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HOMOGENIZATION OF BSDES WITH TWO REFLECTING BARRIERS, VARIATIONAL INEQUALITY AND STOCHASTIC GAME

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ABSTRACT. In this paper, we study the limit of semilinear variational inequality with bilateral constraints and stochastic differential games of mixed type. By a penalization method, we first study homogenization properties for system of two barriers reflected backward stochastic differential equation in the Markovian setting. This result together with certain techniques from stochastic calculus is then applied to show that the unique solution of the homogenized problem is also the value function of certain stochastic differential games of mixed type. Then using standard results from the theory of viscosity solutions, we show that the value function of this stochastic differential game with continuous control is the unique viscosity solution of the corresponding limit semilinear variational inequalities too.

Kery Words and phrases: Backward Stochastic Differential Equation (BSDE), reflecting barriers, homogenization, variational inequality, viscosity solution, stochastic games

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1. INTRODUCTION

Backward stochastic differential equations (BSDE's in short) is an interesting subject in stochastic calculus developed since the pioneering works of Pardoux and Peng [27], [28]. The application of such equations to finance theory and nonlinear partial differential equations has motivated many efforts to establish existence and uniqueness of the solution (see [1], [3], [9], [13], [16], [23], [26] and the references given there). In [11], El Karoui et *al* have introduced the notion of one barrier reflected BSDE, which is a backward equation but the solution is forced to stay above a given continuous obstacle. Moreover, the authors have established the existence and uniqueness of the solution via a penalization as well as Picard's iteration methods. The notion of double barriers reflected BSDE has been introduced by Cvitanic and Karatzas [7] where the solution is forced to remain between two described upper and lower barriers U and L. As is well known, BSDE provide probabilistic formulae for the viscosity solution of semilinear partial differential equations (PDE) (see for instance Pardoux and Peng [28], Pardoux [26] and references therein). In the present work,

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we wish to consider a more general problem, homogenization of two barriers reflected BSDE when the solutions are forced to stay between an upper and lower obstacles, and also provide a probabilistic formulae to limit of system of variational inequalities with two obstacles.

In this paper, we use a penalization method to show the existence and uniqueness of the weak solution for the homogenized problem associated to the reflected BSDE (3.4) when the upper barrier U and the lower barrier L are smooth Itô processes. Such equations appear when one studies the notion of zero-sum mixed problems [15] or American game options [8]. Variational inequality theory was introduced by Hartman and Stampacchia [20], as a tool for the study of partial differential equations with applications principally drawn from mechanics. Such variational inequalities were infinite-dimensional rather than finite-dimensional as we will be studying here. Equilibrium is a central concept in numerous disciplines including economics, management science, operations research, and engineering. Variational inequality theory is a powerful unifying methodology for the study of equilibrium problems. This type of inequalities also arises in zero sum stochastic differential games of mixed type where each player uses both continuous control and stopping times.

A probabilistic approach of the homogenization property has been developed since the early article of Freidlin [12] (see also [5, Chapter 3]) parallel to analytical one. The link between BSDE's and homogenization of semilinear PDE's are given since 1996 by the work of Pardoux [26].

The paper is organized as follows. The BSDE problem with reflecting barriers as well as some preliminary results are described in Section 2. In Section 3, we prove existence and uniqueness of the solution in the Markovian case. The homogenization problem is treated in Section 4 by using the result of Section 3. The last section is devoted to the homogenization problem for variational inequality and stochastic differential games of mixed type.

2. NOTATIONS AND PRELIMINARY RESULTS

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, W_t, t \in [0,T])$ be a complete Wiener space in $\mathbb{R}^d$, i.e. $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $(\mathcal{F}_t, t \in [0,T])$ is a right continuous increasing family of complete sub σ-algebras of $\mathcal{F}$, $(W_t, t \in [0,T])$ is a standard Wiener process in $\mathbb{R}^d$ with respect to $(\mathcal{F}_t, t \in [0,T])$. We assume that

$$\mathcal{F}_t = \sigma\left[W_s, s \leq t\right] \otimes \mathcal{N}$$

where $\mathcal{N}$ denotes the totality of $\mathbb{P}$-null sets and $\sigma_1 \otimes \sigma_2$ denotes the σ-field generated by $\sigma_1 \cup \sigma_2$. Let $\mathcal{P}$ be the σ-field of predictable subsets of Ω x $[0,T]$.

Let us introduce the following spaces:

- L^2 of $\mathcal{F}_T$-measurable, real-valued random variables ξ, $\mathbb{E}\left[|\xi|^2\right] < +\infty$.
- S^2 of continuous $(\mathcal{F}_t)$-adapted real-valued processes $(Y_t)_{t\le T}$, $\mathbb{E}\left[\sup_{t\le T}|Y_t|^2\right] < \infty$.
- $H^{2,k}$ of $(\mathcal{F}_t)$-progressively measurable processes, valued in $\mathbb{R}^k$, $\mathbb{E}\left[\int\limits_0^T |Z_s|^2 ds\right] < \infty$.
- $\mathcal{A}^2$ of continuous, real-valued, increasing, $(\mathcal{F}_t)$-adapted process $(K_t)_{0\le t\le T}$ such that $K(0)=0$ and $\mathbb{E}\left|K_T\right|^2 < +\infty$.

Finally, given:

(A1): a **terminal value** $\xi \in L^2$

(A2): a **coefficient** "f" which is a map $f : \Omega \times [0,T] \times \mathbb{R}^{1+d} \longrightarrow \mathbb{R}$ such that

1. f is $\mathcal{P}\times\mathcal{B}(\mathbb{R}^{1+d})$-measurable and satisfies: $(f(t,0,0))_{t\le T}$ belongs to $L^2(\Omega\times[0,T], dP\otimes dt)$ i.e.,

$$\mathbb{E}\int_0^T |f(t,0,0)|^2\, dt < +\infty \tag{2.1}$$

2. f is uniformly Lipschitz with respect to (y,z), i.e., there exists a constant $k\ge 0$ such that for any y, y', z, $z' \in \mathbb{R}$,

$$\mathbb{P}\text{-a.s.}, \quad |f(\omega,t,y,z) - f(\omega,t,y',z')| \le k(|y-y'| + |z-z'|). \tag{2.2}$$

and

(A3): two **reflecting barriers** L, $U \in S^2$, i.e. real valued and $\mathcal{P}$-measurable processes satisfying

$$\mathbb{E}\left[\sup_{0\le t\le T} {U_t}^2\right] < +\infty \text{ and } \mathbb{E}\left[\sup_{0\le t\le T} {L_t}^2\right] < +\infty, \tag{2.3}$$

we shall always assume that

$$\forall\, 0\le t\le T, \quad L_t \le U_t \quad \text{and} \quad L_T \le \xi \le U_T, \quad \mathbb{P}\text{-a.s.}$$

2.1. **On one barrier.** Let $f : \Omega\times[0,T]\times\mathbb{R}^{1+d} \longrightarrow \mathbb{R}$ which has the property **(A2)**, and only the lower barrier in **(A3)**.

