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IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS WITH CAUSAL OPERATORS

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ABSTRACT. In this paper we present an existence result for a class of impulsive differential equations with causal operators and prove that the solution set is compact in the space of regulated functions. The results are obtained under conditions with respect to the Hausdorff measure of noncompactness. An application from optimal control is given to illustrate our main result.

Key Words and Phrases: Impulsive differential equations, Causal operators, Regulated functions, Measure of noncompactness, Optimal control

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1. Introduction and preliminaries

The study of functional equations with causal operators has recently been developed and some results on existence, stability and control are found in the monographs [7, 14, 23]. The term causal operators or Volterra abstract operator was introduced by Tonelli [39] (see also Tikhonov [38], [40]). The theory of these operators has the advantage of unifying some classes of differential equations as: ordinary differential equations, integrodifferential equations, differential equations with finite or infinite delay, Volterra integral equations, and neutral functional equations, and so on. Many papers in the literature address various aspects of the theory of causal operators. Control problems involving causal operators were studied in [4, 8, 18, 36]. A new class of abstract integral equations has been introduced in [17]. We note that differential equations with causal operators were studied by several authors, see [1], [3], [9]–[12], [19], [25]–[33] and the references therein. The properties of the solutions of the differential equations with causal operators were studied in [2, 21, 34, 35, 40]. The existence of solutions for impulsive differential equations with causal operators in finite dimensional spaces were studied in [19, 26].

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Let E be a real separable Banach space endowed with the norm $\|\cdot\|$. For $x \in E$ and $r > 0$ let $B_r(x) := \{y \in E; \|y - x\| < r\}$ be the open ball centered at x with radius r, and let $B_r[x]$ be its closure. The space of all (classes of) strongly measurable functions $u(\cdot) : [0, b] \to E$ such that

$$\|u(\cdot)\|_p := \left(\int_0^b \|u(t)\|^p\right)^{1/p} < \infty$$

for $1 \leq p < \infty$ and $\|u(\cdot)\|_\infty := \operatorname{ess\,sup}_{t\in[0,b]} \|u(t)\| < \infty$, will be denoted by $L^p([0, b], E)$. This is a Banach space with respect to the norm $\|u(\cdot)\|_p$. We denote by $PC([0, b], E)$ the set of all functions $u : [0, b] \to E$ such that u is continuous at $t \neq t_k$, left continuous at $t = t_k$ and the right limit $u(t_k^+)$ exists for $k = 1, 2, \ldots, m$. Then $PC([0, b], E)$ is a Banach space with respect to the norm $\|u(\cdot)\| = \sup_{0\leq t\leq b} \|u\,(t)\|$.

The following definition of causal operator was given by Tonelli [39].

An operator $Q : PC([0, b], E) \to L^p_{loc}([0, b], E)$ is a *causal operator* or *a Volterra operator* if, for each $\tau \in [0, b)$ and for all $u(\cdot), v(\cdot) \in PC([0, b], E)$ with $u(t) = v(t)$ for every $t \in [0, \tau]$, we have $Qu(t) = Qv(t)$ for a.e. $t \in [0, \tau]$.

In this paper we study the following functional impulsive differential equation:

$$\begin{cases} u'(t) = (Qu(t)), & \text{a.e. } t \in [0, b] \smallsetminus \{t_1, t_2, \ldots, t_m\}, \\ u(t_k^+) = u(t_k^-) + I_k(u(t_k^-)), & k = 1, 2, \ldots, m, \\ u(0) = \xi, \end{cases} \tag{1.1}$$

where $Q : PC([0, b], E) \to L^p([0, b], E)$, $1 \leq p \leq \infty$, is a continuous causal operator, $\xi \in E$, $m \in \mathbb{N}$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = b$ and $I_k : E \to E$ is a continuous operator for each $k = 1, 2, \ldots, m$. Now we provide some examples of impulsive differential equations that can be included in impulsive differential equations with causal operators of the form (1.1). The impulsive differential equation

$$\begin{cases} u'(t) = F(t, u(t), & \text{a.e. } t \in [0, b] \smallsetminus \{t_1, t_2, \ldots, t_m\}, \\ u(t_k^+) = u(t_k^-) + I_k(u(t_k^-)), & k = 1, 2, \ldots, m, \\ u(0) = \xi, \end{cases}$$

can be considered as a causal impulsive differential equations by identifying $F(t, u(t))$ with $(Qu)(t)$. Another example is the general integro-differential equation

$$\begin{cases} u'(t) = F(t, u(t), \int_0^t K(t, s, u(s))ds), & \text{a.e. } t \in [0, b] \smallsetminus \{t_1, t_2, \ldots, t_m\}, \\ u(t_k^+) = u(t_k^-) + I_k(u(t_k^-)), & k = 1, 2, \ldots, m, \\ u(0) = \xi. \end{cases} \tag{1.2}$$

