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VECTOR FRACTIONAL KOROVKIN TYPE APPROXIMATIONS

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ABSTRACT. In this article we study quantitatively with rates the convergence of sequences of general Bochner type integral operators, applied on Banach space valued functions, to function values. The results are mainly pointwise, but in the application to vector Bernstein polynomials we end up to obtain a uniform estimate. To prove our main results we have to build a rich background containing many interesting vector fractional results. Our inequalities are fractional involving the right and left vector Caputo type fractional derivatives, built in vector moduli of continuity. We treat very general classes of Banach space valued functions.

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1. Introduction

In this paper among others we are motivated by the following results.

Theorem 1 (P. P. Korovkin [13], (1960), p. 14)**.** *Let $[a,b]$ be a closed interval in $\mathbb{R}$ and $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators mapping $C([a,b])$ into itself. Suppose that $(L_n f)$ converges uniformly to f for the three test functions $f = 1, x, x^2$. Then $(L_n f)$ converges uniformly to f on $[a,b]$ for all functions $f \in C([a,b])$.*

Let $f \in C([a,b])$ and $0 \leq h \leq b-a$. The first modulus of continuity of f at h is given by

$$\omega_1(f,h) = \sup_{\substack{x,y\in[a,b] \\ |x-y|\leq h}} |f(x) - f(y)|.$$

If $h > b-a$, then we define $\omega_1(f,h) = \omega_1(f,b-a)$.

Another motivation is the following

Theorem 2 (Shisha and Mond [18], (1968))**.** *Let $[a,b] \subset \mathbb{R}$ a closed interval. Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators acting on $C([a,b])$ into itself. For*

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$n=1,\dots,$ suppose $L_n(1)$ is bounded. Let $f\in C([a,b])$. Then for $n=1,2,\dots,$ we have

$$\|L_nf-f\|_\infty\le\|f\|_\infty\|L_n1-1\|_\infty+\|L_n1+1\|_\infty\,\omega_1(f,\mu_n),\tag{1}$$

where

$$\mu_n=\left\|L_n\left((t-x)^2\right)(x)\right\|_\infty^{\frac{1}{2}}$$

and $\|\cdot\|_\infty$ stands for the sup-norm over $[a,b]$.

One can easily see, for $n=1,2,\dots$

$$\mu_n^2\le\left\|L_n\left(t^2;x\right)-x^2\right\|_\infty+2c\,\|L_n(t;x)-x\|_\infty+c^2\,\|L_n(1;x)-1\|_\infty,$$

where $c=\max(|a|,|b|)$.

Thus, given the Korovkin assumptions (see Theorem 1) as $n\to\infty$ we get $\mu_n\to0$, and by (1) that $\|L_nf-f\|_\infty\to0$ for any $f\in C([a,b])$. That is one derives the Korovkin conclusion in a quantitative way and with rates of convergence.

One more motivation follows

Theorem 3 (see Corollary 7.2.2, p. 219, [3])**.** *Consider the positive linear operator*

$$L:C^n([a,b])\to C([a,b]),\quad n\in\mathbb{N}.$$

Let

$$c_k(x)=L\left((t-x)^k,x\right),\quad k=0,1,\dots,n;$$

$$d_n(x)=\left|L\left(|t-x|^n,x\right)\right|^{\frac{1}{n}};\quad c(x)=\max(x-a,b-x)\quad\left(c(x)\ge\frac{b-a}{2}\right).$$

Let $f\in C^n([a,b])$ such that $\omega_1\left(f^{(n)},h\right)\le w$, where w,h are fixed positive numbers, $0<h<b-a$. Then we have the following upper bound

$$|L(f,x)-f(x)|\le|f(x)|\,|c_0(x)-1|+\sum_{k=1}^{n}\frac{\left|f^{(k)}(x)\right|}{k!}|c_k(x)|+R_n.\tag{2}$$

Here

$$R_n=w\phi_n(c(x))\left(\frac{d_n(x)}{c(x)}\right)^n=\frac{w}{n!}\theta_n\left(\frac{h}{c(x)}\right)d_n^n(x),$$

where

$$\theta_n\left(\frac{h}{u}\right)=\frac{n!\phi_n(u)}{u^n},$$

with

$$\phi_n(x)=\int_0^{|x|}\left\lceil\frac{t}{h}\right\rceil\frac{(|x|-t)^{n-1}}{(n-1)!}dt,\quad(x\in\mathbb{R}),$$

$\lceil\cdot\rceil$ is the ceiling of the number.

Inequality (2) is sharp. It is approximately attained by $w\phi_n\left((t-x)_+\right)$ and a measure μ_x supported by $\{x,b\}$ when $x-a\le b-x$, also approximately attained by $w\phi_n\left((x-t)_+\right)$ and a measure μ_x supported by $\{x,a\}$ when $x-a\ge b-x$: in each case with masses $c_0(x)-\left(\frac{d_n(x)}{c(x)}\right)^n$ and $\left(\frac{d_n(x)}{c(x)}\right)^n$, respectively.

Using the last method and its refinements one derives nice and simple results for specific operators.

For example from [3, Corollary 7.3.4, p. 230], we obtain: let $f\in C^1([0,1])$ and consider the Bernstein polynomials

$$(B_nf)(t)=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)\binom{n}{k}t^k(1-t)^{n-k},\quad t\in[0,1],$$

then

$$\|B_nf-f\|_\infty\le\frac{0.78125}{\sqrt{n}}\omega_1\left(f',\frac{1}{4\sqrt{n}}\right).$$

So $B_nf\overset{u}{\to}f$ as $n\to\infty$ with rates.

In [5] we extended the above theory to the fractional level using right and left Caputo fractional derivatives for the first time in the literature, but still for real valued functions.

In this article we present a fractional quantitative Korovkin type approximation theory for linear operators involving Banach space valued functions. We use here vector valued right and left Caputo type fractional derivatives, and these show up in the moduli of continuity appearing on the right hand side of our inequalities. We finish with application of our theory to vector valued Bernstein polynomials. All integrals here are of Bochner type.

In the background section we present many interesting vector fractional results which by themselves have their own merit and appear for the first time.

In approximation theory the involvement of fractional derivatives is very rare, almost nothing exists. The only fractional articles that preexisted author's fractional works are of V. Dzyadyk [11] of 1959, F. Nasibov [16] of 1962, J. Demjanovic [10] of 1975 and of M. Jaskolski [12] of 1989, all regarding estimates to best approximation of functions by algebraic and trigonometric polynomials.

We are also motivated by [4].

2. **Background**

All integrals here are of Bochner type [15]. We need

Definition 4 ([6])**.** Let $[a,b]\subset\mathbb{R}$, X be a Banach space, $\alpha>0$; $m=\lceil\alpha\rceil\in\mathbb{N}$, ($\lceil\cdot\rceil$ is the ceiling of the number), $f:[a,b]\to X$. We assume that $f^{(m)}\in L_1([a,b],X)$.

We call the Caputo-Bochner left fractional derivative of order α:

$$(3)\qquad \left(D_{*a}^{\alpha}f\right)(x) := \frac{1}{\Gamma(m-\alpha)}\int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)\,dt, \quad \forall\, x\in[a,b].$$

If $\alpha\in\mathbb{N}$, we set $D_{*a}^{\alpha}f := f^{(m)}$ the ordinary X-valued derivative (defined similar to numerical one, see [17, p. 83]), and also set $D_{*a}^{0}f := f$.

By [6], $\left(D_{*a}^{\alpha}f\right)(x)$ exists almost everywhere in $x\in[a,b]$ and $D_{*a}^{\alpha}f\in L_1([a,b],X)$.

If $\left\|f^{(m)}\right\|_{L_\infty([a,b],X)} < \infty$, then by [6], $D_{*a}^{\alpha}f \in C([a,b],X)$, hence $\left\|D_{*a}^{\alpha}f\right\| \in C([a,b])$.

We mention

Lemma 5. *Let $\alpha>0$, $\alpha\notin\mathbb{N}$, $m=\lceil\alpha\rceil$, $f\in C^{m-1}([a,b],X)$ and $f^{(m)}\in L_\infty([a,b],X)$. Then $D_{*a}^{\alpha}f(a)=0$.*

Proof. By (3) we get

$$\begin{aligned}\left\|\left(D_{*a}^{\alpha}f\right)(x)\right\| &= \frac{1}{\Gamma(m-\alpha)}\left\|\int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)\,dt\right\| \\ &\leq \frac{1}{\Gamma(m-\alpha)}\int_a^x (x-t)^{m-\alpha-1}\left\|f^{(m)}(t)\right\|dt \\ &\leq \frac{\left\|f^{(m)}\right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)}(x-a)^{m-\alpha}. \qquad (4)\end{aligned}$$

I.e.

$$(5)\qquad \left\|\left(D_{*a}^{\alpha}f\right)(x)\right\| \leq \frac{\left\|f^{(m)}\right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)}(x-a)^{m-\alpha}, \quad \forall\, x\in[a,b].$$

That is $D_{*a}^{\alpha}f(a)=0$. ∎

We mention

Definition 6 ([7])**.** Let $[a,b]\subset\mathbb{R}$, X be a Banach space, $\alpha>0$, $m:=\lceil\alpha\rceil$. We assume that $f^{(m)}\in L_1([a,b],X)$, where $f:[a,b]\to X$. We call the Caputo-Bochner right fractional derivative of order α:

$$(6)\qquad \left(D_{b-}^{\alpha}f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)}\int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\,dz, \quad \forall\, x\in[a,b].$$

We observe that $\left(D_{b-}^{m}f\right)(x) = (-1)^m f^{(m)}(x)$, for $m\in\mathbb{N}$, and $\left(D_{b-}^{0}f\right)(x)=f(x)$.