Definition 2.1. A solution for one barrier reflected BSDE associated with (f,ξ,L) is a $\mathcal{P}$-measurable process $(Y,Z,K) := (Y_t,Z_t,K_t)_{t\le T}$ valued in $\mathbb{R}^{1+d}\times\mathbb{R}^+$ and which satisfies:

$$\begin{cases} (i) & Y\in S^2, \quad Z\in H^{2,d}; \quad K\in\mathcal{A}^2, \\ (ii) & Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + K_T - K_t - \int_t^T Z_s dW_s, \quad t\le T, \\ (iii) & \forall t\le T, Y_t \ge L_t \text{ and } \int_0^T (Y_t - L_t) dK_t = 0 \quad \mathbb{P}\text{-a.s.} \end{cases}$$

The following result established by El Karoui et al. [11] is concerned with the existence and uniqueness of a solution for a single barrier reflected BSDE associated with (f,ξ,L).

Theorem 2.2. *Suppose that the assumptions* **(A1)–(A2)** *and* **(A3)** *hold for* (f, ξ, L), *then there exists a unique* $\mathcal{P}$*-measurable process* (Y, Z, K) *solution of the one barrier reflected BSDE associated with* (f, ξ, L) *has a unique solution. Furthermore the following inequality holds*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t|^2 + \int_0^T |Z_s|^2\,ds + |K_T|^2\right] < \infty \tag{2.4}$$

We need also a comparison result given in the same paper, our assumptions are rather strong than required. Unfortunately we need those assumptions for homogenization purpose.

Theorem 2.3. *Let* (f, ξ, L) *and* (f', ξ', L') *be two sets of data, each one satisfying all assumptions* **(A1)**, **(A2)**, **(A3)** *and such that* $\mathbb{P}$*-a.s.* $\xi \leq \xi'$; $L_t \leq L'_t$; *and* $f(t, y, z) \leq f'(t, y, z)$ $dt \otimes d\mathbb{P}$*-a.e.* $\forall (y, z) \in \mathbb{R}^{1+d}$. *Then* $Y_t \leq Y'_t$, $0 \leq t \leq T$ $\mathbb{P}$*-a.s. and* $dK \geq dK'$.

2.2. **Double barriers reflected BSDE.** Let us now introduce our double barriers reflected BSDE (DRBSDE in short)

Definition 2.4. The process $\left(Y_t, Z_t, K_t^+, K_t^-\right)_{t\leq T}$, with value in $\mathbb{R}^{1+d}\times\mathbb{R}^+\times\mathbb{R}^+$, is called a solution for the double barriers DRBSDE associated to (f, ξ, L, U) if

$$\begin{cases} (i) & Y \in S^2,\ Z \in H^{2,d} \text{ and } K^{\pm} \in \mathcal{A}^2 \\ (ii) & Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s \\ & \qquad +(K_T^+ - K_t^+) - (K_T^- - K_t^-), \quad \forall t \leq T \\ (iii) & \forall t \leq T,\ L_t \leq Y_t \leq U_t \text{ and} \\ & \int_0^T (Y_t - L_t)dK_t^+ = \int_0^T (U_t - Y_t)dK_t^- = 0 \quad \mathbb{P}\text{-a.s.} \end{cases}$$

Remark 2.5. Since our coefficient is Lipschitz in (y, z) uniformly in (ω, t), the existence and uniqueness of a solution was proved (with some additional conditions) in Cvitanic and Karatzas [7] (Corollary 5.5, Theorem 6.1 or Theorem 6.5), see also Hamadène and Hassani [14], Peng and Xu [29].

3. MARKOVIAN CASE

Let $\{X_t^{\epsilon}; t \geq 0\}$ a diffusion process with values in $\mathbb{R}^d$ and generator L^{ε}, such that $X^{\epsilon} \Longrightarrow X$ in $\mathbb{C}([0, T], \mathbb{R}^d)$ equipped with the topology of convergence on compact subsets of $\mathbb{R}_+$, where X itself is a diffusion with generator L^0. We suppose that the martingale problem associated to X is well posed, and there exist $p, q \geq 0$ such that

$$\sup_{\epsilon}\mathbb{E}\left(|X_t^{\epsilon}|^{2p} + \int_0^T |X_s^{\epsilon}|^{2q}\,ds\right) < \infty. \tag{3.1}$$

We assume moreover that

1. $g : \mathbb{R}^d \longrightarrow \mathbb{R}$, $f : [0,T] \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous, which are such that for some $C > 0$, $K > 0$ and for all $x \in \mathbb{R}^d$, y, $y' \in \mathbb{R}$

$$\left\{ \begin{array}{l} |g(x)| \leq C(1 + |x|^p), \\ |f(t,x,0)| \leq C(1 + |x|^q), \text{ and} \\ |f(t,x,y) - f(t,x,y')| \leq K|y - y'|. \end{array} \right. \tag{3.2}$$

2. The barriers

$$(L(s, X_s^\epsilon))_{s \leq T} \quad \text{ and } \quad (U(s, X_s^\epsilon))_{s \leq T} \in S^2.$$

where the function $U(s,x)$ and $L(s,x)$ are in $\mathcal{C}^{1,2}\left([0,T] \times \mathbb{R}^d; \mathbb{R}\right)$ such that for U (resp. L),

$$\left|\frac{\partial U}{\partial t}\right| + \left|\frac{\partial U}{\partial x}\right| + \left|\frac{\partial^2 U}{\partial x^2}\right| \leq C(1 + |x|^p). \tag{3.3}$$

We consider the following double barriers reflected BSDE associated to

$$(f(s, X_s^\epsilon, \cdot),\ g(X_T^\epsilon),\ L(\cdot, X_\cdot^\epsilon),\ U(\cdot, X_\cdot^\epsilon))$$

that is

$$\left\{ \begin{array}{c} Y_s^\epsilon = g(X_T^\varepsilon) + \int_s^T f(r, X_r^\varepsilon, Y_r^\epsilon) dr - \int_s^T Z_r^\epsilon dB_r \\ + \left(K_T^{+\epsilon} - K_s^{+\epsilon}\right) - \left(K_T^{-\epsilon} - K_s^{-\epsilon}\right), \\ \forall t \leq T,\ L(t, X_t^\varepsilon) \leq Y_t^\epsilon \leq U(t, X_t^\varepsilon) \text{ and} \\ \int_0^T (Y_t^\epsilon - L(t, X_t^\varepsilon)) dK_t^{+\varepsilon} = \int_0^T (U(t, X_t^\varepsilon) - Y_t) dK_t^{-\varepsilon} = 0. \end{array} \right. \tag{3.4}$$

We need the following a priori estimates, to prove the existence and uniqueness result for (3.4)

Lemma 3.1. *For each $\varepsilon > 0$,*

$$\mathbb{E}\left[\sup_{t \leq T} |Y_t^\epsilon|^2\right] < \infty \quad \textit{and} \quad \mathbb{E}\left[\sup_{t \leq T} |f(t, X_t^\epsilon, Y_t^\epsilon)|^2\right] < \infty.$$

Proof. The first inequality is a consequence of existence theorem in Lepeltier and San Martin ([22, Theorem 1]), since our assumptions are sufficient (see Remark 1 in [22]). Now using (3.2), we have $|f(t, X_t^\epsilon, Y_t^\epsilon)|^2 \leq C_1(1 + |X_t^\epsilon|^{2q} + |Y_t^\epsilon|^2)$, for some $C_1 > 0$. By (3.1), we obtain the second. □

We can now give existence and uniqueness result for the system (3.4).

