorem, we have

$$x_1(t)-x_2(t) = \exp(\ln x_1(t)) - \exp(\ln x_2(t)) = \xi(t)(\ln x_1(t)-\ln x_2(t)) \tag{5.1}$$

where $\xi(t)$ is between $x_1(t)$ and $x_2(t)$. If t is right scattered, using Young's inequality and (5.1), we obtain,

$$\begin{aligned}
V_1^{\Delta}(t) \le &-\frac{1}{\mu(t)}\left[\frac{1}{\xi(t)}-\left|\frac{1}{\xi(t)}-\mu(t)b(t)\right|\right]|x_1(t)-x_2(t)|\\
&+\left|\frac{c(t)\beta^{1/3}(t)y_1^{1/3}(t)[x_1(t)-x_2(t)]}{9\alpha^{2/3}(t)m^{2/3}(t)x_1^{1/3}(t)x_2^{1/3}(t)y_2^{1/3}(t)}\right|\\
&+\frac{c(t)\alpha^{1/3}(t)|y_2(t)-y_1(t)|}{9\beta^{2/3}(t)m^{2/3}(t)x_1^{1/3}(t)x_2^{1/3}(t)y_1^{1/3}(t)y_2^{1/3}(t)}\\
&+\frac{c(t)\beta^{1/3}(t)x_1^{2/3}(t)|y_2(t)-y_1(t)|}{9\alpha^{2/3}(t)m^{2/3}(t)x_2^{1/3}(t)y_1^{1/3}(t)y_2^{1/3}(t)}
\end{aligned}$$

$$\begin{aligned}
V_1^{\Delta}(t) \le &-\min\left\{b^l,\frac{2}{G_1\mu^u}-b^u\right\}|x_1(t)-x_2(t)| + \frac{c^u(\beta^u)^{1/3}G_2^{1/3}|x_1(t)-x_2(t)|}{9(\alpha^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{1/3}}\\
&+\frac{c^u(\alpha^u)^{1/3}|y_2(t)-y_1(t)|}{9(\beta^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{2/3}}+\frac{c^u(\beta^u)^{1/3}G_1^{2/3}|y_2(t)-y_1(t)|}{9(\alpha^l)^{2/3}(m^l)^{2/3}g_1^{1/3}g_2^{2/3}}.
\end{aligned}$$

If t is right dense, we get,

$$\begin{aligned}
V_1'(t) &= sgn(\ln x_1(t) - \ln x_2(t)) \left(\frac{x_1'(t)}{x_1(t)} - \frac{x_2'(t)}{x_2(t)} \right) \\
&= sgn(\ln x_1(t) - \ln x_2(t)) \Bigg(- b(t)(x_1(t) - x_2(t)) \\
&\quad - \frac{c(t)y_1(t)}{\alpha(t) + \beta(t)x_1(t) + m(t)y_1(t)} + \frac{c(t)y_2(t)}{\alpha(t) + \beta(t)x_2(t) + m(t)y_2(t)} \Bigg)
\end{aligned}$$

Since $sgn(\ln x_1(t) - \ln x_2(t)) = sgn(x_1(t) - x_2(t))$, then

$$\begin{aligned}
V_1'(t) &\leq -b(t)|x_1(t) - x_2(t)| + c(t)\frac{\beta(t)y_1(t)|x_1(t) - x_2(t)|}{AB} \\
&\quad + c(t)\left|\frac{\alpha(t)[y_1(t) - y_2(t)]}{AB}\right| + c(t)\left|\frac{\beta(t)x_1(t)[y_1(t) - y_2(t)]}{AB}\right| \\
&\leq -b^l|x_1(t) - x_2(t)| + \frac{c(t)\beta^{1/3}(t)y_1^{1/3}(t)|x_1(t) - x_2(t)|}{9\alpha^{2/3}(t)m^{2/3}(t)x_1^{1/3}(t)x_2^{1/3}(t)y_2^{1/3}(t)} \\
&\quad + \frac{c(t)\alpha^{1/3}(t)|y_2(t) - y_1(t)|}{9\beta^{2/3}(t)m^{2/3}x_1^{1/3}(t)x_2^{1/3}(t)y_1^{1/3}(t)y_2^{1/3}(t)} + \frac{c(t)\beta^{1/3}(t)x_1^{2/3}(t)|y_2(t) - y_1(t)|}{9\alpha^{2/3}(t)x_2^{1/3}(t)y_1^{1/3}(t)y_2^{1/3}(t)} \\
&\leq -b^l|x_1(t) - x_2(t)| + \frac{c^u(\beta^u)^{1/3}G_2^{1/3}|x_1(t) - x_2(t)|}{9(\alpha^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{1/3}} \\
&\quad + \frac{c^u(\alpha^u)^{1/3}|y_2(t) - y_1(t)|}{9(\beta^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{2/3}} + \frac{c^u(\beta^u)^{1/3}G_1^{2/3}|y_2(t) - y_1(t)|}{9(\alpha^l)^{2/3}(m^l)^{2/3}g_1^{1/3}g_2^{2/3}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
V_1^{\Delta}(t) &\leq -\min\left\{b^l, \frac{2}{G_1\mu^u} - b^u\right\}|x_1(t) - x_2(t)| + \frac{c^u(\beta^u)^{1/3}G_2^{1/3}|x_1(t) - x_2(t)|}{9(\alpha^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{1/3}} \\
&\quad + \frac{c^u(\alpha^u)^{1/3}|y_2(t) - y_1(t)|}{9(\beta^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{2/3}} + \frac{c^u(\beta^u)^{1/3}G_1^{2/3}|y_2(t) - y_1(t)|}{9(\alpha^l)^{2/3}(m^l)^{2/3}g_1^{1/3}g_2^{2/3}}.
\end{aligned}$$