By [7], $\left(D_{b-}^{\alpha}f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha}f\right)\in L_1([a,b],X)$.

If $\left\|f^{(m)}\right\|_{L_\infty([a,b],X)}<\infty$, and $\alpha\notin\mathbb{N}$, by [7], $D_{b-}^{\alpha}f\in C([a,b],X)$, hence $\left\|D_{b-}^{\alpha}f\right\|\in C([a,b])$.

We need

Lemma 7. *Let* $f \in C^{m-1}([a,b], X)$*,* $f^{(m)} \in L_{\infty}([a,b], X)$*,* $m = \lceil \alpha \rceil$*,* $\alpha > 0$*,* $\alpha \notin \mathbb{N}$*. Then* $D_{b-}^{\alpha} f(b) = 0$*.*

Proof. By (6) we get

$$\left\| \left(D_{b-}^{\alpha} f \right)(x) \right\| = \frac{1}{\Gamma(m-\alpha)} \left\| \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz \right\|$$

$$\leq \frac{1}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} \left\| f^{(m)}(z) \right\| dz$$

$$\leq \frac{\left\| f^{(m)} \right\|_{L_{\infty}([a,b],X)}}{\Gamma(m-\alpha+1)} (b-x)^{m-\alpha}, \tag{7}$$

$\forall\, x \in [a,b]$.

Clearly $\left(D_{b-}^{\alpha} f \right)(b) = 0$. □

We mention the left fractional Taylor formula

Theorem 8 ([6])**.** *Let* $m \in \mathbb{N}$ *and* $f \in C^{m-1}([a,b], X)$*, where* $[a,b] \subset \mathbb{R}$ *and* X *is a Banach space, and let* $\alpha > 0 : m = \lceil \alpha \rceil$*. Set*

$$F_x(t) := \sum_{i=0}^{m-1} \frac{(x-t)^i}{i!} f^{(i)}(t), \quad \forall\, t \in [a,x], \tag{8}$$

where $x \in [a,b]$*. Assume that* $f^{(m)}$ *exists outside a* λ*-null Borel set* $B_x \subseteq [a,x]$ *(*λ *is the Lebesgue measure) such that*

$$h_1(F_x(B_x)) = 0, \quad \forall\, x \in [a,b] \tag{9}$$

(h_1 *is the Hausdorff measure of order 1, see* [19]*). We also assume that* $f^{(m)} \in L_1([a,b], X)$*. Then*

$$f(x) = \sum_{i=0}^{m-1} \frac{(x-a)^i}{i!} f^{(i)}(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-z)^{\alpha-1} \left(D_{*a}^{\alpha} f \right)(z)\, dz, \tag{10}$$

$\forall\, x \in [a,b]$.

We also mention the right fractional Taylor formula

Theorem 9 ([7])**.** *Let* $[a,b] \subset \mathbb{R}$*,* X *be a Banach space,* $\alpha > 0$*,* $m = \lceil \alpha \rceil$*,* $f \in C^{m-1}([a,b], X)$*. Set*

$$F_x(t) := \sum_{i=0}^{m-1} \frac{(x-t)^i}{i!} f^{(i)}(t), \quad \forall\, t \in [x,b], \tag{11}$$

where $x \in [a,b]$*.*

Assume that $f^{(m)}$ *exists outside a* λ*-null Borel set* $B_x \subseteq [x,b]$*, such that*

$$h_1(F_x(B_x)) = 0, \quad \forall\, x \in [a,b]. \tag{12}$$

We also assume that $f^{(m)} \in L_1([a,b],X)$. *Then*

$$f(x) = \sum_{i=0}^{m-1} \frac{(x-b)^i}{i!} f^{(i)}(b) + \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} \left(D_{b-}^{\alpha} f\right)(z)\, dz, \tag{13}$$

$\forall\, x \in [a,b]$.

We define the following classes of functions:

Definition 10. We call $(x_0 \in [a,b] \subset \mathbb{R})$

$$H_{x_0}^{(1)} := \{ f \in C^{m-1}([a,b],X) : [a,b] \subset \mathbb{R}, (X, \|\cdot\|) \tag{14}$$

is a Banach space, $\alpha > 0 : m = \lceil \alpha \rceil$; $f^{(m)} \in L_\infty([a,b],X)$; $F_x^{(1)}(t) := \sum_{i=0}^{m-1} \frac{(x-t)^i}{i!} f^{(i)}(t)$ is defined $\forall\, t \in [x, x_0]$,with $x \in [a, x_0]$ and $f^{(m)}$ exists outside a λ-null Borel set $B_x^{(1)} \subseteq [x, x_0]$, such that $h_1\left(F_x^{(1)}\left(B_x^{(1)}\right)\right) = 0$, $\forall\, x \in [a, x_0]$; $F_x^{(2)}(t) := \sum_{i=0}^{m-1} \frac{(x-t)^i}{i!} f^{(i)}(t)$ is defined $\forall\, t \in [x_0, x]$, with $x \in [x_0, b]$ and $f^{(m)}$ exists outside a λ-null Borel set $B_x^{(2)} \subseteq [x_0, x]$, such that $h_1\left(F_x^{(2)}\left(B_x^{(2)}\right)\right) = 0$, $\forall\, x \in [x_0, b]\}$,

$$H^{(2)} := \{ f \in C^m([a,b],X) : [a,b] \subset \mathbb{R}, \tag{15}$$

X is a Banach space, $\alpha > 0 : m = \lceil \alpha \rceil \}$.

Notice that

$$H^{(2)} \subset H_{x_0}^{(1)}, \quad \forall\, x_0 \in [a,b]. \tag{16}$$

Let $([a,b], \Sigma, \mu)$ be a complete measure space, where μ is a positive and finite measure and $(X, \|\cdot\|)$ be a Banach space. Let $f \in C([a,b],X)$, then, by [14], we have that f is strongly μ-measurable. Clearly $\|f\| \in C([a,b])$ and $\|f\|$ is measurable, hence $\int_{[a,b]} \|f(t)\|\, d\mu(t) < \infty$ iff f is Bochner-integrable, see [14]. I.e. the Bochner integral $\int_{[a,b]} f(t)\, d\mu(t)$ exists.

We need

Lemma 11. *Let* $([a,b], \Sigma, \mu)$ *be a complete measure space, where* μ *is a positive and finite measure and* $f \in H_{x_0}^{(1)}$, $x_0 \in [a,b]$. *Then*

$$\begin{aligned} E_{x_0} &:= \int_{[a,b]} f(x)\, d\mu(x) - \sum_{i=0}^{m-1} \frac{f^{(i)}(x_0)}{i!} \int_{[a,b]} (x-x_0)^i\, d\mu(x) \\ &= \frac{1}{\Gamma(\alpha)} \left\{ \int_{[a,x_0]} \left(\int_x^{x_0} (z-x)^{\alpha-1} \left(\left(D_{x_0-}^{\alpha} f\right)(z) - \left(D_{x_0-}^{\alpha} f\right)(x_0) \right) dz \right) d\mu(x) \right. \\ &\quad \left. + \int_{(x_0,b]} \left(\int_{x_0}^x (x-z)^{\alpha-1} \left(\left(D_{*x_0}^{\alpha} f\right)(z) - \left(D_{*x_0}^{\alpha} f\right)(x_0) \right) dz \right) d\mu(x) \right\}. \end{aligned} \tag{17}$$

All the above integrals are of Bochner type.

Proof. By Theorem 9 we get

$$f(x)=\sum_{i=0}^{m-1}\frac{(x-x_0)^i}{i!}f^{(i)}(x_0)+\frac{1}{\Gamma(\alpha)}\int_x^{x_0}(z-x)^{\alpha-1}\left(D_{x_0-}^{\alpha}f\right)(z)\,dz, \tag{18}$$

$\forall\ x\in[a,x_0]$. The remainder is a continuous function in x.

By Theorem 8 we get

$$f(x)=\sum_{i=0}^{m-1}\frac{(x-x_0)^i}{i!}f^{(i)}(x_0)+\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}(z-x)^{\alpha-1}\left(D_{*x_0}^{\alpha}f\right)(z)\,dz, \tag{19}$$

$\forall\ x\in[x_0,b]$. The remainder is a continuous function in x.