Proposition 3.1. *For each $\varepsilon > 0$, the DRBSDE (3.4) has a unique solution $(Y^\varepsilon, Z^\varepsilon, K^{+\varepsilon}, K^{-\varepsilon})$, also $K^{+\varepsilon}$ and $K^{-\varepsilon}$ are absolutely continuous with respect to the Lebesgue measure.*

Proof. Recall that the barriers are smooth Itô processes (in S^2) by assumption and the process $(f(s, X_s^\epsilon, Y_s^{\epsilon,n}))_{0 \leq s \leq T} \in S^2$ from Lemma 3.1. The result follows from Theorem 6.1 in [7]. □

3.1. **Connection with Dynkin games.** As shown by Cvitanic and Karatzas ([7, Theorems 4.1 and 6.5]) see also Hamadène and Hassani [14], the existence of a solution $(Y^\varepsilon, Z^\varepsilon, K^{+\varepsilon}, K^{-\varepsilon})$ to (3.4) implies that Y^ε is the value of a certain stochastic game of stopping. Let us recall the corresponding stochastic game.

We define

$$\mathcal{M}_{t,\theta} := \{\tau \in \mathcal{M} / t \leq \tau \leq \theta \text{ a.s.}\} \text{ for } 0 \leq t \leq \theta \leq T$$

where $\mathcal{M}$ is the class of $\mathcal{F}$-stopping times $\tau : \Omega \longrightarrow [0, T]$, and let $h(t) := f(X_t^\epsilon, Y_t^\epsilon)$. For any $\varepsilon > 0$, $0 \leq t \leq T$ and any two stopping times $\sigma, \tau \in \mathcal{M}_{t,T}$, consider the payoff (3.5)

$$R_t^\varepsilon(\sigma, \tau) := \int_t^{\sigma\wedge\tau} h(s)ds + g(X_T^\epsilon)1_{\{\sigma\wedge\tau = T\}} + L(\tau, X_\tau^\epsilon)1_{\{\tau < T, \tau \leq \sigma\}} + U(\sigma, X_\sigma^\epsilon)1_{\{\sigma<\tau\}},$$

as well as the upper and the lower values, respectively,

$$\begin{aligned} \overline{V}_t^\varepsilon &= \operatorname{ess\,sup}_{\sigma\in\mathcal{M}_{t,T}} \operatorname{ess\,inf}_{\tau\in\mathcal{M}_{t,T}} \mathbb{E}\left[R_t^\varepsilon(\sigma,\tau)/\mathcal{F}_t\right], \\ \underline{V}_t^\varepsilon &= \operatorname{ess\,inf}_{\tau\in\mathcal{M}_{t,T}} \operatorname{ess\,sup}_{\sigma\in\mathcal{M}_{t,T}} \mathbb{E}\left[R_t^\varepsilon(\sigma,\tau)/\mathcal{F}_t\right] \end{aligned} \tag{3.6}$$

of the corresponding stochastic game. This game has value V_t^ε, given by the state-process Y^ϵ the first component of the solution to DRBSDE (3.4), that is,

$$V_t^\varepsilon = \overline{V}_t^\varepsilon = \underline{V}_t^\varepsilon = Y_t^\epsilon \text{ a.s. } \forall 0 \leq t \leq T, \tag{3.7}$$

as well as saddle-point $(\widehat{\sigma}, \widehat{\tau}) \in \mathcal{M}_{t,T} \times \mathcal{M}_{t,T}$ given by

$$\begin{aligned} \widehat{\sigma}_t &:= \inf\{s \in [t,T)/Y_s^\epsilon = U(s, X_s^\epsilon)\} \wedge T, \\ \widehat{\tau}_t &:= \inf\{s \in [t,T)/Y_s^\epsilon = L(s, X_s^\epsilon)\} \wedge T, \end{aligned} \tag{3.8}$$

namely

$$\begin{aligned} \mathbb{E}\left[R_t^\varepsilon(\widehat{\sigma}_t, \tau)/\mathcal{F}_t\right] &\leq \mathbb{E}\left[R_t^\varepsilon(\widehat{\sigma}_t, \widehat{\tau}_t)/\mathcal{F}_t\right] \\ &= Y_t^\epsilon \leq \mathbb{E}\left[R_t^\varepsilon(\sigma, \widehat{\tau}_t)/\mathcal{F}_t\right] \text{ a.s.} \end{aligned} \tag{3.9}$$

for every $(\sigma, \tau) \in \mathcal{M}_{t,T} \times \mathcal{M}_{t,T}$ (see [7, Theorem 4.1]).

3.2. **Link with variational inequality.** Let us recall some known results about this link.

Let $x \in \mathbb{R}^d$ and $\{X_s^{t,x,\epsilon}; 0 \leq t \leq s \leq T\}$ the diffusion process defined as above, starting at x at time t. We denote by $(\{Y_s^{t,x,\epsilon}, Z_s^{t,x,\epsilon}, K_s^{t,x,+\epsilon}, K_s^{t,x,-\epsilon}\}; 0 \leq t \leq s \leq T)$ be the unique solution associated to DRBSDE $(f(s, X_s^{t,x,\epsilon}, \cdot),\ g(X_T^{t,x,\epsilon}),\ L(\cdot, X_\cdot^{t,x,\epsilon}),\ U(\cdot, X_\cdot^{t,x,\epsilon}))$. We assume moreover the polynomial growth condition on the barriers

$$|U(t,x)| + |L(t,x)| \leq C(1 + |x|^p) \tag{3.10}$$

for any $t \in [0,T]$ and the constants C and p are already used in (3.2). Let us consider the following double obstacles variational inequality:

(3.11)
$$\begin{cases} \min\left((u^\epsilon - L)\,;\max\left[(u^\epsilon - U)\,;\left(-\frac{\partial u^\epsilon}{\partial t} - L_t^\epsilon u^\epsilon\right) - f(\cdot,\cdot,u^\epsilon)\right]\right)(t,x) &= 0 \\ u^\epsilon(T,x) &= g(x). \end{cases}$$

Definition 3.2. Let u^ϵ be a function which belongs to $\mathcal{C}\left([0,T]\times\mathbb{R}^d;\mathbb{R}\right)$; u^ϵ is said to be a viscosity

(i) subsolution of (3.11) if $u^\epsilon(T,\cdot) \leq g(\cdot)$ and for any $\phi \in \mathcal{C}^{1,2}\left([0,T]\times\mathbb{R}^d;\mathbb{R}\right)$ and any local maximum point $(t,x)\in[0,T]\times\mathbb{R}^d$ of $u^\epsilon - \phi$, we have

$$\min\left((u^\epsilon - L)\,;\ \max\left[(u^\epsilon - U)\,;\left(-\frac{\partial\phi}{\partial t} - L_t^\epsilon\phi\right) - f(\cdot,\cdot,u^\epsilon)\right]\right)(t,x) \leq 0$$

(ii) supersolution of (3.11) if $u^\epsilon(T,\cdot) \geq g(\cdot)$ and for any $\phi \in \mathcal{C}^{1,2}\left([0,T]\times\mathbb{R}^d;\mathbb{R}\right)$ and any local minimum point $(t,x)\in[0,T]\times\mathbb{R}^d$ of $u^\epsilon - \phi$, we have

$$\min\left((u^\epsilon - L)\,;\max\left[(u^\epsilon - U)\,;\left(-\frac{\partial\phi}{\partial t} - L_t^\epsilon\phi\right) - f(\cdot,\cdot,u^\epsilon)\right]\right)(t,x) \geq 0$$

(iii) solution of (3.11) if it is both a viscosity subsolution and supersolution.