Also, the differential equation with "maxima":

$$\begin{cases} u'(t) = F\left(t, u(t), \max\limits_{0\le s\le t} u(s)\right), & \text{a.e. } t \in [0,b] \setminus \{t_1, t_2, \ldots, t_m\}, \\ u(t_k^+) = u(t_k^-) + I_k(u(t_k^-)), & k = 1, 2, \ldots, m, \\ u(0) = \xi, \end{cases}$$

is another example of a causal impulsive differential equation. Finally, we remark that the Fredholm operator, given by

$$(Qu)(t) = \int_0^a K(t, s, u(s))ds,$$

is a causal operator if and only if $K(t, s, u) \equiv 0$ for $t < s < a$.

We denote by $\chi(A)$ the Hausdorff measure of non-compactness of a nonempty bounded set $A \subset E$, and it is defined by ([13], [20]):

$$\chi(A) = \inf\{\varepsilon > 0; A \text{ admits a finite cover by balls of radius } \le \varepsilon\}.$$

This is equivalent to the measure of non-compactness introduced by Kuratowski (see [13], [20]).

If $\dim(A) = \sup\{\|x - y\|; x, y \in A\}$ is the diameter of the bounded set A, then we have that $\chi(A) \le \dim(A)$ and $\chi(A) \le 2d$ if $\sup_{x\in A} \|x\| \le d$. We recall some properties of χ (see [13], [20]). If A, B are bounded subsets of E and $\overline{A}$ denotes the closure of A, then

(1) $\chi(A) = 0$ if and only if $\overline{A}$ is compact;
(2) $\chi(A) = \chi(\overline{A}) = \chi(\overline{co}(A))$;
(3) $\chi(\lambda A) = |\lambda|\chi(A)$ for every $\lambda \in \mathbb{R}$;
(4) $\chi(A) \le \chi(B)$ if $A \subset B$;
(5) $\chi(A + B) = \chi(A) + \chi(B)$;
(6) If $T : E \to E$ is a bounded linear operator, then $\gamma(TA) \le \|T\|\,\gamma(A)$.

If $V \subset PC([0, b], E)$ is equicontinuous, then

$$\chi_{PC}(V) := \sup_{t\in[0,b]} \chi(V(t)),$$

where $V(t) := \{u(t) : u(\cdot) \in V\}$, is the Hausdorff measure of non-compactness in the space $PC([0, b], E)$ (see [13]).

We recall the following lemma due to Kisielewicz [22, Lemma 2.2].

Lemma 1.1. *Let $\{u_n(\cdot); n \ge 1\}$ be a subset in $L^1([0, b], E)$ for which there exists $m(\cdot) \in L^1([0, b], \mathbb{R}_+)$ such that $\|u_n(t)\| \le m(t)$ for each $n \ge 1$ and for a.e. $t \in [0, b]$. Then the function $t \mapsto \chi(t) := \chi(\{u_n(t); n \ge 1\})$ is integrable on $[0, b]$ and, for each $t \in [0, b]$, we have*

$$\chi\left(\left\{\int_0^t u_n(t)dt; n \ge 1\right\}\right) \le \int_0^t \chi(t)dt.$$

2. An existence result

Consider the following functional impulsive differential equation:

$$
\begin{cases} u'(t) = (Qu(t)), & \text{a.e. } t \in [0,b] \setminus \{t_1, t_2, \dots, t_m\}, \\ u(t_k^+) = u(t_k^-) + I_k(u(t_k^-)), & k = 1, 2, \dots, m, \\ u(0) = \xi, \end{cases} \tag{2.1}
$$

where $Q : PC([0,b],E) \to L^p([0,b],E)$, $1 \le p \le \infty$, is a continuous causal operator, $\xi \in E$, $m \in \mathbb{N}$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = b$, $I_k : E \to E$ is continuous for each $k = 1, 2, \dots, m$.

We consider the following assumptions:

(H$_1$) $Q : PC([0,b],E) \to L^p([0,b],E)$, $1 \le p \le \infty$, is a continuous causal operator, and $I_k : E \to E$ is continuous for each $k = 1, 2, \dots, m$.

(H$_2$) For each $r > 0$ there exist $\psi, \eta \in L^p([0,b],\mathbb{R}_+)$ such that, for each $u(\cdot) \in PC([0,b],E)$ with $\sup\limits_{0 \le t \le b} \|u(t)\| \le r$, we have

$$
\|(Qu)(t)\| \le \psi(t) \quad \text{for a.e. } t \in [0,b],
$$

$$
\sum_{s < t_k < t} \left\| I_k(u(t_k^-)) \right\| \le \int_s^t \eta(\tau) d\tau \quad \text{for } s, t \in [0,b] \text{ with } s < t.
$$

(H$_3$) For each bounded subsets $A \subset PC([0,b],E)$ and $B \subset E$ there exist constants γ_A, $\delta_B^k > 0$ $(k = 1, 2, \dots, m)$ such that

$$
\chi((QA)(t)) \le \gamma_A \chi(A(t)), \tag{2.2}
$$

and

$$
\chi(I_k(B)) \le \delta_B^k \chi(B), \quad k = 1, 2, \dots, m,
$$

for all $t \in [0,b]$, where $(QA)(t) := \{(Qu)(t) : u(\cdot) \in A\}$.