Consequently we find

$$\begin{aligned}\int_{[a,b]}f(x)\,d\mu(x)&=\int_{[a,x_0]}f(x)\,d\mu(x)+\int_{(x_0,b]}f(x)\,d\mu(x)\\&=\sum_{i=0}^{m-1}\frac{f^{(i)}(x_0)}{i!}\int_{[a,b]}(x-x_0)^i\,d\mu(x)\\&\quad+\frac{1}{\Gamma(\alpha)}\left\{\int_{[a,x_0]}\left(\int_x^{x_0}(z-x)^{\alpha-1}\left(D_{x_0-}^{\alpha}f\right)(z)\,dz\right)d\mu(x)\right.\\&\quad\left.+\int_{(x_0,b]}\left(\int_{x_0}^{x}(x-z)^{\alpha-1}\left(D_{*x_0}^{\alpha}f\right)(z)\,dz\right)d\mu(x)\right\}.\end{aligned} \tag{20}$$

Notice also that $D_{x_0-}^{\alpha}f(x_0)=D_{*x_0}^{\alpha}f(x_0)=0$. The claim is proved. ∎

Convention 12. We assume that

$$D_{*x_0}^{\alpha}f(x)=0,\quad \text{for } x<x_0, \tag{21}$$

and

$$D_{x_0-}^{\alpha}f(x)=0,\quad \text{for } x>x_0, \tag{22}$$

for all $x,x_0\in[a,b]$.

We need

Definition 13. Let $f\in C([a,b],X)$, $[a,b]\subset\mathbb{R}$, $(X,\|\cdot\|)$ a Banach space. We define the first modulus of continuity of f as

$$\omega_1(f,\delta):=\sup_{\substack{x,y\in[a,b]:\\|x-y|\le\delta}}\|f(x)-f(y)\|,\quad 0<\delta\le b-a. \tag{23}$$

If $\delta>b-a$, then $\omega_1(f,\delta)=\omega_1(f,b-a)$.

Notice $\omega_1(f,\delta)$ is increasing in $\delta>0$.

Clearly f is uniformly continuous, see [2, p. 53]. Also easily we see that $\omega_1(f,\delta)<\infty$, see again [2, p. 52]. For $f\in B([a,b],X)$ (bounded functions) $\omega_1(f,\delta)$ is defined the same way.

Lemma 14. *We have* $\omega_1(f,\delta) \to 0$ *as* $\delta \downarrow 0$ *iff* $f \in C([a,b],X)$.

Proof. ($\Longrightarrow$) Let $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$. Then $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ with $\omega_1(f,\delta) \leq \varepsilon$. I.e. $\forall\, x,y \in [a,b] : |x-y| \leq \delta$ we get $\|f(x) - f(y)\| \leq \varepsilon$. That is $f \in C([a,b],X)$.

($\Longleftarrow$) Let $f \in C([a,b],X)$. Then $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$: whenever $|x-y| \leq \delta$, $x,y \in [a,b]$, it implies $\|f(x) - f(y)\| \leq \varepsilon$. I.e. $\forall\, \varepsilon > 0$, $\exists\, \delta > 0 : \omega_1(f,\delta) \leq \varepsilon$. That is $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$. ∎

We mention

Proposition 15. *Let* $f \in C^n([a,b],X)$, $n = \lceil \nu \rceil$, $\nu > 0$. *Then* $D^{\nu}_{*a}f(x)$ *is continuous in* $x \in [a,b]$.

Proof. We notice that (see [15, p. 116])

$$D^{\nu}_{*a}f(x) = \frac{1}{\Gamma(n-\nu)} \int_0^{x-a} z^{n-\nu-1} f^{(n)}(x-z)\,dz, \tag{24}$$

and

$$D^{\nu}_{*a}f(y) = \frac{1}{\Gamma(n-\nu)} \int_0^{y-a} z^{n-\nu-1} f^{(n)}(y-z)\,dz.$$

Here $a \leq x \leq y \leq b$, and $0 \leq x-a \leq y-a$.

Hence it holds ([8] and [1, p. 426, Theorem 11.43])

$$\begin{aligned} D^{\nu}_{*a}f(y) - D^{\nu}_{*a}f(x) = \frac{1}{\Gamma(n-\nu)} \Bigg[&\int_0^{x-a} z^{n-\nu-1} \left(f^{(n)}(y-z) - f^{(n)}(x-z) \right) dz \\ &+ \int_{x-a}^{y-a} z^{n-\nu-1} f^{(n)}(y-z)\,dz \Bigg]. \end{aligned} \tag{25}$$

We have that

$$\begin{aligned} \|D^{\nu}_{*a}f(y) - D^{\nu}_{*a}f(x)\| &\leq \frac{1}{\Gamma(n-\nu)} \Bigg[\frac{(x-a)^{n-\nu}}{(n-\nu)} \omega_1\left(f^{(n)}, |y-x|\right) \\ &\quad + \frac{\left\|f^{(n)}\right\|_{L_\infty([a,b],X)}}{(n-\nu)} \left((y-a)^{n-\nu} - (x-a)^{n-\nu} \right) \Bigg] \\ &\leq \frac{1}{\Gamma(n-\nu+1)} \Big[(b-a)^{n-\nu} \omega_1\left(f^{(n)}, |y-x|\right) \\ &\quad + \left\|f^{(n)}\right\|_{L_\infty([a,b],X)} \left((y-a)^{n-\nu} - (x-a)^{n-\nu} \right) \Big]. \end{aligned}$$

So as $y \to x$ the last expression goes to zero. As a result,

$$D^{\nu}_{*a}f(y) \to D^{\nu}_{*a}f(x), \tag{26}$$

proving the claim. ∎

Proposition 16. *Let* $f \in C^m([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$. *Then* $D^{\nu}_{b-}f(x)$ *is continuous in* $x \in [a,b]$.

Proof. As in Proposition 15. ∎

We also mention

Proposition 17. *Let* $f \in C^{m-1}([a,b],X)$, $f^{(m)} \in L_{\infty}([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$ *and*

$$D_{*x_0}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{27}$$

for all $x, x_0 \in [a,b] : x \geq x_0$.

Then $D_{*x_0}^{\alpha} f(x)$ *is continuous in* x_0.

Proof. Fix $x : x \geq y_0 \geq x_0$; $x, x_0, y_0 \in [a,b]$. Then

$$\left\| D_{*x_0}^{\alpha} f(x) - D_{*y_0}^{\alpha} f(x) \right\| = \frac{1}{\Gamma(m-\alpha)} \left\| \int_{x_0}^{y_0} (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt \right\|$$

$$\leq \frac{\left\| f^{(m)} \right\|_{L_{\infty}([a,b],X)}}{\Gamma(m-\alpha)} \left(\int_{x_0}^{y_0} (x-t)^{m-\alpha-1} dt \right)$$

$$= \frac{\left\| f^{(m)} \right\|_{L_{\infty}([a,b],X)}}{\Gamma(m-\alpha+1)} \left((x-y_0)^{m-\alpha} - (x-x_0)^{m-\alpha} \right) \to 0, \tag{28}$$

as $y_0 \to x_0$, proving continuity of $D_{*x_0}^{\alpha} f$ in $x_0 \in [a,b]$. ☐

Proposition 18. *Let* $f \in C^{m-1}([a,b],X)$, $f^{(m)} \in L_{\infty}([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$ *and*

$$D_{x_0-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{x}^{x_0} (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta)\, d\zeta, \tag{29}$$

for all $x, x_0 \in [a,b] : x_0 \geq x$.

Then $D_{x_0-}^{\alpha} f(x)$ *is continuous in* x_0.

Proof. As in Proposition 17. ☐

We need

Proposition 19. *Let* $g \in C([a,b],X)$, $0 < c < 1$, $x, x_0 \in [a,b]$. *Define*

$$L(x,x_0) = \int_{x_0}^{x} (x-t)^{c-1} g(t)\, dt, \quad \text{for } x \geq x_0, \tag{30}$$

and $L(x,x_0) = 0$, *for* $x < x_0$.

Then L *is jointly continuous in* (x,x_0) *on* $[a,b]^2$.

Proof. We notice that $L(x_0,x_0) = 0$.

Assume $x \geq x_0$, then

$$L(x,x_0) = \int_0^{x-x_0} z^{c-1} g(x-z)\, dz = \int_0^{b-a} \chi_{[0,x-x_0]}(z)\, z^{c-1} g(x-z)\, dz, \tag{31}$$

where χ is the characteristic function.

Let $x_N \to x$, $x_{0N} \to x_0$, $N \in \mathbb{N}$ and assume without loss of generality that $x_N \geq x_{0N}$.