Since in our setting, the generator f do not depend on z, by virtue of Hamadène and Hassani ([14, Theorem 7.2]), the function $u^\epsilon : [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}$ defined by $u^\epsilon(t,x) = Y_t^{t,x,\epsilon}$, is a viscosity solution of (3.11).

4. THE DRBSDE HOMOGENIZATION RESULT

Let $\{(Y_s^\epsilon, Z_s^\epsilon, K_s^{+\epsilon}, K_s^{-\epsilon}); 0 \leq s \leq t\}$ the unique solution of double barriers reflected BSDE (3.4). We want to prove that $(X^\epsilon, Y^\epsilon, Z^\epsilon, K^{+\epsilon}, K^{-\epsilon})$ converge in law to (X, Y, Z, K^+, K^+) where (Y, Z, K^+, K^+) is the unique solution of double barriers reflected BSDE.

$$(4.1)\quad \begin{cases} Y_s = g(X_T) + \int_s^T f(r,X_r,Y_r)dr - \int_s^T Z_r dB_r + \left(K_T^+ - K_s^+\right) - \left(K_T^- - K_s^-\right) \\ \forall t \leq T, \quad L(t,X_t) \leq Y_t \leq U(t,X_t) \\ \text{and } \int_0^T (Y_t - L(t,X_t))dK_t^+ = \int_0^T (U(t,X_t) - Y_t)dK_t^- = 0 \ \mathbb{P}\text{-a.s.} \end{cases}$$

We give now a useful Lemmas, the first give us some tightness criteria for sequence of quasi-martingales.

Lemma 4.1 (See Meyer-Zheng [25] or Kurtz [24])**.** *The sequence of quasi-martingale $\{V_s^n; 0 \leq s \leq T\}$ defined on the filtered probability space $\{\Omega;\mathcal{F}_s, 0 \leq s \leq T;\mathbb{P}\}$ is tight whenever*

$$\sup_n\left(\sup_{0\leq s\leq T}\mathbb{E}|V_s^n| + CV_T(V^n)\right) < +\infty,$$

where $CV_T(V^n)$, denotes the "conditional variation of V^n on $[0,T]$" defined by

$$CV_T(V^n) = \sup \mathbb{E}\left(\sum_i |\mathbb{E}\big(V^n_{t_{i+1}} - V^n_{t_i}/\mathcal{F}_{t_i})|\right),$$

with "sup" meaning that the supremum is taken over all partitions of the interval $[0,T]$.

We use the penalization method to prove the convergence result. This second Lemma is the core, which explain each step.

Lemma 4.2. *Let U^ϵ be a family of random variables defined on the same probability spaces. For each $\epsilon \geq 0$, we assume the existence of a family of random variables $(U^{\epsilon,n})_n$, such that*

- $U^{\epsilon,n} \Longrightarrow U^{0,n}$ *as ϵ goes to zero.*
- $U^{\epsilon,n} \Longrightarrow U^{\epsilon}$ *as $n \to +\infty$, uniformly in ϵ.*
- $U^{0,n} \Longrightarrow U^{0}$ *as $n \to +\infty$*

then, U^ϵ converge in distribution to U^0.

Proof. This lemma is a simplified version of Theorem 3.2 in Billingsley [2, p. 28]. □

We put

$$M^\epsilon_t = -\int_0^t Z^\epsilon_s dB_r.$$

We denote by

- $\mathbb{C}([0,T],\mathbb{R}^d)$ the space of functions of $[0,T]$ with values in $\mathbb{R}^d$ equipped with the topology of uniform convergence.
- $\mathbb{D}([0,T],\mathbb{R}^k)$ is the space of càdlàg functions of $[0,T]$ with values in $\mathbb{R}^k$ equipped with the topology of Meyer-Zheng.

Theorem 4.3. *Under the above conditions, as ϵ tends to zero, the family of processes $(X^\epsilon, Y^\epsilon, M^\epsilon, K^{+\epsilon}, K^{-\epsilon})$ converge in law to the processes (X,Y,M,K^+,K^-) in $\mathbb{C}([0,t],\mathbb{R}^d) \times \mathbb{D}([0,t],\mathbb{R}^2) \times (\mathbb{C}[0,t],\mathbb{R}) \times (\mathbb{C}[0,t],\mathbb{R})$.*

The proof of this theorem follow the above Lemma. We first need some extra lemmas.

Let define

$$k^{+\varepsilon,n}_t(y) = n(L(t,X^\varepsilon_t) - y_t)^+ k^{-\varepsilon,n}_t(y) = n(y_t - U(t,X^\varepsilon_t))^+$$
$$k^{+0,n}_t(y) = n(L(t,X_t) - y_t)^+ k^{-0,n}_t(y) = n(y_t - U(t,X_t))^+$$

and

$$K^{\pm\varepsilon,n}_t(y) = \int_0^t k^{\pm\varepsilon,n}_s(y) ds K^{\varepsilon,n} = K^{+\varepsilon,n} - K^{-\varepsilon,n} k^{\varepsilon,n} = k^{+\varepsilon,n} - k^{-\varepsilon,n}$$

$$K_t^{\pm 0,n}(y) = \int_0^t k_s^{\pm 0,n}(y) ds K^{0,n} = K^{+0,n} - K^{-0,n} k^{0,n} = k^{+0,n} - k^{-0,n}.$$

Consider the backward stochastic differential equation

$$Y_s^{\epsilon,n} = g(X_T^\epsilon) + \int_s^T f(r, X_r^\epsilon, Y_r^{\epsilon,n}) dr - \int_s^T Z_r^{\epsilon,n} dB_r + \int_s^t k_r^{\varepsilon,n}(Y^{\epsilon,n}) dr, \tag{4.2}$$

and let (Y^n, Z^n), be the unique solution of the backward stochastic differential equation

$$Y_s^n = g(X_T) + \int_s^T f(r, X_r, Y_r^n) dr - \int_s^T Z_r^n dB_r + \int_s^T k_r^{0,n}(Y^n) dr. \tag{4.3}$$