By solution of (2.1) we mean a function $u(\cdot) : [0,b] \to E$ such that $u(0) = \xi$, $u(\cdot)$ is continuous on (t_k, t_{k+1}) for $k = 1, 2, \dots, m$, $u'(t) = (Qu)(t)$ for a.e. $t \in [0,b] \setminus \{t_1, t_2, \dots, t_m\}$, and $u(t_k^+) = u(t_k^-) + I_k(u(t_k^-))$, $k = 1, 2, \dots, m$.

It is easy to show that (see [13]) a function $u(\cdot) \in PC([0,b],E)$ is a solution for (2.1) on $[0,b]$, if and only if

$$
u(t) = u(0) + \int_0^t (Qu)(s) ds + \sum_{0 < t_k < t} I_k(u(t_k^-)) \text{ for } t \in [0,b]. \tag{2.3}
$$

For a fixed $\xi \in E$, by $\mathcal{S}_T(\xi)$ we denote the set of solutions $u(\cdot)$ of Cauchy problem (2.1) on an interval $[0,T]$ with $T \in (0,b]$.

Theorem 2.1. *Let $Q : PC([0,b],E) \to L^p([0,b],E)$ be a causal operator such that conditions* (H$_1$)–(H$_3$) *hold. Then, for every $\xi \in E$, there exists $T \in (0,b]$ such that the set $\mathcal{S}_T(\xi)$ is nonempty and compact set in $PC([0,T],E)$.*

Proof. First we shall show that there exists $T \in (0, b]$ such that the set $S_T(\xi)$ is nonempty. Let $\delta > 0$ be any number and let $r := \|\xi\| + \delta$. We choose $T \in (0, b]$ such that

$$\int_0^T \psi(s)ds \leq \delta/4 \text{ and } \int_0^T \eta(s)ds \leq \delta/4$$

and we consider the set B defined as follows

$$B = \{u \in PC([0,T], E);\ \|u(t) - \xi\| \leq \delta\}.$$

Further on, we consider the integral operator $\Lambda : B \to PC([0,T], E)$ given by

$$(\Lambda u)(t) = \xi + \int_0^t (\mathfrak{C}u)(s)ds + \sum_{0<t_k<t} I_k(u(t_k^-)), \quad \text{for } t \in [0,T]$$

and we prove that this is a continuous operator from B into B. First, we observe that if $u(\cdot) \in B$, then $\sup_{0\leq t\leq b} \|u(t)\| < r$, and so $\|(\mathfrak{C}u)(t)\| \leq \psi(t)$ for a.e. $t \in [0,T]$. Hence, for each $u(\cdot) \in B$, we have

$$\begin{aligned}
\|(\Lambda u)(t) - \xi\| &\leq \int_0^t \|(\mathfrak{C}u)(s)\| \, ds + \sum_{0<t_k<t} \left\|I_k(u(t_k^-))\right\| \\
&\leq \int_0^T \|(\mathfrak{C}u)(s)\| \, ds + \sum_{0<t_k<T} \left\|I_k(u(t_k^-))\right\| \\
&\leq \int_0^T [\psi(t) + \eta(t)] \, dt \leq \delta
\end{aligned}$$

and thus, $\Lambda(B) \subset B$. Further on, let $u_n(\cdot) \to u(\cdot)$ in B. Then we have

$$\begin{aligned}
\|(\Lambda u_n)(t) - (\Lambda u)(t)\| &\leq \int_0^t \|(\mathfrak{C}u_n)(s) - (\mathfrak{C}u)(s)\| ds \\
&\quad + \sum_{0<t_k<t} \left\|I_k(u_n(t_k^-)) - I_k(u(t_k^-))\right\| \\
&\leq \int_0^T \|(\mathfrak{C}u_n)(s) - (\mathfrak{C}u)(s)\| ds \\
&\quad + \sum_{0<t_k<T} \left\|I_k(u_n(t_k^-)) - I_k(u(t_k^-))\right\| \\
&\leq T^{1/q} \left(\int_0^T \|(\mathfrak{C}u_n)(s) - (\mathfrak{C}u)(s)\|^p ds\right)^{1/p} \\
&\quad + \sum_{0<t_k<T} \left\|I_k(u_n(t_k^-)) - I_k(u(t_k^-))\right\|
\end{aligned}$$