So we have again

$$L(x_N, x_{0N}) = \int_0^{x_N - x_{0N}} z^{c-1} g(x_N - z)\, dz$$

$$= \int_0^{b-a} \chi_{[0, x_N - x_{0N}]}(z)\, z^{c-1} g(x_N - z)\, dz. \tag{32}$$

The integrands above are Bochner integrable functions.

We have

$$\chi_{[0, x_N - x_{0N}]}(z) \to \chi_{[0, x - x_0]}(z), \text{ a.e.,} \tag{33}$$

and (see also [9, p. 88]) the function below on the right is strongly measurable

$$\chi_{[0, x_N - x_{0N}]}(z)\, z^{c-1} g(x_N - z) \to \chi_{[0, x - x_0]}(z)\, z^{c-1} g(x - z), \text{ a.e.} \tag{34}$$

Notice that

$$\chi_{[0, x_N - x_{0N}]}(z)\, z^{c-1} \|g(x_N - z)\| \leq z^{c-1} \|\|g\|\|_\infty, \tag{35}$$

which is an integrable function.

Thus by Dominated Convergence theorem, [8], we obtain

$$L(x_N, x_{0N}) \to L(x, x_0), \quad \text{as } N \to \infty. \tag{36}$$

Clearly now $L(x, x_0)$ is jointly continuous on $[a, b]^2$. □

We mention

Proposition 20. *Let $g \in C([a, b])$, $0 < c < 1$, $x, x_0 \in [a, b]$. Define*

$$K(x, x_0) = \int_x^{x_0} (\zeta - x)^{c-1} g(\zeta)\, d\zeta, \quad \text{for } x \leq x_0, \tag{37}$$

and $K(x, x_0) = 0$, for $x > x_0$.

Then $K(x, x_0)$ is jointly continuous from $[a, b]^2$ into $\mathbb{R}$.

Proof. As in Proposition 19. □

Based on Propositions 19, 20 we derive

Corollary 21. *Let $f \in C^m([a, b], X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a, b]$. Then $D^a_{*x_0} f(x)$, $D^a_{x_0 -} f(x)$ are jointly continuous functions in (x, x_0) from $[a, b]^2$ into X, X is a Banach space.*

We need

Theorem 22. *Let $f : [a,b]^2 \to X$ be jointly continuous, X is a Banach space. Consider*

$$G(x) = \omega_1 (f(\cdot, x), \delta, [x,b]), \tag{38}$$

$\delta > 0$, $x \in [a,b]$.

Then G is continuous on $[a,b]$.

Proof. (i) Let $x_n \to x$, $a \le x_n \le x$, and $0 < \delta \le b - x$ first. (The case when $x_n \to x$ with $x_n \ge x$ is similar.) Then we can write

$$G(x_n) = \max(A, B, C), \tag{39}$$

where

$$A = \sup \{ \| f(u, x_n) - f(v, x_n) \| ; u, v \in [x,b], |u - v| \le \delta \}, \quad 0 < \delta \le b - x,$$

$$B = \sup \{ \| f(u, x_n) - f(v, x_n) \| ; u \in [x_n, x], v \in [x,b], |u - v| \le \delta \}, \tag{40}$$

$$C = \sup \{ \| f(u, x_n) - f(v, x_n) \| ; u, v \in [x_n, x], |u - v| \le \delta \}.$$

Now, when $x_n \to x$, then $A \to G(x)$, $B \to K(x) \le G(x)$, $C \to 0$ (since also u converges to v).

In conclusion, $G(x_n) \to \max \{ G(x), K(x), 0 \} = G(x)$.

(ii) If $\delta > b - x$, then $\omega_1 (f(\cdot, x), \delta, [x,b]) = \omega_1 (f(\cdot, x), b - x, [x,b])$, a case covered by (i).

That is proving the claim. □

Theorem 23. *Let $f : [a,b]^2 \to X$ be jointly continuous, X is a Banach space. Then*

$$H(x) = \omega_1 (f(\cdot, x), \delta, [a,x]), \tag{41}$$

$x \in [a,b]$, is continuous in $x \in [a,b]$, $\delta > 0$.

Proof. As in Theorem 22. □

We make

Remark 24. Let $f \in C^{n-1}([a,b])$, $f^{(n)} \in L_\infty([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$, $\nu \notin \mathbb{N}$. Then as in the proof of Lemma 5, we have

$$\| D_{*a}^{\nu} f(x) \| \le \frac{\| f^{(n)} \|_{L_\infty([a,b],X)}}{\Gamma(n - \nu + 1)} (x - a)^{n-\nu}, \quad \forall\, x \in [a,b]. \tag{42}$$

Thus we observe

$$\omega_1 (D_{*a}^{\nu} f, \delta) = \sup_{\substack{x,y \in [a,b] \\ |x-y| \le \delta}} \| D_{*a}^{\nu} f(x) - D_{*a}^{\nu} f(y) \|$$

$$\leq \sup_{\substack{x,y\in[a,b] \\ |x-y|\leq\delta}} \left(\frac{\left\|f^{(n)}\right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)} (x-a)^{n-\nu} + \frac{\left\|f^{(n)}\right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)} (y-a)^{n-\nu} \right)$$

$$\leq \frac{2\left\|f^{(n)}\right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}. \tag{43}$$

Consequently

$$\omega_1\left(D_{*a}^{\nu}f, \delta\right) \leq \frac{2\left\|f^{(n)}\right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}. \tag{44}$$

Similarly, let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil\alpha\rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, then

$$\omega_1\left(D_{b-}^{\alpha}f, \delta\right) \leq \frac{2\left\|f^{(m)}\right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \tag{45}$$

So for $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil\alpha\rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, we find

$$\sup_{x_0\in[a,b]} \omega_1\left(D_{*x_0}^{\alpha}f, \delta\right)_{[x_0,b]} \leq \frac{2\left\|f^{(m)}\right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}, \tag{46}$$

and

$$\sup_{x_0\in[a,b]} \omega_1\left(D_{x_0-}^{\alpha}f, \delta\right)_{[a,x_0]} \leq \frac{2\left\|f^{(m)}\right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \tag{47}$$

We make

Remark 25. Let $([a,b], \Sigma, \mu)$ be a complete measure space, with μ a positive finite measure.

Let $\alpha > 0$, then by Hölder's inequality we obtain

$$\int_{[a,x_0]} (x_0-x)^{\alpha} d\mu(x) \leq \left(\int_{[a,x_0]} (x_0-x)^{\alpha+1} d\mu(x)\right)^{\frac{\alpha}{(\alpha+1)}} \mu([a,x_0])^{\frac{1}{(\alpha+1)}}, \tag{48}$$

and

$$\int_{(x_0,b]} (x-x_0)^{\alpha} d\mu(x) \leq \left(\int_{(x_0,b]} (x-x_0)^{\alpha+1} d\mu(x)\right)^{\frac{\alpha}{(\alpha+1)}} \mu((x_0,b])^{\frac{1}{(\alpha+1)}}. \tag{49}$$

Let now $m = \lceil\alpha\rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, $k = 1, \ldots, m-1$. Then by applying again Hölder's inequality we obtain

$$\int_{[a,b]} |x-x_0|^{k} d\mu(x) \leq \left(\int_{[a,b]} |x-x_0|^{\alpha+1} d\mu(x)\right)^{\frac{k}{(\alpha+1)}} \mu([a,b])^{\frac{\alpha+1-k}{(\alpha+1)}}. \tag{50}$$

We need

Lemma 26 ([3, p. 208, Lemma 7.1.1])**.** *Let $f \in B([a,b], X)$, $(X, \|\cdot\|)$ is a Banach space. Then*

$$\|f(x) - f(x_0)\| \leq \omega_1(f,h) \left\lceil \frac{|x-x_0|}{h} \right\rceil \leq \omega_1(f,h)\left(1 + \frac{|x-x_0|}{h}\right), \tag{51}$$

$\forall\ x, x_0 \in [a,b]$, $h > 0$.

We give

Definition 27. Let $([a,b], \Sigma, \mu_N)$ be a complete measure space, with μ_N a positive finite measure, $\forall\ N \in \mathbb{N}$. We define the linear operators

$$L_N(f) = \int_{[a,b]} f(t)\,\mu_N(dt), \quad \forall\ N \in \mathbb{N}, \tag{52}$$

$\forall\ f \in C([a,b], X)$, where $(X, \|\cdot\|)$ is a Banach space.

Remark 28 (on Definition 27)**.** Actually it is $L_N : C([a,b], X) \to X$, and $L_N(f)$ exists as a Bochner integral. If $c \in X$, then

$$L_N(c) = c\mu_N([a,b]) \tag{53}$$

and for $\overrightarrow{i} \in X : \left\|\overrightarrow{i}\right\| = 1$ we get

$$L_N\left(\overrightarrow{i}\right) = \overrightarrow{i}\mu_N([a,b]). \tag{54}$$

Denote $\mu_N([a,b]) =: M_N$.