We set

$$M_t^{\epsilon,n} = -\int_0^t Z_r^{\epsilon,n} dB_r, \quad \text{and} \quad M_t^n = -\int_s^t Z_r^n dB_r.$$

Lemma 4.4. *Under the above conditions, we have for each $n > 0$*

$$\sup_\epsilon \mathbb{E}\left[\sup_{t \le T} |Y_t^{\epsilon,n}|^2\right] < \infty \quad \textit{and} \quad \sup_\epsilon \mathbb{E}\left[\sup_{t \le T} |f(t, X_t^\epsilon, Y_t^{\epsilon,n})|^2\right] < \infty.$$

Proof. Let $(\underline{Y}^{\epsilon,n}, \underline{Z}_r^{\epsilon,n})$ and $(\overline{Y}^{\epsilon,n}, \overline{Z}_r^{\epsilon,n})$ be the solution of the following BSDE

$$\underline{Y}_s^{\epsilon,n} = g(X_T^\epsilon) + \int_s^T f(r, X_r^\epsilon, \underline{Y}_r^{\epsilon,n}) dr - \int_s^T k_r^{-\varepsilon,n}(\underline{Y}^{\epsilon,n}) dr - \int_s^T \underline{Z}_r^{\epsilon,n} dB_r$$

$$\overline{Y}_s^{\epsilon,n} = g(X_T^\epsilon) + \int_s^T f(r, X_r^\epsilon, \overline{Y}_r^{\epsilon,n}) dr + \int_s^T k_r^{+\varepsilon,n}(\overline{Y}^{\epsilon,n}) dr - \int_s^T \overline{Z}_r^{\epsilon,n} dB_r$$

then by the comparison theorem for BSDE we have for all $n > 0$, $\varepsilon > 0$,

$$\underline{Y}^{\epsilon,n} \le Y^{\epsilon,n} \le \overline{Y}^{\epsilon,n}.$$

Since $f_n(t,x,y) \in \{f(t,x,y) - n(y - U(t,y))^+, f(t,x,y) + n(L(t,x) - y)^+\}$ is Lipschitz in y uniformly in (ω, t, x), so using (3.1), we have $\sup_\epsilon \mathbb{E}\left[\sup_{t \le T} |\underline{Y}_t^{\epsilon,n}|^2\right] < \infty$ and $\sup_\epsilon \mathbb{E}\left[\sup_{t \le T} |\overline{Y}_t^{\epsilon,n}|^2\right] < \infty$. Hence

$$\sup_\epsilon \mathbb{E}\left[\sup_{t \le T} |Y_t^{\epsilon,n}|^2\right] \le \max\left\{\sup_\epsilon \mathbb{E}\left[\sup_{t \le T} |\underline{Y}_t^{\epsilon,n}|^2\right], \sup_\epsilon \mathbb{E}\left[\sup_{t \le T} |\overline{Y}_t^{\epsilon,n}|^2\right]\right\} < \infty.$$

From (3.2) and (3.1) we have $\sup_\epsilon \mathbb{E}[\sup_{t \le T} |f(t, X_t^\epsilon, Y_t^{\epsilon,n})|^2] < \infty$. □

Lemma 4.5. *There exists a constant C such that for each $n > 0$,*

$$\mathbb{E}\left[\sup_{0 \le r \le T} k_r^{+\varepsilon,n}(Y^{\epsilon,n})^2\right] \le C \quad \textit{and} \quad \mathbb{E}\left[\sup_{0 \le r \le T} k_r^{-\varepsilon,n}(Y^{\epsilon,n})^2\right] \le C.$$

Proof. For proof, see the proof of the following lemma. □

Lemma 4.6. *Under the above conditions, we have for each $n > 0$,*

$$\sup_\varepsilon \mathbb{E}\left[(K_T^{+\varepsilon,n}(Y^{\epsilon,n}))^2 + (K_T^{-\varepsilon,n}(Y^{\epsilon,n}))^2\right] < \infty.$$

Proof. Recall that the barriers are smooth Itô processes (in S^2) and from Lemma 4.4 $(f(s,X_s^{\epsilon},Y_s^{\epsilon,n}))_{0\le s\le T}\in S^2$. For each $n\ge 0$, let $\overline{Y}^{\varepsilon,n}:=Y^{\varepsilon,n}-U$, $f^*(s):=f(s,X_s^{\varepsilon},Y_s^{\varepsilon,n})$ and $U_t:=U_0+\int_0^t u_s ds+\int_0^t v_s dW_s$, recall that

$$\exists M>0/\mathbb{E}\left[\sup_{0\le t\le T}|u_s|+\int_0^T |v_s|^2\,ds\right]<M,$$

then

$$\begin{aligned}\overline{Y}_t^{\varepsilon,n}&=g(X_T^{\varepsilon})-U(T,X_T^{\varepsilon})+\int_t^T (f^*(s)-u_s)\,ds-\int_t^T (Z_s^{\varepsilon,n}-v_s)\,dW_s\\&\quad-n\int_t^T \left(\overline{Y}_s^{\varepsilon,n}\right)^+ds+n\int_t^T \left(\overline{Y}_s^{\varepsilon,n}-(L(s,X_s^{\varepsilon})-U(s,X_s^{\varepsilon}))\right)^-ds.\end{aligned}$$

For each $n\in\mathbb{N}$, let $\mathcal{D}^n$ the class of $\mathcal{P}$-measurable processes $z:\Omega\times[0,T]\longrightarrow[0,n]$. For $\nu\in\mathcal{D}^n$ and $\mu\in\mathcal{D}^n$, by applying Itô's formula to the product of $\overline{Y}^{\varepsilon,n}$ and $\exp\left(-\int_0^{\cdot}(\mu(r)+\nu(r))dr\right)$ and using the same arguments as in Cvitanic and Karatzas [7] (see also Matoussi et *al* [16]), one can show that

$$\begin{aligned}\overline{Y}_t^{\varepsilon,n}&=\operatorname*{ess\,sup}_{\mu\in\mathcal{D}^n}\operatorname*{ess\,inf}_{\nu\in\mathcal{D}^n}\mathbb{E}\Big\{(\xi-U_T)\exp\left(-\int_t^T(\mu(r)+\nu(r))dr\right)\\&\quad+\int_t^T\exp\left(-\int_t^s(\mu(r)+\nu(r))dr\right)[f^*(s)-u_s+\mu(s)(L_s-U_s)]\,ds/\mathcal{F}_t\Big\}\text{ a.s.}\end{aligned}$$