if $1 \leq p < \infty$ and $1/p + 1/q = 1$, and

$$\begin{aligned}
\sup_{0\leq t\leq T} \|(Pu_n)(t) - (Pu)(t)\| &\leq T \operatorname*{ess\,sup}_{0\leq t\leq T} \|(\mathfrak{C}u_n)(s) - (\mathfrak{C}u)(s)\| \\
&\quad + \sum_{0<t_k<T} \left\|I_k(u_n(t_k^-)) - I_k(u(t_k^-))\right\|
\end{aligned}$$

if $p=\infty$. Using Lemma 1.15 from [30], by (H_1) and (H_2) it follows that

$$\sup_{0\le t\le T}\|(\Lambda u_m)(t)-(\Lambda u)(t)\|\to 0 \quad \text{as } m\to\infty,$$

so that $\Lambda : B\to B$ is a continuous operator. Moreover, it follows that $\Lambda(B)$ is bounded. Further on, we show that $\Lambda(B)$ is equicontinuous on $[0,T]$. Let $\varepsilon>0$. On the closed set $[0,T]$, the functions $t\mapsto\int_0^t\psi(s)ds$ and $t\mapsto\int_0^t\eta(s)ds$, are uniformly continuous, and so there exist $\eta>0$ such that

$$\left|\int_s^t\psi(\tau)d\tau\right|\le\varepsilon/4 \text{ and } \left|\int_s^t\eta(\tau)d\tau\right|\le\varepsilon/4$$

for every $t,s\in[0,T]$ with $|t-s|<\eta$. Let $t,s\in[0,T]$ are such that $|t-s|\le\eta$. If we suppose that $0\le s\le t\le T$ then, for each $u(\cdot)\in B$, we have

$$\begin{aligned}
&\|(\Lambda u)(t)-(\Lambda u)(s)\|\\
&\quad=\left\|\int_0^t(\mathfrak{C}u)(\tau)d\tau+\sum_{0<t_k<t}I_k(u(t_k^-))-\int_0^s(\mathfrak{C}u)(\tau)d\tau-\sum_{0<t_k<s}I_k(u(t_k^-))\right\|\\
&\quad\le\int_s^t\|(\mathfrak{C}u)(\tau)\|d\tau+\sum_{s<t_k<t}\left\|I_k(u(t_k^-))\right\|\le\int_s^t[\psi(\tau)+\eta(\tau)]\,d\tau\le\varepsilon.
\end{aligned}$$

Therefore, we conclude that $\Lambda(B)$ is uniformly equicontinuous on $[0,T]$. Next, we construct a sequence $\{u_n(\cdot)\}_{n\ge1}$ of continuous functions $u_n(\cdot):[0,T]\to E$ as follows. Let $n\in\mathbb{N}$. For $i=1,2,\dots,n$, we define $u_n^1(t)=\xi$, $t\in[0,T]$ and

$$u_n^i(t)=\begin{cases}u_n^{i-1}(t), & \text{if } t\in[0,(i-1)T/n]\\ \xi+\int_0^{t-T/n}(\mathfrak{C}u_n^{i-1})(s)ds & \\ \qquad+\sum\limits_{0<t_k<t-T/n}I_k(u_n^{i-1}(t_k^-)), & \text{if } t\in[(i-1)T/n,iT/n],\end{cases}$$

for $i>1$. It is easy to see that if $i\in\{1,2,\dots,n-1\}$ and $\|u_n^i(t)\|\le r$ for $t\in[0,iT/n]$, then $\|u_n^{i+1}(t)\|\le r$ for $t\in[0,iT/n]$ and, by (H$_2$), $\|(\mathfrak{C}u_n^i)(t)\|\le\psi(t)$ for a.e. $t\in[0,iT/n]$ and

$$\sum_{0<t_k<t-T/n}\|I_k(u_n^{i-1}(t_k^-))\|\le\int_0^{t-T/n}\eta(s)ds$$

for $t\in[0,iT/n]$. It follows that

$$\begin{aligned}
\|u_n^{i+1}(t)-\xi\|&=\left\|\int_0^{t-T/n}(\mathfrak{C}u_n^i)(s)ds+\sum_{0<t_k<t-T/n}I_k(u_n^i(t_k^-))\right\|\\
&\le\int_0^{t-T/n}\|(\mathfrak{C}u_n^i)(s)\|ds+\sum_{0<t_k<t-T/n}\left\|I_k(u_n^i(t_k^-))\right\|\\
&\le\int_0^{t-T/n}[\psi(\tau)+\eta(\tau)]\,d\tau<\delta,
\end{aligned}$$

for all $t \in [0,(i+1)T/n]$. Since $\|u_n^1(t)\| \leq r$ for $t \in [0, T/n]$, then by induction on k we have that $\|u_n^i(t)\| \leq r$ for all $k = 1, 2, \dots, n$, $t \in [0, iT/n]$. In the following, to simplify the notation, we put $u_n(\cdot) = u_n^n(\cdot)$, $n \in \mathbb{N}$. Since $u_n^n(s) = u_n^{n-1}(s)$ for all $s \in [0,(n-1)T/n]$ and $\mathfrak{C}$ is a causal operator, then

$$(\mathfrak{C}u_n^n)(s) = (\mathfrak{C}u_n^{n-1})(s) \quad \text{for all } s \in [0,(n-1)T/n].$$