If additionally X is a Banach lattice and $f, g \in C([a,b], X)$ are such that $f(t) \leq g(t)$, $\forall\ t \in [a,b]$, in the X-lattice order, then (by [1, p. 426, Theorem 11.43])

$$L_N(f) \leq L_N(g), \tag{55}$$

in the X-lattice order.

Thus L_N is a positive linear operator in the X-lattice order.

We further notice that

$$\|L_N(f)\| = \left\|\int_{[a,b]} f(t)\,\mu_N(dt)\right\| \leq \int_{[a,b]} \|f(t)\|\,\mu_N(dt) \leq M_n \|\|f\|\|_\infty, \tag{56}$$

i.e.

$$\|L_N(f)\| \leq M_n \|\|f\|\|_\infty, \quad \forall\ N \in \mathbb{N}, \tag{57}$$

so that L_N is a bounded linear operator, $\forall\ N \in \mathbb{N}$.

3. Main Results

We present our first main result

Theorem 29. *Let $([a,b], \Sigma, \mu)$ be a complete measure space with μ a positive finite measure. Let $f \in H_{x_0}^{(1)}$, $x_0 \in [a,b]$; $r_1, r_2 > 0$, $0 < \alpha \notin \mathbb{N}$. Then*

$$\left\|\int_{[a,b]} f(x)\,d\mu(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \int_{[a,b]} (x - x_0)^k\,d\mu(x)\right\|$$
$$\leq \frac{1}{\Gamma(\alpha+1)} \left\{ \left[(\mu([a,x_0]))^{\frac{1}{\alpha+1}} + \frac{1}{(\alpha+1)\,r_1} \right] \right.$$

$$\times\,\omega_1\left(D^{\alpha}_{x_0-}f, r_1\left(\int_{[a,x_0]}(x_0-x)^{\alpha+1}d\mu(x)\right)^{\frac{1}{(\alpha+1)}}\right)_{[a,x_0]}$$

$$\times\left(\int_{[a,x_0]}(x_0-x)^{\alpha+1}d\mu(x)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}+\left[(\mu((x_0,b]))^{\frac{1}{(\alpha+1)}}+\frac{1}{(\alpha+1)r_2}\right]$$

$$\times\,\omega_1\left(D^{\alpha}_{*x_0}f, r_2\left(\int_{(x_0,b]}(x-x_0)^{\alpha+1}d\mu(x)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,b]}$$

$$\times\left(\int_{(x_0,b]}(x-x_0)^{\alpha+1}d\mu(x)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}\Bigg\}. \tag{58}$$

Proof. By (17) we obtain

$$\|E_{x_0}\| = \left\|\int_{[a,b]}f(x)\,d\mu(x)-\sum_{i=0}^{m-1}\frac{f^{(i)}(x_0)}{i!}\int_{[a,b]}(x-x_0)^i d\mu(x)\right\|$$

$$=\frac{1}{\Gamma(\alpha)}\Bigg\|\Bigg\{\int_{[a,x_0]}\left(\int_x^{x_0}(z-x)^{\alpha-1}\left(D^{\alpha}_{x_0-}f(z)-\left(D^{\alpha}_{x_0-}f\right)(x_0)\right)dz\right)d\mu(x)$$

$$+\int_{(x_0,b]}\left(\int_{x_0}^{x}(x-z)^{\alpha-1}\left(\left(D^{\alpha}_{*x_0}f\right)(z)-\left(D^{\alpha}_{*x_0}f\right)(x_0)\right)dz\right)d\mu(x)\Bigg\}\Bigg\| \tag{59}$$

$$\leq\frac{1}{\Gamma(\alpha)}\Bigg\{\left\|\int_{[a,x_0]}\left(\int_x^{x_0}(z-x)^{\alpha-1}\left(D^{\alpha}_{x_0-}f(z)-\left(D^{\alpha}_{x_0-}f\right)(x_0)\right)dz\right)d\mu(x)\right\|$$

$$+\left\|\int_{(x_0,b]}\left(\int_{x_0}^{x}(x-z)^{\alpha-1}\left(\left(D^{\alpha}_{*x_0}f\right)(z)-\left(D^{\alpha}_{*x_0}f\right)(x_0)\right)dz\right)d\mu(x)\right\|\Bigg\}$$

$$\leq\frac{1}{\Gamma(\alpha)}\Bigg\{\left(\int_{[a,x_0]}\left\|\int_x^{x_0}(z-x)^{\alpha-1}\left(D^{\alpha}_{x_0-}f(z)-\left(D^{\alpha}_{x_0-}f\right)(x_0)\right)dz\right\|d\mu(x)\right)$$

$$+\left(\int_{(x_0,b]}\left\|\int_{x_0}^{x}(x-z)^{\alpha-1}\left(\left(D^{\alpha}_{*x_0}f\right)(z)-\left(D^{\alpha}_{*x_0}f\right)(x_0)\right)dz\right\|d\mu(x)\right)\Bigg\} \tag{60}$$

$$\leq\frac{1}{\Gamma(\alpha)}\Bigg\{\left(\int_{[a,x_0]}\left(\int_x^{x_0}(z-x)^{\alpha-1}\left\|D^{\alpha}_{x_0-}f(z)-\left(D^{\alpha}_{x_0-}f\right)(x_0)\right\|dz\right)d\mu(x)\right)$$

$$+\left(\int_{(x_0,b]}\left(\int_{x_0}^{x}(x-z)^{\alpha-1}\left\|\left(D^{\alpha}_{*x_0}f\right)(z)-\left(D^{\alpha}_{*x_0}f\right)(x_0)\right\|dz\right)d\mu(x)\right)\Bigg\}$$

$$\overset{(h_1,h_2>0)}{\leq}\frac{1}{\Gamma(\alpha)}\Bigg\{\left[\int_{[a,x_0]}\left(\int_x^{x_0}(z-x)^{\alpha-1}\left(1+\frac{x_0-z}{h_1}\right)dz\right)d\mu(x)\right]$$

$$\times\,\omega_1\left(D^{\alpha}_{x_0-}f, h_1\right)_{[a,x_0]}$$

$$+\left[\int_{(x_0,b]}\left(\int_{x_0}^{x}(x-z)^{\alpha-1}\left(1+\frac{z-x_0}{h_2}\right)dz\right)d\mu(x)\right]\omega_1\left(D^{\alpha}_{*x_0}f, h_2\right)_{[x_0,b]}\Bigg\}. \tag{61}$$

I.e. it holds

$$\|E_{x_0}\|\leq\frac{1}{\Gamma(\alpha)}$$

$$\times\left\{\left[\int_{[a,x_0]}\left(\int_x^{x_0}(z-x)^{\alpha-1}\left(1+\frac{x_0-z}{h_1}\right)dz\right)d\mu(x)\right]\omega_1\left(D_{x_0-}^{\alpha}f,h_1\right)_{[a,x_0]}\right.$$

$$\left.+\left[\int_{(x_0,b]}\left(\int_{x_0}^{x}(x-z)^{\alpha-1}\left(1+\frac{z-x_0}{h_2}\right)dz\right)d\mu(x)\right]\omega_1\left(D_{*x_0}^{\alpha}f,h_2\right)_{[x_0,b]}\right\} \tag{62}$$

$$=\frac{1}{\Gamma(\alpha)}\left\{\left[\int_{[a,x_0]}\left(\frac{(x_0-x)^{\alpha}}{\alpha}+\frac{1}{h_1}\left(\int_x^{x_0}(x_0-z)^{2-1}(z-x)^{\alpha-1}dz\right)\right)d\mu(x)\right]\right.$$

$$\times\,\omega_1\left(D_{x_0-}^{\alpha}f,h_1\right)_{[a,x_0]}$$

$$+\left[\int_{(x_0,b]}\left(\frac{(x-x_0)^{\alpha}}{\alpha}+\frac{1}{h_2}\left(\int_{x_0}^{x}(x-z)^{\alpha-1}(z-x_0)^{2-1}dz\right)\right)d\mu(x)\right]$$

$$\left.\omega_1\left(D_{*x_0}^{\alpha}f,h_2\right)_{[x_0,b]}\right\} \tag{63}$$

$$=\frac{1}{\Gamma(\alpha)}\left\{\left[\int_{[a,x_0]}\left(\frac{(x_0-x)^{\alpha}}{\alpha}+\frac{1}{h_1}\frac{(x_0-x)^{\alpha+1}}{\alpha(\alpha+1)}\right)d\mu(x)\right]\omega_1\left(D_{x_0-}^{\alpha}f,h_1\right)_{[a,x_0]}\right.$$

$$\left.+\left[\int_{(x_0,b]}\left(\frac{(x-x_0)^{\alpha}}{\alpha}+\frac{1}{h_2}\frac{(x-x_0)^{\alpha+1}}{\alpha(\alpha+1)}\right)d\mu(x)\right]\omega_1\left(D_{*x_0}^{\alpha}f,h_2\right)_{[x_0,b]}\right\}.$$