Let $V_2(t) = |\ln y_1(t) - \ln y_2(t)|$. By using mean value theorem, we get

$$y_1(t) - y_2(t) = \exp(\ln y_1(t)) - \exp(\ln y_2(t)) = \xi_2(t)(\ln y_1(t) - \ln y_2(t)) \tag{5.2}$$

where $\xi_2(t)$ lies between $y_1(t)$ and $y_2(t)$. If t is a right scattered point, by (5.2) we have

$$\begin{aligned}
V_2^{\Delta}(t) &\leq -\frac{1}{\mu(t)}\left[\frac{1}{\xi_2(t)} - \left|\frac{1}{\xi_2(t)} - \mu(t)\frac{f(t)x_1(t)m(t)}{AB}\right|\right]|y_1(t) - y_2(t)| \\
&\quad + \frac{f(t)\alpha^{1/3}(t)|x_1(t) - x_2(t)|}{9\beta^{2/3}(t)m^{2/3}(t)x_1^{1/3}(t)x_2^{1/3}(t)y_1^{1/3}(t)y_2^{1/3}(t)} \\
&\quad + \frac{f(t)m^{1/3}(t)y_1^{2/3}(t)|x_1(t) - x_2(t)|}{9\alpha^{2/3}(t)\beta^{2/3}(t)x_1^{1/3}(t)x_2^{1/3}(t)y_2^{1/3}(t)}
\end{aligned}$$

$$\leq -\min\left\{\frac{f^l g_1 m^l}{A^u B^u}, \frac{2}{G_2\mu^u} - \frac{f^u G_1 m^u}{A^l B^l}\right\}$$
$$+ \frac{f^u(\alpha^u)^{1/3}|x_1(t)-x_2(t)|}{9(\beta^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{2/3}} + \frac{f^u(m^u)^{1/3}G_2^{2/3}|x_1(t)-x_2(t)|}{9(\alpha^l)^{2/3}(\beta^l)^{2/3}g_1^{2/3}g_2^{1/3}}.$$

If t is right dense, then

$$V_2'(t) = sgn(\ln y_1(t) - \ln y_2(t))\left(\frac{y_1'(t)}{y_1(t)} - \frac{y_2'(t)}{y_2(t)}\right)$$
$$= sgn(\ln y_1(t)$$
$$- \ln y_2(t))\left(\frac{f(t)x_1(t)}{\alpha(t)+\beta(t)x_1(t)+m(t)y_1(t)} - \frac{f(t)x_2(t)}{\alpha(t)+\beta(t)x_2(t)+m(t)y_2(t)}\right)$$
$$\leq -\frac{f(t)x_1(t)m(t)}{AB}|y_1(t)-y_2(t)| + \frac{f(t)\alpha^{1/3}(t)|x_1(t)-x_2(t)|}{9\beta^{2/3}(t)m^{2/3}(t)x_1^{1/3}(t)x_2^{1/3}(t)y_1^{1/3}(t)y_2^{1/3}(t)}$$
$$+ \frac{f(t)m^{1/3}(t)y_1^{2/3}(t)|x_1(t)-x_2(t)|}{9\alpha^{2/3}(t)\beta^{2/3}(t)x_1^{1/3}(t)x_2^{1/3}(t)y_2^{1/3}(t)}$$
$$\leq -\frac{f^l g_1 m^l}{A^u B^u}|y_1(t)-y_2(t)| + \frac{f^u(\alpha^u)^{1/3}|x_1(t)-x_2(t)|}{9(\beta^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{2/3}} + \frac{f^u(m^u)^{1/3}G_2^{2/3}|x_1(t)-x_2(t)|}{9(\alpha^l)^{2/3}(\beta^l)^{2/3}g_1^{2/3}g_2^{1/3}}$$