Therefore

$$\begin{aligned}\overline{Y}_t^{\varepsilon,n}&=\operatorname*{ess\,sup}_{\mu\in\mathcal{D}^n}\operatorname*{ess\,inf}_{\nu\in\mathcal{D}^n}\mathbb{E}\left\{\int_t^T\exp\left(-\int_t^s(\mu(r)+\nu(r))dr\right)|f^*(s)-u_s|ds/\mathcal{F}_t\right\}\\&\le\operatorname*{ess\,sup}_{\mu\in\mathcal{D}^n}\mathbb{E}\left\{\int_t^T\exp\left(-\int_t^s(\mu(r)+n)dr\right)|f^*(s)-u_s|ds/\mathcal{F}_t\right\}\\&\le\mathbb{E}\left\{\int_t^T\exp\left(-n(s-t)\right)|f^*(s)-u_s|ds/\mathcal{F}_t\right\}\\&\le\frac{1}{n}\mathbb{E}\left\{\int_t^T\sup_{0\le s\le T}|f^*(s)-u_s|ds/\mathcal{F}_t\right\},\end{aligned}$$

since $\sup_{0\le s\le T}|f^*(s)-u_s|\in L^2$, from Doob's maximal inequality, we have

$$\mathbb{E}\sup_{0\le t\le T}\left(\left[\overline{Y}_t^{\varepsilon,n}\right]^+\right)^2=\mathbb{E}\sup_{0\le t\le T}\left[\frac{k_t^{-\varepsilon,n}(Y^{\varepsilon,n})}{n}\right]^2\le\frac{C}{n^2}$$

where C is a constant which is independent in ε and can change line by line, so

$$\sup_{\varepsilon}\mathbb{E}\left[K_T^{-\varepsilon,n}(Y^{\varepsilon,n})\right]^2\le CT.$$

A similar analysis yields

$$\mathbb{E}\sup_{0\le s\le T}k_t^{+\varepsilon,n}(Y^{\varepsilon,n})^2\le C,\quad\text{ and so }\quad\sup_{\varepsilon}\mathbb{E}\left[K_T^{+\varepsilon,n}(Y^{\varepsilon,n})\right]^2\le CT.$$

Thus

$$\sup_{\varepsilon}\mathbb{E}\left[K_T^{-\varepsilon,n}(\underline{Y}^{\epsilon,n})\right]^2+\sup_{\varepsilon}\mathbb{E}\left[K_T^{+\varepsilon,n}(\underline{Y}^{\epsilon,n})\right]^2\le C.$$

□

The above proof, give us also the following

Lemma 4.7. *There exists a constant $C > 0$ such that for all $n > 0$,*

$$\sup_{\varepsilon} \mathbb{E}\left(\sup_{0\le t\le T} |Y_t^{\epsilon,n}|^2 + \mathbb{E}\int_0^T |Z_s^{\epsilon,n}|^2\, ds + K_T^{-\varepsilon,n}(Y^{\epsilon,n})^2 + K_T^{+\varepsilon,n}(Y^{\epsilon,n})^2 \right) < C.$$

This (uniform in ε) a priori estimate lead to

Proposition 4.1. *Under the above conditions, the family of processes $(Y^{\epsilon,n}, M^{\epsilon,n})$ which converge in law to the the family of processes (Y^n, M^n) on $\mathbb{D}([0,T],\mathbb{R})^2$.*

Proof. **Step 1: Tightness.**

Clearly

$$CV_T(Y^{\epsilon,n}) \le \mathbb{E}\left[\int_0^T \left(|f(r, X_r^\epsilon, Y_r^{\epsilon,n})| + k_r^{+\varepsilon,n}(Y^{\epsilon,n}) + k_r^{-\varepsilon,n}(Y^{\epsilon,n}) \right) dr \right] < C,$$

and it follows from Lemma 4.4 and assumptions that

$$\sup_{\epsilon} \left(CV_T(Y^{\epsilon,n}) + \mathbb{E} \sup_{0\le s\le T} |Y_s^{\epsilon,n}|^2 + \int_0^T |Z_r^{\epsilon,n}|^2\, dr \right) < C,$$

hence the sequence $\{(Y_s^{\epsilon,n}, M_s^{\epsilon,n}); 0 \le s \le T\}$ satisfy Meyer-Zheng's tightness criterion for quasi-martingales under $\mathbb{P}$.

Step 2: Convergence in law.

By tighness, there exists a subsequence (which we still denote $(Y^{\epsilon,n}, M^{\epsilon,n})$) such that

$$(X^\epsilon, Y^{\epsilon,n}, M^{\epsilon,n}) \Longrightarrow (X, Y^n, M^n),$$

on $\mathbb{C}([0,T],\mathbb{R}^d) \times (\mathbb{D}([0,T],\mathbb{R}))^2$, where the first factor is equipped with the topology of uniform convergence, and the second with the topology of convergence in ds measure. Clearly, for each $0 \le s \le T$, $(x,y) \longrightarrow \int_s^T f(x(r), y(r))dr$ is continuous for $\mathbb{C}([0,T],\mathbb{R}^d) \times \mathbb{D}([0,T],\mathbb{R})^2$ equipped with the same topology as above, and $y \longrightarrow \int_s^T \left(k_r^{+\varepsilon,n} - k_r^{-\varepsilon,n}\right)(y)dr$ is continuous in $\mathbb{D}([0,T],\mathbb{R})$ as ϵ goes to 0. We can now take the limit in (4.2), yielding

$$Y_s^n = g(X_s) + \int_s^T f(r, X_r, Y_r^n)dr + M_t^n - M_s^n + \int_s^T \left(k_r^{+0,n} - k_r^{-0,n}\right)(Y^n)dr.$$

Moreover, for any $0 \le s_1 < s_2 \le T$, $\phi \in \mathbb{C}_b^\infty$ and ψ_s a functional of $X_r^\epsilon, Y_r^{\epsilon,n}, K_r^{\epsilon,n}$ $0 \le r \le T$, bounded and continuous in $\mathbb{C}([0,T],\mathbb{R}^d) \times \mathbb{D}([0,T],\mathbb{R}^k) \times \mathbb{C}([0,T],\mathbb{R}^d)$, we have

$$\mathbb{E}\left(\psi_{s_1}(X^\epsilon, Y^{\epsilon,n})\phi(X_{s_2}^\epsilon) - \phi(X_{s_1}^\epsilon) - \int_{s_1}^{s_2} L\phi(X_r^\epsilon)dr \right) \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0,$$

and for each $\varepsilon > 0$,

$$\mathbb{E}(\psi_{s_1}(X^\epsilon, Y^{\epsilon,n}) \int_0^\alpha (M_{s_2+r}^{\epsilon,n} - M_{s_1+r}^{\epsilon,n})dr) = 0.$$

From weak convergence, the fact that $\mathbb{E}(\sup_{0\le s\le T}|M_s^{\epsilon,n}|^2)<+\infty$, dividing the second identity by α and for $\alpha\to 0$, we have

$$\mathbb{E}\left[\psi_{s_1}(X,Y^n)\left(\phi(X_{s_2})-\phi(X_{s_1})-\int_{s_1}^{s_2}L\phi(X_r)dr\right)\right]=0,$$

$$\mathbb{E}\left(\psi_{s_1}(X,Y^n)(M_{s_2}^n-M_{s_1}^n)\right)=0.$$

Therefore, both M^n and M^X (the martingale part of X) are $\mathcal{F}_t^{X,Y^n}$ martingales. It follows from the first statement that the process X satisfies the martingale problem with respect to the filtration $\mathcal{F}_t^{X,Y^n}$, hence M^X is $\mathcal{F}_t^{X,Y^n}$-martingale.