Therefore it holds

$$\|E_{x_0}\|\leq\frac{1}{\Gamma(\alpha)}$$

$$\times\left\{\left[\frac{1}{\alpha}\int_{[a,x_0]}(x_0-x)^{\alpha}d\mu(x)+\frac{1}{h_1\alpha(\alpha+1)}\int_{[a,x_0]}(x_0-x)^{\alpha+1}d\mu(x)\right]\right. \tag{64}$$

$$\omega_1\left(D_{x_0-}^{\alpha}f,h_1\right)_{[a,x_0]}$$

$$+\left[\frac{1}{\alpha}\int_{(x_0,b]}(x-x_0)^{\alpha}d\mu(x)+\frac{1}{h_2\alpha(\alpha+1)}\int_{(x_0,b]}(x-x_0)^{\alpha+1}d\mu(x)\right]$$

$$\left.\omega_1\left(D_{*x_0}^{\alpha}f,h_2\right)_{[x_0,b]}\right\}.$$

Momentarily we assume positive choices of

$$h_1=r_1\left(\int_{[a,x_0]}(x_0-x)^{\alpha+1}d\mu(x)\right)^{\frac{1}{(\alpha+1)}}>0, \tag{65}$$

and

$$h_2=r_2\left(\int_{(x_0,b]}(x-x_0)^{\alpha+1}d\mu(x)\right)^{\frac{1}{(\alpha+1)}}>0. \tag{66}$$

Consequently we obtain

$$\|E_{x_0}\|\leq\frac{1}{\Gamma(\alpha)}\left\{\left[\mu([a,x_0])^{\frac{1}{(\alpha+1)}}+\frac{1}{(\alpha+1)r_1}\right]\omega_1\left(D_{x_0-}^{\alpha}f,h_1\right)_{[a,x_0]}\left(\frac{h_1}{r_1}\right)^{\alpha}\right.$$

$$\left.+\left[(\mu((x_0,b]))^{\frac{1}{(\alpha+1)}}+\frac{1}{r_2(\alpha+1)}\right]\omega_1\left(D_{*x_0}^{\alpha}f,h_2\right)_{[x_0,b]}\left(\frac{h_2}{r_2}\right)^{\alpha}\right\}, \tag{67}$$

proving (58).

Next we examine special cases. If $\int_{(x_0,b]} (x-x_0)^{\alpha+1} d\mu(x) = 0$, then $(x-x_0)=0$, a.e. on $(x_0,b]$, that is $x=x_0$ a.e. on $(x_0,b]$, more precisely $\mu\{x \in (x_0,b] : x \neq x_0\} = 0$, hence $\mu(x_0,b]=0$.

Therefore μ concentrates on $[a,x_0]$.

In that case inequality (58) is written and holds as

$$\left\| \int_{[a,x_0]} f(x)\, d\mu(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \int_{[a,x_0]} (x-x_0)^k d\mu(x) \right\| \leq \frac{1}{\Gamma(\alpha+1)} \left\{ \left[(\mu([a,x_0]))^{\frac{1}{(\alpha+1)}} + \frac{1}{(\alpha+1) r_1} \right] \omega_1 \left(D^{\alpha}_{x_0-} f, r_1 \left(\int_{[a,x_0]} (x_0-x)^{\alpha+1} d\mu(x) \right)^{\frac{1}{(\alpha+1)}} \right)_{[a,x_0]} \left(\int_{[a,x_0]} (x_0-x)^{\alpha+1} d\mu(x) \right)^{\left(\frac{\alpha}{\alpha+1}\right)} \right\}. \tag{68}$$

Since $(b,b]=\emptyset$ and $\mu(\emptyset)=0$, in the case of $x_0=b$, we get again (68) written for $x_0=b$. So inequality (68) is a valid inequality when $\int_{[a,x_0]} (x_0-x)^{\alpha+1} d\mu(x) \neq 0$.

If additionally we assume that $\int_{[a,x_0]} (x_0-x)^{\alpha+1} d\mu(x) = 0$, then $(x_0-x)=0$, a.e. on $[a,x_0]$, that is $x=x_0$ a.e. on $[a,x_0]$, which means $\mu\{x\in[a,x_0] : x\neq x_0\}=0$. Hence $\mu=\delta_{x_0}M$, where δ_{x_0} is the unit Dirac measure and $M=\mu([a,b])>0$.

In the last case we obtain that L.H.S.(68)=R.H.S.(68)=0, that is (68) is valid trivially.

Finally let us go the other way around. Let us assume that $\int_{[a,x_0]}(x_0-x)^{\alpha+1}d\mu(x) = 0$, then reasoning similarly as before we get that μ over $[a,x_0]$ concentrates at x_0. That is $\mu=\delta_{x_0}\mu([a,x_0])$, on $[a,x_0]$.

In the last case (58) is written and it holds as

$$\left\| \int_{(x_0,b]} f(x)\, d\mu(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \int_{(x_0,b]} (x-x_0)^k d\mu(x) \right\| \leq \frac{1}{\Gamma(\alpha+1)} \left\{ \left[(\mu((x_0,b]))^{\frac{1}{(\alpha+1)}} + \frac{1}{(\alpha+1) r_2} \right] \omega_1 \left(D^{\alpha}_{*x_0} f, r_2 \left(\int_{(x_0,b]} (x-x_0)^{\alpha+1} d\mu(x) \right)^{\frac{1}{(\alpha+1)}} \right)_{[x_0,b]} \left(\int_{(x_0,b]} (x-x_0)^{\alpha+1} d\mu(x) \right)^{\frac{\alpha}{(\alpha+1)}} \right\}. \tag{69}$$

If $x_0=a$ then (69) can be redone and rewritten, just replace $(x_0,b]$ by $[a,b]$ all over.

So inequality (69) is valid when

$$\int_{(x_0,b]} (x-x_0)^{\alpha+1} \, d\mu(x) \neq 0.$$

If additionally we assume that $\int_{(x_0,b]} (x-x_0)^{\alpha+1} \, d\mu(x) = 0$, then as before $\mu(x_0,b] = 0$. Hence (69) is trivially true, in fact L.H.S.(69)=R.H.S.(69)=0.

The proof of (58) now has been completed in all possible cases. □

We continue in a special case.

In the assumptions of Theorem 29, when $r = r_1 = r_2 > 0$, and by calling $M = \mu([a,b]) \geq \mu([a,x_0]), \mu((x_0,b])$, we get

Corollary 30. *It holds*

$$\left\| \int_{[a,b]} f(x)\, d\mu(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \int_{[a,b]} (x-x_0)^k \, d\mu(x) \right\|$$
$$\leq \frac{1}{\Gamma(\alpha+1)} \left[\left[M^{\frac{1}{(\alpha+1)}} + \frac{1}{(\alpha+1)r} \right] \right.$$
$$\left[\omega_1 \left(D^{\alpha}_{x_0-} f, r \left(\int_{[a,x_0]} (x_0-x)^{\alpha+1} \, d\mu(x) \right)^{\frac{1}{(\alpha+1)}} \right)_{[a,x_0]} \right.$$
$$\left(\int_{[a,x_0]} (x_0-x)^{\alpha+1} \, d\mu(x) \right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$
$$+ \omega_1 \left(D^{\alpha}_{*x_0} f, r \left(\int_{[x_0,b]} (x-x_0)^{\alpha+1} \, d\mu(x) \right)^{\frac{1}{(\alpha+1)}} \right)_{[x_0,b]} \tag{70}$$
$$\left. \left(\int_{[x_0,b]} (x-x_0)^{\alpha+1} \, d\mu(x) \right)^{\left(\frac{\alpha}{\alpha+1}\right)} \right].$$

We need

Definition 31. Let $([a,b], \Sigma, \mu_N)$ be a complete measure space, with μ_N a positive finite measure, $\forall\, N \in \mathbb{N}$. Consider the positive linear functional

$$\widetilde{L_N}(f) = \int_{[a,b]} f(t)\, \mu_N(dt), \ \ \forall\, N \in \mathbb{N}, \tag{71}$$

$\forall\, f \in C([a,b])$.

Notice that $\widetilde{L_N}(1) = \mu_N([a,b]) =: M_N$. Let the constant $c \in X$ a Banach space, then $L_N(cf) = c\widetilde{L_N}(f)$, $\forall\, f \in C([a,b])$. We may use formula (71) for $f \in L_1([a,b])$ with respect to μ_N.