Thus

$$V_2^{\Delta}(t) \leq -\min\left\{\frac{f^l g_1 m^l}{A^u B^u}, \frac{2}{G_2\mu^u} - \frac{f^u G_1 m^u}{A^l B^l}\right\}$$
$$+ \frac{f^u(\alpha^u)^{1/3}|x_1(t)-x_2(t)|}{9(\beta^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{2/3}} + \frac{f^u(m^u)^{1/3}G_2^{2/3}|x_1(t)-x_2(t)|}{9(\alpha^l)^{2/3}(\beta^l)^{2/3}g_1^{2/3}g_2^{1/3}}.$$

Let us define a Lyapunov function as $V(t) := a_1V_1(t) + a_2V_2(t),\ a_1, a_2 \in (0,1)$. $V^{\Delta}(t) = a_1V_1^{\Delta}(t) + a_2V_2^{\Delta}(t)$.

$$V^{\Delta}(t) \leq -\Bigg(a_1 \min\left\{b^l, \frac{2}{G_1\mu^u} - b^u\right\}$$
$$- \Bigg(a_1\frac{c^u(\beta^u)^{1/3}G_2^{1/3}}{9(\alpha^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{1/3}} + a_2\frac{f^u(\alpha^u)^{1/3}}{9(\beta^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{2/3}}$$
$$+ a_2\frac{f^u(m^u)^{1/3}G_2^{2/3}}{9(\alpha^l)^{2/3}(\beta^l)^{2/3}g_1^{2/3}g_2^{1/3}}\Bigg)\Bigg)|x_1(t)-x_2(t)|$$
$$- \Bigg(a_2\min\left\{\frac{f^l g_1 m^l}{A^u B^u}, \frac{2}{G_2\mu^u} - \frac{f^u G_1 m^u}{A^l B^l}\right\}$$
$$- \Bigg(a_1\frac{c^u(\alpha^u)^{1/3}}{9(\beta^l)^{2/3}(m^l)^{2/3}g_1^{2/3}g_2^{2/3}} + a_1\frac{c^u(\beta^u)^{1/3}G_1^{2/3}}{9(\alpha^l)^{2/3}(m^l)^{2/3}g_1^{1/3}g_2^{2/3}}\Bigg)\Bigg)|y_2(t)-y_1(t)|.$$

By assumption

$$V^{\Delta}(t) < -\delta[|x_1(t)-x_2(t)| + |y_2(t)-y_1(t)|].$$

Integrating both sides of the above inequality from t_1 to t, we get

$$\int_{t_1}^{t} V^{\Delta}(s)\Delta s < -\delta \int_{t_1}^{t} [|x_1(s) - x_2(s)| + |y_2(s) - y_1(s)|]\Delta s,$$

$$\int_{t_1}^{t} [|x_1(s) - x_2(s)| + |y_2(s) - y_1(s)|]\Delta s < \frac{V_1(t_1)}{\delta}.$$

Then,

$$\int_{t_1}^{t} [|x_1(s) - x_2(s)| + |y_2(s) - y_1(s)|]\Delta s < +\infty,$$

$$\lim_{t\to\infty} [|x_1(t) - x_2(t)| + |y_2(t) - y_1(t)|] = 0,$$

$\lim_{t\to\infty}[|x_1(t) - x_2(t)|] = 0$ and $\lim_{t\to\infty}[|y_1(t) - y_2(t)|] = 0$.

Hence, we get the desired result. $\square$

Corollary 5.3. *In addition to conditions of Lemma* 4.4 *and Lemma* 4.6 *if* $a_1, a_2 \in (0,1)$, $\alpha(t) = 0$, $\delta > 0$ *and*

$$\Bigg(a_1 \min\left\{ b^l, \frac{2}{G_1\mu^u} - b^u \right\} - \left[a_1 \frac{c^u G_2^{1/2}}{4m^l g_1 g_2^{1/2}} + a_2 \frac{f^u G_2^{1/2}}{4\beta^l g_1 g_2^{1/2}} \right],$$

$$a_2 \min\left\{ \frac{f^l g_1 m^l}{A^u B^u}, \frac{2}{G_2\mu^u} - \frac{f^u G_1 m^u}{A^l B^l} \right\} - \left[a_1 \frac{c^u G_1^{1/2}}{4m^l g_1^{1/2} g_2} \right] \Bigg) > \delta,$$

then, system (4.1) *is globally attractive.*

Example 5.4. $\mathbb{T} = [2k, 2k+1]$, $k \in \mathbb{N}$ k start with 0.

$$x^{\Delta}(t) = (0.5 - \frac{0.1}{t+1}) - \exp(x) - \frac{0.01\exp(y)}{\exp(x) + \exp(y)},$$

$$y^{\Delta}(t) = -0.1 + \frac{0.2\exp(x)}{\exp(x) + \exp(y)}.$$

Example 5.4 satisfies Corollary 5.3, therefore solution of this system is globally attractive.