Moreover, we have that $u_n^n(t_k^-) = u_n^{n-1}(t_k^-)$ and so $I_k\left(u_n^n(t_k^-)\right) = I_k\left(u_n^{n-1}(t_k^-)\right)$ for $0 < t_k < t - T/n$ with $t \in [T/n,(n-1)T/n]$. Next, if $t \in [(n-1)T/n, T]$, then $t - T/n \leq (n-1)T/n$ and consequently

$$\int_0^{t-T/n} (\mathfrak{C}u_n^n)(s)ds = \int_0^{t-T/n} (\mathfrak{C}u_n^{n-1})(s)ds$$

for $t \in [(n-1)T/n, T]$. It follows that the sequence $\{u_n(\cdot)\}_{n\geq 1}$ can be written as

$$u_n(t) = \begin{cases} \xi & \text{for } t \in [0, T/n] \\ \xi + \int_0^{t-T/n} (\mathfrak{C}u_n)(s)ds + \sum\limits_{0<t_k<t-T/n} I_k(u_n(t_k^-)) & \text{for } t \in [T/n, T], \end{cases}$$

for every $n \in \mathbb{N}$. Moreover, it is easy to see that $u_n(\cdot) \in PC([0,T],E)$ for all $n \geq 1$. Further, if $0 \leq t \leq T/n$, then we have

$$\begin{aligned} \|(\Lambda u_n)(t) - u_n(t)\| &= \left\| \int_0^t (\mathfrak{C}u_n)(s)ds \right\| \\ &\leq \int_0^{T/n} \|S(t-s)(\mathfrak{C}u_n)(s)\| ds \\ &\leq \int_0^{T/n} \psi(s)ds. \end{aligned}$$

If $T/n \leq t \leq T$, then we have

$$\begin{aligned} \|(\Lambda u_n)(t) - u_n(t)\| &= \Bigg\| \int_0^t (\mathfrak{C}u_n)(s)ds + \sum_{0<t_k<t} I_k(u_n(t_k^-)) \\ &\qquad - \int_0^{t-T/n} (\mathfrak{C}u_n)(s)ds - \sum_{0<t_k<t-T/n} I_k(u_n(t_k^-)) \Bigg\| \\ &\leq \int_{t-T/n}^t \|(\mathfrak{C}u_n)(s)\| ds + \sum_{t-T/n<t_k<t} \left\| I_k(u_n(t_k^-)) \right\| \\ &\leq \int_{t-T/n}^t \left[\psi(\tau) + \eta(\tau)\right] d\tau. \end{aligned}$$

Therefore, it follows that

$$\sup_{0\leq t\leq T} \|(\Lambda u_n)(t) - u_n(t)\| \to 0 \text{ as } m \to \infty. \tag{2.4}$$

Let $A = \{u_n(\cdot); n \geq 1\}$. Denote by I the identity mapping on B. From (2.4) it follows that $(I - \Lambda)(A)$ is a equicontinuous subset of B. Since $A \subset (I - \Lambda)(A) + \Lambda(A)$ and the set $\Lambda(A)$ is equicontinuous, then we infer that the set A is also equicontinuous on $[0,T]$. Set $A(t)=\{u_n(t);n\geq 1\}$ for $t\in[0,T]$. Then, by Lemma 1.1 and the properties of the measure of non-compactness we have

$$\begin{aligned}\chi(A(t))\leq \chi\left(\int_0^t(\mathfrak{C}A)(s)ds\right)+\chi\left(\int_{t-T/n}^t(\mathfrak{C}A)(s)ds\right)\\ +\chi\left(\sum_{0<t_k<t-T/n}I_k(A(t_k^-))\right).\end{aligned}$$

Note that, given $\varepsilon>0$, we can find $n(\varepsilon)>0$ such that $\int_{t-T/n}^t\psi(s)ds<\varepsilon/2$ for $t\in[0,T]$ and $n\geq n(\varepsilon)$. Hence we have that

$$\begin{aligned}\chi\left(\int_{t-T/n}^t(\mathfrak{C}A)(s)ds\right)&=\chi\left(\left\{\int_{t-T/n}^t(\mathfrak{C}u_m)(s)ds;n\geq n(\varepsilon)\right\}\right)\\ &\leq 2\sup_{n\geq n(\varepsilon)}\int_{t-T/n}^t\psi(s)ds<\varepsilon.\end{aligned}$$