Based on Theorem 29 and Corollary 30 we give

Theorem 32. *Let* $([a,b],\Sigma,\mu_N)$ *be a complete measure space with* μ_N *a positive finite measure,* $\forall\ N\in\mathbb{N}$. *Let* $f\in H_{x_0}^{(1)}$, $x_0\in[a,b]$; $r>0$, $0<\alpha\notin\mathbb{N}$. *Let the linear operators* L_N *as in* (52) *and the linear functionals* $\widetilde{L_N}$ *as in* (71). *Then*

$$\left\|L_n(f)-\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}\widetilde{L_N}\left((x-x_0)^k\right)\right\|$$

$$\leq\frac{1}{\Gamma(\alpha+1)}\left[\left(\widetilde{L_N}(1)\right)^{\frac{1}{\alpha+1}}+\frac{1}{(\alpha+1)r}\right]$$

$$\left[\omega_1\left(D_{x_0-}^{\alpha}f,r\left(\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\chi_{[a,x_0]}(x)\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[a,x_0]}\right. \tag{72}$$

$$\left(\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\chi_{[a,x_0]}(x)\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$+\,\omega_1\left(D_{*x_0}^{\alpha}f,r\left(\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\chi_{[x_0,b]}(x)\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,b]}$$

$$\left.\left(\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\chi_{[x_0,b]}(x)\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}\right]$$

$$\leq\frac{1}{\Gamma(\alpha+1)}\left[\left(\widetilde{L_N}(1)\right)^{\frac{1}{\alpha+1}}+\frac{1}{(\alpha+1)r}\right]$$

$$\left[\omega_1\left(D_{x_0-}^{\alpha}f,r\left(\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[a,x_0]}\right. \tag{73}$$

$$\left.\omega_1\left(D_{*x_0}^{\alpha}f,r\left(\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,b]}\right]\left(\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)},$$

$\forall\ N\in\mathbb{N}$, *where in the above* χ *stands for the characteristic function.*

We make

Definition 33. We call $(x_0\in[a,b]\subset\mathbb{R})$

$$\widetilde{H}_{x_0}^{(1)}:=\{f\in C([a,b],X):[a,b]\subset\mathbb{R},(X,\|\cdot\|) \tag{74}$$

is a Banach space, $0<\alpha<1$; $f'\in L_\infty([a,b],X)$; f' exists outside a λ-null Borel set $B_x^{(1)}\subseteq[x,x_0]$, such that $h_1\left(f\left(B_x^{(1)}\right)\right)=0$, $\forall\ x\in[a,x_0]$; f' exists outside a λ-null Borel set $B_x^{(2)}\subseteq[x_0,x]$, such that $h_1\left(f\left(B_x^{(2)}\right)\right)=0$, $\forall\ x\in[x_0,b]\}$.

Notice that $C^1([a,b],X)\subset\widetilde{H}_{x_0}^{(1)}$, $\forall\ x_0\in[a,b]$.

The last Definition 33 simplifies a lot Definition 10 when $m=1$.

Because h_1 is an outer measure on the power set $\mathcal{P}(X)$ we can further simplify Definition 33, based on $f(\emptyset)=\emptyset$, $h_1(\emptyset)=0$, and $A\subset B$ implies $h_1(A)\leq h_1(B)$, as follows:

Remark 34. Let $x_0 \in [a,b] \subset \mathbb{R}$. We have that

$$\widetilde{H}_{x_0}^{(1)} := \{f \in C([a,b],X) : (X, \|\cdot\|) \tag{75}$$

is a Banach space, $0<\alpha<1$; $f' \in L_\infty([a,b],X)$; f' exists outside a λ-null Borel set $B_a \subseteq [a,x_0]$, such that $h_1(f(B_a)) = 0$; f' exists outside a λ-null Borel set $B_b \subseteq [x_0,b]$, such that $h_1(f(B_b)) = 0\}$.

We make

Remark 35. In the setting of Theorem 32 we observe:

$$\begin{aligned} L_N(f) - f(x_0) &= L_N(f) - f(x_0)\widetilde{L_N}(1) + f(x_0)\widetilde{L_N}(1) - f(x_0) \\ &= \left(L_N(f) - f(x_0)\widetilde{L_N}(1)\right) + f(x_0)\left(\widetilde{L_N}(1) - 1\right). \end{aligned} \tag{76}$$

Hence it holds

$$\|L_N(f) - f(x_0)\| \le \left\|L_N(f) - f(x_0)\widetilde{L_N}(1)\right\| + \|f(x_0)\|\left|\widetilde{L_N}(1) - 1\right|. \tag{77}$$

Next we apply Theorem 32 when $0<\alpha<1$, i.e. $m = \lceil \alpha \rceil = 1$.

Theorem 36. *Let $([a,b],\Sigma,\mu_N)$ be a complete measure space with μ_N a positive finite measure, $\forall\ N \in \mathbb{N}$. Let $f \in \widetilde{H}_{x_0}^{(1)}$, $x_0 \in [a,b]$; $r>0$, $0<\alpha<1$. Let L_N as in* (52) *and $\widetilde{L_N}$ as in* (71)*. Then*

$$\begin{aligned} &\|L_N(f) - f(x_0)\| \le \|f(x_0)\|\left|\widetilde{L_N}(1) - 1\right| \\ &+ \frac{1}{\Gamma(\alpha+1)}\left[\left(\widetilde{L_N}(1)\right)^{\frac{1}{\alpha+1}} + \frac{1}{(\alpha+1)r}\right] \\ &\left[\omega_1\left(D_{x_0-}^{\alpha}f, r\left(\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\chi_{[a,x_0]}(x)\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[a,x_0]}\right. \\ &\left(\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\chi_{[a,x_0]}(x)\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)} \\ &+ \omega_1\left(D_{*x_0}^{\alpha}f, r\left(\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\chi_{[x_0,b]}(x)\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,b]} \\ &\left.\left(\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\chi_{[x_0,b]}(x)\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}\right] \end{aligned} \tag{78}$$

$$\begin{aligned} &\le \|f(x_0)\|\left|\widetilde{L_N}(1) - 1\right| + \frac{1}{\Gamma(\alpha+1)}\left[\left(\widetilde{L_N}(1)\right)^{\frac{1}{\alpha+1}} + \frac{1}{(\alpha+1)r}\right] \\ &\left[\omega_1\left(D_{x_0-}^{\alpha}f, r\left(\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[a,x_0]}\right. \\ &\left.+ \omega_1\left(D_{*x_0}^{\alpha}f, r\left(\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,b]}\right]\left(\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}, \end{aligned} \tag{79}$$

$\forall\ N \in \mathbb{N}$.

We conclude the following convergence result (Korovkin type):

Corollary 37. *All as in Theorem* 36. *Assume that* $\widetilde{L_N}(1) \to 1$, *as* $N \to \infty$, *and* $\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\right) \to 0$, *as* $N \to \infty$. *Then* $L_N(f) \overset{\|\cdot\|}{\to} f(x_0)$, *as* $N \to \infty$.

Proof. By inequalities (78)–(79). Notice also the fact

$$\widetilde{L_N}(1) \leq \left|\widetilde{L_N}(1) - 1\right| + 1 \leq K + 1, \;\; K > 0, \tag{80}$$

because $\left|\widetilde{L_N}(1) - 1\right| \leq K$, as $\left|\widetilde{L_N}(1) - 1\right| \to 0$, with $N \to \infty$.

That is $\widetilde{L_N}(1)$ is bounded. □

It follows:

Theorem 38. *Here all as in Theorem* 32. *Then*

$$\begin{aligned} \|L_N(f) - f(x_0)\| &\leq \|f(x_0)\| \left|\widetilde{L_N}(1) - 1\right| \\ &+ \sum_{k=1}^{m-1} \frac{\left\|f^{(k)}(x_0)\right\|}{k!} \left|\widetilde{L_N}\left((x-x_0)^k\right)\right| \\ &+ \frac{1}{\Gamma(\alpha+1)} \left[\left(\widetilde{L_N}(1)\right)^{\frac{1}{\alpha+1}} + \frac{1}{(\alpha+1)r}\right] \\ &\left[\omega_1\left(D_{x_0-}^{\alpha} f, r\left(\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[a,x_0]}\right. \\ &\left.+\, \omega_1\left(D_{*x_0}^{\alpha} f, r\left(\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,b]}\right] \left(\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}, \end{aligned} \tag{81}$$

$\forall \; N \in \mathbb{N}$.