Step 3: Identification of the limit.

Since the martingale problem in Step 2 is well-posed, let $(\overline{Y}^n,\overline{U}^n)$ denote the unique solution of the BSDE

$$\overline{Y}_s^n=g(X_t)+\int_s^t f(r,X_r,\overline{Y}_r^n)dr-\int_s^t\overline{U}_r^n dM_r^X+\int_s^t k_r^{0,n}(\overline{Y}^n)dr,$$

satisfying $\mathbb{E}Tr\int_s^t\overline{U}_r^n\langle M^X\rangle_r\overline{U}_r^n<+\infty$, and let $\widetilde{M}_s^n=\int_0^s\overline{U}_r^n dM_r^X$. Since $\overline{Y}^n$ and $\overline{U}^n$ are $\mathcal{F}_t^X$ adapted, and M^X is $\mathcal{F}_t^{X,Y^n}$ martingale, $\widetilde{M}^n$ is $\mathcal{F}_t^{X,Y^n}$ martingale. It follows from Itô formula that

$$\begin{aligned}
&\mathbb{E}|\overline{Y}_s^n-Y_s^n|^2+\mathbb{E}[M^n-\widetilde{M}^n]_t-\mathbb{E}[M^n-\widetilde{M}^n]_s\\
&=2\int_s^t\langle f(r,X_r,Y_r^n)-f(r,X_r,\overline{Y}_r^n),\overline{Y}_r^n-Y_r^n\rangle dr\\
&\quad+2\int_s^t\langle k_r^{0,n}(Y^n)-k_r^{0,n}(\overline{Y}^n),\overline{Y}_r^n-Y_r^n\rangle dr\\
&\le C_n\mathbb{E}\int_s^t|\overline{Y}_r^n-Y_r^n|^2dr
\end{aligned}$$

(we use the fact that the operator is n-Lipschitz). Hence from Gronwall's lemma $\overline{Y}_r^n=Y_r^n$, $0\le s\le t$, and $M^n=\widetilde{M}^n$, and so all sequence converge. □

Now we deal with the uniform convergence of the process $(Y^{\epsilon,n},M^{\epsilon,n},K^{\epsilon,n})_n$ to $(Y^\epsilon,M^\epsilon,K^\epsilon)$ as n goes to $+\infty$.

Proposition 4.2. *The family of processes $(Y^{\epsilon,n},M^{\epsilon,n},K^{\epsilon,n})_n$ converges uniformly of $\epsilon\in[0,1]$ in probability to the family of processes $(Y^\epsilon,M^\epsilon,K^\epsilon)$ as n goes to $+\infty$.*

Proof. By the same proof as in Lemma 4.4, we have (since one can choose all constants independently on ε and n),

$$\sup_\varepsilon\sup_n\mathbb{E}\left(\sup_{0\le t\le T}|Y_t^{\epsilon,n}|^2+\mathbb{E}\int_0^T|Z_s^{\epsilon,n}|^2\,ds+K_T^{-\varepsilon,n}(Y^{\epsilon,n})^2+K_T^{+\varepsilon,n}(Y^{\epsilon,n})^2\right)<\infty.$$

Now, let us prove the convergence of $(Y^{\epsilon,n}, Z^{\epsilon,n})_n$ for every $(n,m) \in \mathbb{N}^* \times \mathbb{N}^*$. By Itô's formula, one has

$$\begin{aligned}
&|Y_s^{\epsilon,n} - Y_s^{\epsilon,m}|^2 + \int_s^T |Z_r^{\epsilon,n} - Z_r^{\epsilon,m}|^2 dr \\
&\quad = 2\int_s^T (Y_r^{\epsilon,n} - Y_r^{\epsilon,m})(f(r, X_r^\epsilon, Y_r^{\epsilon,n} - f(r, X_r^\epsilon, Y_r^{\epsilon,m}))dr \\
&\qquad + 2\int_s^T (Y_r^{\epsilon,n} - Y_r^{\epsilon,m})(Z_r^{\epsilon,n} - Z_r^{\epsilon,m})dB_r \\
&\qquad - 2\int_s^T (Y_r^{\epsilon,n} - Y_r^{\epsilon,m})(k_r^{\varepsilon,n}(Y^{\epsilon,n}) - k_r^{\varepsilon,m}(Y^{\epsilon,m}))dr,
\end{aligned}$$

using (3.2), we deduce that

$$\begin{aligned}
&\mathbb{E}|Y_s^{\epsilon,n} - Y_s^{\epsilon,m}|^2 + \mathbb{E}\int_s^T |Z_r^{\epsilon,n} - Z_r^{\epsilon,m}|^2 dr \\
&\quad \leq 2K\int_s^T |Y_r^{\epsilon,n} - Y_r^{\epsilon,m}|^2 dr - 2\mathbb{E}\int_s^T (Y_r^{\epsilon,n} - Y_r^{\epsilon,m})(k_r^{\varepsilon,n}(Y^{\epsilon,n}) - k_r^{\varepsilon,m}(Y^{\epsilon,m}))dr.
\end{aligned}$$

Using Schwartz inequality, we have

$$\begin{aligned}
&|\mathbb{E}\int_s^T (Y_r^{\epsilon,n} - Y_r^{\epsilon,m})(k_r^{\varepsilon,n}(Y^{\epsilon,n}) - k_r^{\varepsilon,m}(Y^{\epsilon,m}))dr| \\
&\quad \leq \left(\mathbb{E}\int_s^T (Y_r^{\epsilon,n} - Y_r^{\epsilon,m})^2\right)^{1/2} \left(\mathbb{E}\int_s^T (k_r^{\varepsilon,n}(Y^{\epsilon,n}) - k_r^{\varepsilon,m}(Y^{\epsilon,m}))^2\right)^{1/2} \\
&\quad \leq \sqrt{T}\left(\mathbb{E}\sup_{0\leq r\leq T}(Y_r^{\epsilon,n} - Y_r^{\epsilon,m})^2\right)^{1/2} \left(\mathbb{E}\int_s^T (k_r^{\varepsilon,n}(Y^{\epsilon,n}) + k_r^{\varepsilon,m}(Y^{\epsilon,m}))^2\right)^{1/2} \\
&\quad \leq C\left[\left(\mathbb{E}\sup_{0\leq r\leq T}(\overline{Y}_r^{\varepsilon,n})^2\right)^{1/2} + \left(\mathbb{E}\sup_{0\leq r\leq T}(\overline{Y}_r^{\varepsilon,m})^2\right)^{1/2}\right] \\
&\quad \leq C\left(\frac{1}{n} + \frac{1}{m}\right).
\end{aligned}$$

We let C to be a constant changing line by line and we have used the facts that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, $(Y_r^{\epsilon,n} - Y_r^{\epsilon,m}) = \overline{Y}_r^{\varepsilon,n} - \overline{Y}_r^{\varepsilon,m}$, $2ab \leq a^2 + b^2$ and Lemma 4.5.