Using the last inequality, we obtain that

$$\chi(A(t))\leq\chi\left(\int_0^t(\mathfrak{C}A)(s)ds\right)+\sum_{0<t_k<t-T/n}\chi\left((I_kA)(t_k^-)\right).$$

Since for every $t\in[0,T]$, $A(t)$ is bounded then, by Lemma 1.1, (H_3) and the and properties of the measure of non-compactness we have that

$$\begin{aligned}\chi(A(t))&\leq\int_0^t\chi\left((\mathfrak{C}A)(s)\right)ds+\sum_{k=1}^m\delta_r^k\chi(I_k(A(t_k^-)))\\ &\leq\int_0^t\gamma_r\chi(A(s))ds+\sum_{k=1}^m\delta_r^k\chi(A(t_k^-))\end{aligned}$$

for every $t\in[0,T]$. Therefore, if we put $m(t):=\chi(A(t))$, $t\in[0,T]$, then we infer that

$$m(t)\leq\int_0^t\gamma_r m(s)ds+\sum_{k=1}^m\delta_r^k m(t_k^-),$$

for every $t\in[0,T]$. Then, by Gronwall's lemma for impulsive integral inequalities (see [13, Theorem 1.5.1]), we must have that $m(t)=\chi(A(t))=0$ for every $t\in[0,T]$. Moreover, since (see [13]) $\chi_{PC}(A)=\sup_{0\leq t\leq T}\chi(A(t))$ we deduce that $\chi_{PC}(A)=0$. Therefore, A is relatively compact subset of $PC([0,T],E)$. Then, by Arzela-Ascoli theorem (see [13, Theorem 1.1.5]), and extracting a subsequence if necessary, we may assume that the sequence $\{u_n(\cdot)\}_{m\geq 1}$ converges on $[0,T]$ to a function $u(\cdot)\in B$. Therefore, since

$$\begin{aligned}\sup_{0\leq t\leq T}\|(\Lambda u)(t)-u(t)\|\leq&\sup_{0\leq t\leq T}\|(\Lambda u)(t)-(\Lambda u_n)(t)\|\\ &+\sup_{0\leq t\leq T}\|(\Lambda u_n)(t)-u_n(t)\|+\sup_{0\leq t\leq T}\|u_n(t)-u(t)\|\end{aligned}$$

then, by (2.4) and by the fact that Λ is a continuous operator, we obtain that sup)$0 \le t \le T\|(\Lambda u)(t) - u(t)\| = 0$. It follows that

$$u(t) = (\Lambda u)(t) = \xi + \int_0^t (\mathfrak{C}u)(s))ds + \sum_{0<t_k<t} I_k(u(t_k^-))$$

for every $t \in [0, T]$, that is $u(\cdot) = \Lambda u(\cdot)$. Hence

$$u(t) = \xi + \int_0^t (\mathfrak{C}u)(s)ds + \sum_{0<t_k<t} I_k(u(t_k^-)), \quad \text{for } t \in [0, T]$$

solve the Cauchy problem (2.1), that is, $u(\cdot) \in S_T(\xi)$ and so $S_T(\xi)$ is a nonempty set. Since Λ is continuous, then $S_T(\xi)$ is a closed subset in $PC([0, T], E)$. Moreover, since $S_T(\xi) = \Lambda(S_T(\xi))$ it follows that $\chi((S_T(\xi))(t)) = \chi(\Lambda(S_T(\xi))(t))$ for every $t \in [0, T]$, where $(S_T(\xi))(t) := \{u(t); u \in S_T(\xi)\}$. Therefore, following the same argument as above, we obtain that relatively compact subset of $PC([0, T], E)$. Since $S_T(\xi)$ is a closed subset in $PC([0, T], E)$ it follows that $S_T(\xi)$ is a compact subset in $PC([0, T], E)$. □

Remark 2.2. The conclusion of Theorem 2.1 is also true if we replace the condition (2.2) with the condition:

($\mathbf{H}_3'$) For each bounded subsets $A \subset PC([0, b], E)$ there exists $\gamma_A > 0$ such that

$$\chi((\mathfrak{C}A)(t)) \le \gamma_A \sup_{0 \le s \le t} \chi(A(s)) \quad \text{for every } t \in [0, b]. \tag{2.5}$$

3. An optimal control problem

In the following, we shall establish necessary conditions for the existence of an optimal solution for the control problem:

$$\begin{cases} u'(t) = (Qu)(t), & \text{for a.e. } t \in [0, T] \setminus \{t_1, t_2, \ldots, t_m\} \\ u(t_k^+) = u(t_k^-) + I_k(u(t_k^-)), & k = 1, 2, \ldots, m, \\ u(0) = \xi \text{ minimize } g(u(T)), \end{cases} \tag{3.1}$$

where $g(\cdot) : E \to \mathbb{R}$ is a given function. For this aim, it will need to establish some preliminary results. For a fixed $\xi \in E$ we denote by $\mathcal{A}_T(\xi)$ the attainable set of Cauchy problem (2.1); that is, $\mathcal{A}_T(\xi) = \{u(T); u(\cdot) \in \mathcal{S}_T(\xi)\}$.