Proof. We may write:

$$\begin{aligned} L_N(f) - f(x_0) &= L_N(f) - f(x_0) + \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \widetilde{L_N}\left((x-x_0)^k\right) \\ &- \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \widetilde{L_N}\left((x-x_0)^k\right) \\ &= \left(L_N(f) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \widetilde{L_N}\left((x-x_0)^k\right)\right) - f(x_0) \\ &+ f(x_0)\widetilde{L_N}(1) + \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} \widetilde{L_N}\left((x-x_0)^k\right) \end{aligned}$$

$$= \left[L_N(f) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \widetilde{L_N}\left((x-x_0)^k\right)\right] \tag{82}$$

$$+ f(x_0)\left(\widetilde{L_N}(1) - 1\right) + \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} \widetilde{L_N}\left((x-x_0)^k\right).$$

Hence it holds

$$\|L_N(f) - f(x_0)\| \leq \|f(x_0)\| \left|\left(\widetilde{L_N}(1) - 1\right)\right|$$
$$\sum_{k=1}^{m-1} \frac{\left\|f^{(k)}(x_0)\right\|}{k!} \left|\widetilde{L_N}\left((x-x_0)^k\right)\right|$$
$$+ \left\| L_N(f) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \widetilde{L_N}\left((x-x_0)^k\right) \right\|$$
$$\overset{\text{(bt (72)-(73))}}{\leq} \|f(x_0)\| \left|\widetilde{L_N}(1) - 1\right| + \sum_{k=1}^{m-1} \frac{\left\|f^{(k)}(x_0)\right\|}{k!} \left|\widetilde{L_N}\left((x-x_0)^k\right)\right|$$
$$+ \frac{1}{\Gamma(\alpha+1)} \left[\left(\widetilde{L_N}(1)\right)^{\frac{1}{\alpha+1}} + \frac{1}{(\alpha+1) r} \right] \tag{83}$$
$$\left[\omega_1 \left(D_{x_0-}^{\alpha} f, r \left(\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\right)\right)^{\frac{1}{(\alpha+1)}} \right)_{[a,x_0]} \right.$$
$$\left. + \omega_1 \left(D_{*x_0}^{\alpha} f, r \left(\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\right)\right)^{\frac{1}{(\alpha+1)}} \right)_{[x_0,b]} \right] \left(\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)},$$

$\forall\ N \in \mathbb{N}$. ∎

We make

Remark 39. By (50) we have that

$$\left|\widetilde{L_N}\left((x-x_0)^k\right)\right| \leq \widetilde{L_N}\left(|x-x_0|^k\right)$$
$$\leq \left(\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\right)\right)^{\frac{k}{\alpha+1}} \left(\widetilde{L_N}(1)\right)^{\left(\frac{\alpha+1-k}{\alpha+1}\right)}, \tag{84}$$

for $k = 1, \ldots, m-1$; $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$.

Next we use (81).

If clear if $\widetilde{L_N}(1) \to 1$ and $\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\right) \to 0$, as $N \to \infty$, we obtain again that $L_N(f) \overset{\|\cdot\|}{\to} f(x_0)$, as $N \to \infty$.

We state now the following convergence theorem (Korovkin type):

Theorem 40. *Let $([a,b], \Sigma, \mu_N)$ be a complete measure space with μ_N a positive finite measure, $\forall\ N \in \mathbb{N}$. Let $x_0 \in [a,b]$, $0 < \alpha \notin \mathbb{N}$; L_N as in (52) and $\widetilde{L_N}$ as in (71), $\forall\ N \in \mathbb{N}$. Assume that $\widetilde{L_N}(1) \to 1$ and $\widetilde{L_N}\left(|x-x_0|^{\alpha+1}\right) \to 0$, as $N \to \infty$. Then $L_N(f) \overset{\|\cdot\|}{\to} f(x_0)$, as $N \to \infty$, $\forall\ f \in H_{x_0}^{(1)}$.*

4. **Application**

Here $[a,b]=[0,1]$.

Consider $g\in C([0,1])$ and the classic Bernstein polynomials

$$\left(\widetilde{B_N}g\right)(t)=\sum_{k=0}^{N} g\left(\frac{k}{N}\right)\binom{N}{k} t^k(1-t)^{N-k},\quad \forall\, t\in[0,1],\quad N\in\mathbb{N}. \tag{85}$$

Let $x_0\in[0,1]$ be fixed, then

$$\left(\widetilde{B_N}g\right)(x_0)=\sum_{k=0}^{N} g\left(\frac{k}{N}\right)\binom{N}{k} x_0^k(1-x_0)^{N-k}. \tag{86}$$

We have that $\left(\widetilde{B_N}1\right)=1$, and $\widetilde{B_N}$ are positive linear operators. The last means $\left(\widetilde{B_N}1\right)(x_0)=1$.

Let $(X,\|\cdot\|)$ be a Banach space, and $f\in H_{x_0}^{(1)}$; $r>0$, $0<\alpha<1$.

We consider the vector valued in X Bernstein linear operators

$$(B_Nf)(x_0)=\sum_{k=0}^{N} f\left(\frac{k}{N}\right)\binom{N}{k} x_0^k(1-x_0)^{N-k},\quad N\in\mathbb{N}. \tag{87}$$

That is $(B_Nf)(x_0)\in X$.

Applying Theorem 36 we get

Corollary 41. *It holds*

$$\begin{aligned}\|(B_Nf)(x_0)-f(x_0)\| &\le \frac{1}{\Gamma(\alpha+1)}\left[1+\frac{1}{(\alpha+1)r}\right]\\ &\left[\omega_1\left(D_{x_0-}^{\alpha}f, r\left(\left(\widetilde{B_N}\left(|x-x_0|^{\alpha+1}\right)\right)(x_0)\right)^{\frac{1}{(\alpha+1)}}\right)_{[0,x_0]}\right.\\ &\left.+\,\omega_1\left(D_{*x_0}^{\alpha}f, r\left(\left(\widetilde{B_N}\left(|x-x_0|^{\alpha+1}\right)\right)(x_0)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,1]}\right]\\ &\left(\left(\widetilde{B_N}\left(|x-x_0|^{\alpha+1}\right)\right)(x_0)\right)^{\left(\frac{\alpha}{\alpha+1}\right)},\quad \forall\, N\in\mathbb{N}.\end{aligned} \tag{88}$$

Next let $\alpha=\frac{1}{2}$, $r=\frac{1}{\alpha+1}$, that is $r=\frac{2}{3}$. Notice $\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2}$.

Corollary 42. *It holds*

$$\begin{aligned}&\|(B_Nf)(x_0)-f(x_0)\|\\ &\le \frac{4}{\sqrt{\pi}}\left[\omega_1\left(D_{x_0-}^{\frac{1}{2}}f,\frac{2}{3}\left(\left(\widetilde{B_N}\left(|x-x_0|^{\frac{3}{2}}\right)\right)(x_0)\right)^{\frac{2}{3}}\right)_{[0,x_0]}\right.\\ &\left.+\,\omega_1\left(D_{*x_0}^{\frac{1}{2}}f,\frac{2}{3}\left(\left(\widetilde{B_N}\left(|x-x_0|^{\frac{3}{2}}\right)\right)(x_0)\right)^{\frac{2}{3}}\right)_{[x_0,1]}\right]\end{aligned}$$

$$\left(\left(\widetilde{B_N} \left(|x - x_0|^{\frac{3}{2}} \right) \right) (x_0) \right)^{\frac{1}{3}}, \quad \forall\ N \in \mathbb{N}. \tag{89}$$

We have that (see [5])

$$\left(\widetilde{B_N} \left(|x - x_0|^{\frac{3}{2}} \right) \right) (x_0) \leq \frac{1}{(4N)^{\frac{3}{4}}}, \quad \forall\ x_0 \in [0,1]. \tag{90}$$

We have proved

Corollary 43. *Here* $[a,b] = [0,1]$, $x_0 \in [0,1]$. *Let* $f \in \widetilde{H}_{x_0}^{(1)}$, $\alpha = \frac{1}{2}$, $N \in \mathbb{N}$. *Then*

$$\begin{aligned} &\| (B_N f)(x_0) - f(x_0) \| \\ &\leq \frac{2^{\frac{3}{2}}}{\sqrt{\pi} \sqrt[4]{N}} \left[\omega_1 \left(D_{x_0-}^{\frac{1}{2}} f, \frac{1}{3\sqrt{N}} \right)_{[0,x_0]} + \omega_1 \left(D_{*x_0}^{\frac{1}{2}} f, \frac{1}{3\sqrt{N}} \right)_{[x_0,1]} \right]. \end{aligned} \tag{91}$$

Notice that $\frac{2^{\frac{3}{2}}}{\sqrt{\pi}} \approx 1.59$.

So as $N \to \infty$ *we derive that* $(B_N f)(x_0) \overset{\|\cdot\|}{\to} f(x_0)$, *quantitatively, where* $x_0 \in [0,1]$.

We finish with

Corollary 44. *Let* $f \in C^1([0,1], X)$, $(X, \|\cdot\|)$ *is a Banach space. Then*

$$\begin{aligned} &\| \| (B_N f)(x_0) - f(x_0) \| \|_{\infty,[0,1]} \\ &\leq \frac{2^{\frac{3}{2}}}{\sqrt{\pi} \sqrt[4]{N}} \left[\sup_{x_0 \in [0,1]} \omega_1 \left(D_{x_0-}^{\frac{1}{2}} f, \frac{1}{3\sqrt{N}} \right)_{[0,x_0]} + \sup_{x_0 \in [0,1]} \omega_1 \left(D_{*x_0}^{\frac{1}{2}} f, \frac{1}{3\sqrt{N}} \right)_{[x_0,1]} \right], \end{aligned} \tag{92}$$

$\forall\ N \in \mathbb{N}$.

So as $N \to \infty$, *we derive that* $\|B_N f - f\| \to 0$ *uniformly with rates.*

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