Hence, from Gronwall's lemma and above Lemmas, we deduce that

$$\sup_{0\leq s\leq T}\mathbb{E}\left(|Y_s^{\epsilon,n} - Y_s^{\epsilon,m}|^2 + \int_0^T |Z_r^{\epsilon,n} - Z_r^{\epsilon,m}|^2 dr\right) \leq C\left(\frac{1}{n} + \frac{1}{m}\right).$$

Using Bulkholder-Davis-Gundy inequality, we obtain

$$\sup_\epsilon \mathbb{E}\left(\sup_{0\leq s\leq T}|Y_s^{\epsilon,n} - Y_s^{\epsilon,m}|^2 + \int_0^T |Z_r^{\epsilon,n} - Z_r^{\epsilon,m}|^2 dr\right) \leq C\left(\frac{1}{n} + \frac{1}{m}\right).$$

We set

$$\lim_{n\to+\infty} Y^{\epsilon,n} = \overline{Y}^\epsilon, \quad \lim_{n\to+\infty} Z^{\epsilon,n} = \overline{Z}^\epsilon.$$

If we return to the equation satisfied by the $(Y^{\epsilon,n}, Z^{\epsilon,n})$, we find also that $(K^{\epsilon,n})_n$ converges uniformly in $\mathbb{L}^2(\Omega)$ to the limit $\overline{K}^\epsilon$ where

$$\overline{K}_t^\epsilon = \lim_n \int_0^t k_r^{\varepsilon,n}(Y^{\epsilon,n})dr.$$

We have shown

$$\sup_{\epsilon,n} \mathbb{E}\|K^{\epsilon,n}\|^2_{H^1([0,T],\mathbb{R})} < \infty,$$

where $H^1([0,T],\mathbb{R}^d)$ is the Sobolev space. Hence the sequence $(K^{\epsilon,n})_n$ is bounded independently of ϵ in $\mathbb{L}^2(\Omega; H^1([0,T],\mathbb{R}^d))$ and there exist a subsequence of $(K^{\epsilon,n})_n$ which converges weakly. The limiting process $\overline{K}^\epsilon$ belong to $\mathbb{L}^2(\Omega; H^1([0,T],\mathbb{R}^d))$, hence $\overline{K}^\epsilon$ is absolutely continuous. By uniqueness of solution of the reflected BSDE, we can find that $\overline{Y}^\epsilon = Y^\epsilon$, $\overline{Z}^\epsilon = Z^\epsilon$, $\overline{K}^\epsilon = K^\epsilon$. □

Proposition 4.3. *Under the assumption of the above lemma, the family of processes* (Y^n, M^n, K^n) *converge in probability to* (Y, M, K) *as* n *goes to* $+\infty$.

Proof. Similar to the above one. □

Proof of Theorem 4.3. Combining the above lemmas, we find that $(X^\epsilon, Y^\epsilon, M^\epsilon, K^\epsilon)$ converge in law to (X, Y, M, K) in the sense defined as above, where

$$Y_s = g(X_T) + \int_s^T f(r, X_r, Y_r)dr - \int_s^T Z_r dB_r + K_T - K_s.$$

Corollary 4.1. *Under the assumptions of theorem,* $\{Y_0^\epsilon\}$ *converge to* Y_0 *as* ϵ *goes to* 0.

Proof. Since Y_0^ϵ is deterministic, we have

$$Y_0^\epsilon = \mathbb{E}\left[g(X_T^\epsilon) + \int_0^T f(s, X_s^\epsilon, Y_s^\epsilon)ds + K_T^\epsilon\right].$$

Put

$$A_\epsilon = g(X_T^\epsilon) + \int_0^T f(s, X_s^\epsilon, Y_s^\epsilon)ds + K_T^\epsilon,$$

we have

$$\mathbb{E}\,|A_\epsilon|^2 \le (C+K)\mathbb{E}\left(1 + |X_T^\epsilon|^{2p} + \int_0^T \left[|Y_s^\epsilon|^2 + |X_s^\epsilon|^{2q}\right] ds\right) + \mathbb{E}\,|K_T^\epsilon|^2,$$

by above assumptions and estimates, we have

$$\sup_\epsilon \mathbb{E}\,|A_\epsilon|^2 < \infty,$$

and A_ϵ converge in law, as ϵ goes to 0, toward

$$g(X_T) + \int_0^T f(r, X_r, Y_r)dr + K_T,$$

the uniform integrability of A_ϵ implies that

$$\lim_{\epsilon\longrightarrow 0}\mathbb{E}(A_\epsilon)=\mathbb{E}\left(\lim_{\epsilon\longrightarrow 0}A_\epsilon\right),$$

this means that Y_0^ε converge to

$$Y_0=g(X_T)+\int_0^T f(r,X_r,Y_r)dr+K_T.$$

□

We applied the previous results to the homogenization of a class of variational inequality as well as to stochastic game.

5. APPLICATIONS

Let $u^\epsilon:[0,T]\times\mathbb{R}\longrightarrow\mathbb{R}$ be a solution of the variational inequality (3.11) The homogenization problem consists in computing the limit as $\epsilon\downarrow 0$ of $u_\epsilon(t,x)$.

Theorem 5.1. *If the barriers satisfies also the polynomial growth conditions* (3.10), *then*

$$u^\epsilon(t,x)\longrightarrow u(t,x),\quad \textit{as } \epsilon \textit{ goes to } 0,$$

where u is the solution of the solution (in the viscosity sense) of the variational inequality

$$\text{(5.1)}\quad \left\{\begin{array}{rcl}\min\left((u-L)\,;\max\left[(u-U)\,;\left(-\frac{\partial u}{\partial t}-L_t^0u\right)-f(\cdot,\cdot,u)\right]\right)(t,x) & = & 0,\\ u(T,x) & = & g(x).\end{array}\right.$$

For any $\varepsilon>0$, $0\le t\le T$ and any two stopping times $\sigma,\tau\in\mathcal{M}_{t,T}$, consider the pay off of the stochastic game $R_t^\varepsilon(\sigma,\tau)$, its value function V_t^ε (see (3.5), (3.7)). The homogenization problem consists also in computing the limit as $\epsilon\downarrow 0$,

Theorem 5.2. *For any $0\le t\le T$ and any two stopping times $\sigma,\tau\in\mathcal{M}_{t,T}$*

$$\begin{array}{llll}R_t^\varepsilon(\sigma,\tau)\longrightarrow R_t(\sigma,\tau) & \textit{as} \quad \epsilon & \textit{goes to} & 0,\\ V_t^\varepsilon\longrightarrow V_t & \textit{as} \quad \epsilon & \textit{goes to} & 0,\end{array}$$

where

$$R_t(\sigma,\tau)=\int_t^{\sigma\wedge\tau}h(s)ds+g(X_T)1_{\{\sigma\wedge\tau=T\}}+L(\tau,X_\tau)1_{\{\tau<T,\tau\le\sigma\}}+U(\sigma,X_\sigma)1_{\{\sigma<\tau\}}$$

is the payoff of a stochastic game, which value function V_t is given by the state-process Y given in the Theorem 4.3.

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