Lemma 3.1. *Assume that $Q : PC([0, b], E) \to L^p([0, b], E)$ is a causal operator such that the condition* (H_1)–(H_3) *hold. Then the multifunction $S_T : E \to PC([0, T], E)$ is upper semicontinuous.*

Proof. Let $\mathcal{K}$ be a closed set in $PC([0,T],E)$ and $\mathcal{G}=\{\xi\in E;\mathcal{S}_T(\xi)\cap\mathcal{K}\neq\emptyset\}$. We must show that $\mathcal{G}$ is closed in E. For this, let $\{\xi_n\}_{n\geq 1}$ be a sequence in $\mathcal{G}$ such that $\xi_n\to\xi$. Further on, for each $n\geq 1$, let $u_n(\cdot)\in\mathcal{S}_T(\xi_n)\cap\mathcal{K}$. Then

$$u_n(t)=\xi_n+\int_0^t (Qu_n)(s)ds+\sum_{0<t_k<t} I_k(u_n(t_k^-))$$

for every $t\in(0,T]$. As in proof of Theorem 1.1 we can show that $\{u_n(\cdot)\}_{n\geq 1}$ converges uniformly on $[0,T]$ to a continuous function $u(\cdot)\in\mathcal{K}$. Since

$$u(t)=\lim_{n\to\infty} u_n(t)=\xi+\int_0^t (Qu)(s)ds+\sum_{0<t_k<t} I_k(u(t_k^-))$$

for every $t\in[0,T]$, we deduce that $u(\cdot)\in\mathcal{S}_T(\xi)\cap\mathcal{K}$. This prove that $\mathcal{G}$ is closed and so $\xi\mapsto\mathcal{S}_T(\xi)$ is upper semicontinuous. □

Corollary 3.2. *Assume that $Q:PC([0,b],E)\to L^p([0,b],E)$ is a causal operator such that conditions* (H$_1$)–(H$_3$) *hold. Then, for any $\xi\in E$ and any $t\in[0,T]$ the attainable set $\mathcal{A}_t(\xi)$ is compact in $C([0,t],E)$and the multifunction $(t,\xi)\to\mathcal{A}_t(\xi)$is jointly upper semicontinuous.*

Theorem 3.3. *Let K_0 be a compact set in E and let $g(\cdot):E\to\mathbb{R}$ be a lower semicontinuous function. If $Q:PC([0,b],E)\to L^p([0,b],E)$ is a causal operator such that the condition* (H$_1$)–(H$_3$) *hold, then the control problem* (3.1) *has an optimal solution; that is, there exists $\xi_0\in K_0$ and $u_0(\cdot)\in\mathcal{S}_T(\xi_0)$ such that $g(u_0(T))=\inf\{g(u(T));u(\cdot)\in\mathcal{S}_T(\xi_0),\xi_0\in K_0\}$.*

Proof. From Corollary 3.2 we deduce that the attainable set $\mathcal{A}_T(\xi)$ is upper semicontinuous. Then the set $\mathcal{A}_T(K_0)=\{u(T);u(\cdot)\in\mathcal{S}_T(\xi),\xi\in K_0\}=\cup_{\xi\in K_0}\mathcal{A}_T(\xi)$ is compact in E and so, since $g(\cdot)$ is lower semicontinuous, there exists $\xi_0\in K_0$ such that $g(u_0(T))=\inf\{g(u(T));u(\cdot)\in\mathcal{S}_T(\xi_0),\xi_0\in K_0\}$. □

4. **An example**

Consider the following impulsive differential equation:

$$\begin{cases} u'(t)=g(t)+\int_0^t K(t,s)f(s,u(s))ds, & \text{a.e. } t\in[0,b]\setminus\{t_1,t_2,\dots,t_m\},\\ u(t_k^+)=u(t_k^-)+\left(\int_{t_k}^{t_{k+1}}\lambda(t)dt\right)u(t_k^-), & k=1,2,\dots,m,\\ u(0)=\xi, \end{cases} \tag{4.1}$$