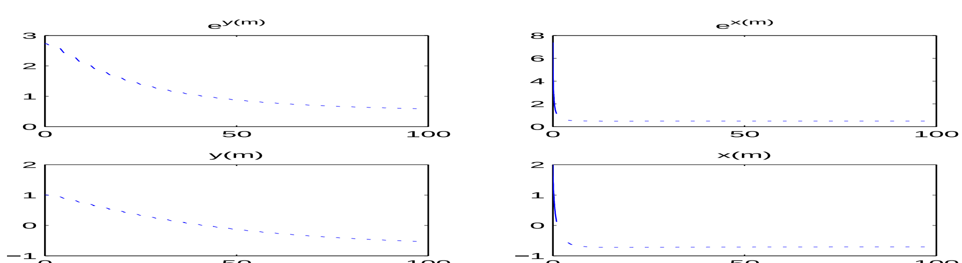

FIGURE 2. Initial conditions in this example are $x(0) = 3$, $y(0) = 2$.

Although we take several different initial conditions, the solutions $(\exp(x(t)), \exp(y(t)))$ approaches to 0.5 in each case. Therefore numeric solution of Example 5.4 shows the global attractivity.

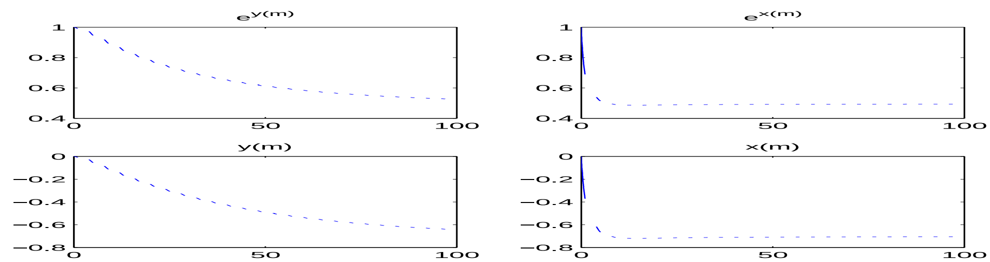

FIGURE 3. Initial conditions in this example are $x(0) = 0$, $y(0) = 0$.

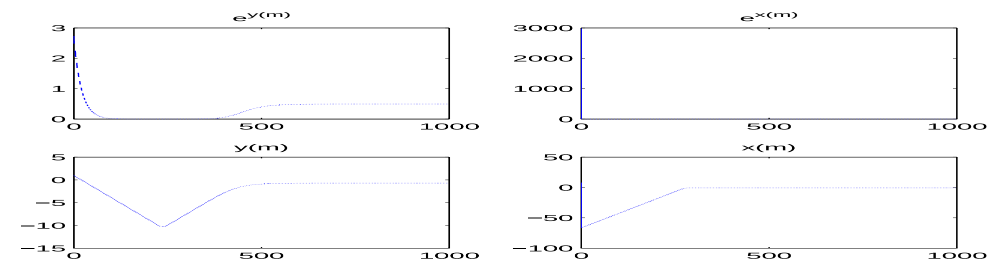

FIGURE 4. Initial conditions in this example are $x(0) = 8$, $y(0) = 1$.

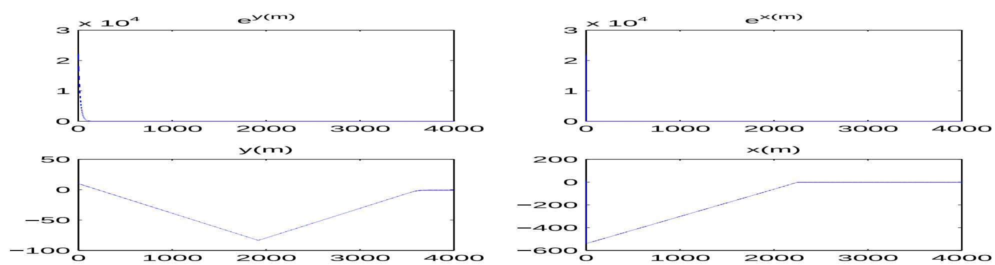

FIGURE 5. Initial conditions in this example are $x(0) = 10$, $y(0) = 10$.

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