where $g(\cdot),\lambda(\cdot)\in L^p([0,b],E)$, $p\geq 1$, and $K:[0,b]\times[0,b]\to\mathcal{L}(E)$ is strongly continuous. Let $M:=\sup_{s,t\in[0,b]}\|K(t,s)\|$. Assume that (**f1**) $f : [0,b] \times E \to E$ is a Carathéodory function; that is, $t \mapsto f(t,u)$ is strongly measurable for all $u \in E$, $u \mapsto f(t,u)$ is continuous for a.e. $t \in [0,b]$, and there $c(\cdot) \in L^P([0,b],\mathbb{R}_+)$

$$\|f(t,u)\| \le c(t), \quad t \in [0,b],\ u \in E,$$

(**f2**) For each bounded set $A \subset E$ there exist $l_A > 0$ such that

$$\chi(f(s,A)) \le l_A \chi(A) \quad \text{for every } t \in [0,b].$$

If we put

$$(\mathfrak{C}u)(t) := g(t) + \int_0^t K(t,s) f(s,u(s)) ds, \quad t \in [0,b],$$

and

$$I_k(u) := \left(\int_{t_k}^{t_{k+1}} \lambda(t) dt \right) u, \quad k = 1,2,\ldots,m,$$

for all $u \in CP([0,b],E)$, then equations (4.1) can be written in abstract form (2.1). It is easy to see that $\mathfrak{C} : PC([0,b],E) \to L^p([0,b],E)$, $1 \le p \le \infty$, is a continuous causal operator, and $I_k : E \to E$ is continuous for each $k = 1,2,\ldots,m$. Next, by (**f1**) we have that

$$\begin{aligned} \|(\mathfrak{C}u)(t)\| &\le \|g(t)\| + \int_0^t \|K(t,s)\| \cdot \|f(s,u(s))\| \, ds \\ &\le \|g(t)\| + M \int_0^t c(s) ds \le \|g(t)\| + M b^{1/p'} \|c\|_p , \end{aligned}$$

so that $\psi(\cdot) := \|g(\cdot)\| + M b^{1/p'} \|c\|_p \in L^p([0,b],\mathbb{R}_+)$ and

$$\|(\mathfrak{C}u)(t)\| \le \psi(t) \quad \text{for a.e. } t \in [0,b].$$

Now, let $s,t \in [0,b]$ be such that $s < t$ and let $\{t_\nu, t_{\nu+1}, \ldots, t_r\} \subset \{t_1, t_2, \ldots, t_m\}$ be such that $s < t_\nu < t_{\nu+1} < \cdots < t_r < t$. Then,

$$\begin{aligned} \sum_{k=\nu}^{r} \left\| I_k(u(t_k^-)) \right\| &\le \sum_{k=\nu}^{r} \left(\int_{t_k}^{t_{k+1}} \lambda(t) dt \right) \left\| u(t_k^-) \right\| \le \left(\int_{t_\nu}^{t_r} \lambda(t) dt \right) \|u\|_{CP} \\ &\le \int_s^t \|u\|_{CP} \lambda(t) dt, \end{aligned}$$

that is,

$$\sum_{k=\nu}^{r} \left\| I_k(u(t_k^-)) \right\| \le \int_s^t \eta(\tau) \, d\tau,$$

where $\eta(\cdot) := \|u\|_{CP} \lambda(\cdot) \in L^p([0,b],\mathbb{R}_+)$. Therefore, ($H_2$) is verified as true. Next, if $A \subset PC([0,b],E)$ is a bounded set, then using the properties of noncompactness measure, Mean Value Theorem (see [23]) and (**f2**), we have

$$\begin{aligned}
\chi((\mathfrak{C}A)(t)) &= \chi\left(\left\{g(t) + \int_0^t K(t,s)f(s,u(s))ds; u \in A\right\}\right) \\
&= \chi\left(\left\{\int_0^t K(t,s)f(s,u(s))ds; u \in A\right\}\right) \\
&\leq M\chi\left(\left\{\int_0^t f(s,u(s))ds; u \in A\right\}\right) \\
&= Mt\chi\left(\overline{conv}\left\{f(s,u(s)), 0 \leq s \leq t, u \in A\right\}\right) \\
&= Mb\chi\left(\left\{f(s,A(s)), 0 \leq s \leq t\right\}\right) \\
&\leq Mbl_A \sup_{0\leq s\leq t} \chi(A(s)),
\end{aligned}$$

that is,

$$\chi((\mathfrak{C}A)(t)) \leq \gamma_A \sup_{0\leq s\leq t} \chi(A(s)) \quad \text{for every } t \in [0,b].$$

Also, it is easy to see that

$$\chi(I_k(B)) \leq \delta_B^k \chi(B), \quad k = 1, 2, \ldots, m,$$

for each bounded set $B \subset E$, where $\delta_B^k := \int_{t_k}^{t_{k+1}} \lambda(t)dt$, $k = 1, 2, \ldots, m$. Consequently, all the hypothesis of Theorem 2.1 are satisfied (see also Remark 2.2), so that (4.1) has a solution on $[0,b]$.

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