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ASYMPTOTICS FOR THE RUIN PROBABILITIES OF A TWO-DIMENSIONAL RENEWAL RISK MODEL

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ABSTRACT. In this paper, we mainly consider the uniform asymptotic behavior for the finite-time ruin probabilities of a two-dimensional insurance model. In this model, the insurance company operates two classes of insurance business, whose claims occur in pairs and share a same claim arrival process. In the obtained result, the claim distributions cover most of the common subexponential distributions. The risk model with dependent inter-arrival times as well as the one with Brownian motion diffusions are also investigated.

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1. INTRODUCTION

The multi-dimensional risk models were initially investigated by Hult et al. (2005). Since then, many researchers have been devoted to the research of this field and have obtained many meaningful results, see, for instance, Yuen et al. (2006), Li et al. (2007), Chen et al. (2011), Zhang and Wang (2012), Chen et al. (2013), Gao and Yang (2014), Yang and Li (2014), Jiang et al. (2015), Lu and Zhang (2016) and Yang and Yuen (2016), and so on.

Among the above-mentioned results, Chen et al. (2011) considered a two-dimensional insurance model satisfying the following assumptions.

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Assumption 1.1. The claims come in pairs and the claim amounts $\{X_{in}, n \geq 1\}$, $i = 1, 2$ form two sequences of independent, identically distributed (i.i.d.) and non-negative random variables (r.v.s) with finite means and distributions F_1 and F_2 respectively.

Assumption 1.2. The claims of the two classes of insurance business share the same claim arrival process

$$N(t) = \inf\left\{n : \sum_{i=1}^{n} \theta_i \leq t\right\}, \quad t \geq 0, \tag{1.1}$$

where the claim inter-arrival times $\{\theta_n, n \geq 1\}$ are i.i.d. r.v.s with a common finite positive mean λ^{-1}. Namely, $\{N(t), t \geq 0\}$ is a renewal counting process. We always assume that $\{X_{1n}, n \geq 1\}$, $\{X_{2n}, n \geq 1\}$ and $\{N(t), t \geq 0\}$ are mutually independent.

Assumption 1.3. The premium rates of the two classes of insurance business are two positive constants c_1 and c_2 such that the following safety loading conditions hold:

$$\mu_i = \lambda^{-1} c_i - EX_{i1} > 0, \quad i = 1, 2.$$

With the initial surplus vector of the insurance company $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} > \vec{0}$, the ruin probability by time t, $t \geq 0$ can be defined as

$$\psi_a(\vec{x}, t) = P\left(\bigcap_{i=1}^{2}\left\{\sup_{0\leq s\leq t} L_i(s) > x_i\right\}\right)$$

or

$$\psi_b(\vec{x}, t) = P\left(\bigcup_{i=1}^{2}\left\{\sup_{0\leq s\leq t} L_i(s) > x_i\right\}\right),$$

where

$$\begin{pmatrix} L_1(t) \\ L_2(t) \end{pmatrix} = \sum_{j=1}^{N(t)} \begin{pmatrix} X_{1j} \\ X_{2j} \end{pmatrix} - \begin{pmatrix} c_1 t \\ c_2 t \end{pmatrix}, \quad t \geq 0 \tag{1.2}$$

is the aggregate loss process of the insurance company. See Lu and Zhang (2016) for some other definitions of the ruin probabilities.

Before introducing a brief review and our main result on the two-dimensional ruin probabilities, we need some notions and notation. For a r.v. X with a distribution F, we denote its tail by $\overline{F}$, namely, $\overline{F}(x) := P(X > x)$ for any $-\infty < x < \infty$. We say that X is bounded above if there exists some $-\infty < C < \infty$ such that $F(C) = 1$, and unbounded above otherwise. Hereafter, we always suppose that X is unbounded above. We say that X (or F) is long-tailed, denoted by $F \in \mathcal{L}$, if $\lim_{x\to\infty} \frac{\overline{F}(x-1)}{\overline{F}(x)} = 1$; we say that X (or F) is dominatedly-varying tailed, denoted by $F \in \mathcal{D}$, if $\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty$ for some $0 < y < 1$; we say that X (or F) is consistently-varying tailed, denoted by $F \in \mathcal{C}$, if $\lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)} = 1$; we say that X (or F) is subexponential, denoted by $F \in \mathcal{S}$, if $F \in \mathcal{L}$ and $\lim_{x\to\infty}\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 2$, where F^{*2} is the two-fold convolution of F; We say that F is strongly subexponential, denoted by $F \in \mathcal{S}_*$, if $\int_0^\infty \overline{F}(x)dx < \infty$ and $\limsup_{x\to\infty}\sup_{1\le u<\infty}\left|\frac{\overline{F_u^{*2}}(x)}{\overline{F_u}(x)} - 1\right| = 0$, where

$$\overline{F_u}(x) = \begin{cases} \min\{1, \int_x^{x+u}\overline{F}(t)dt\}, & \text{if } x > 0; \\ 1, & \text{if } x \le 0. \end{cases}$$

If $\int_0^\infty \overline{F}(x)dx < \infty$, then we have the following inclusion relationships:

$$\mathcal{C} \subseteq \mathcal{L}\cap\mathcal{D} \subseteq \mathcal{S}_* \subseteq \mathcal{S} \subseteq \mathcal{L},$$

see, for instance, Cline et al. (1994), Korshunov (2002) and Denisov et al. (2004), among many others.

Now we return to the main topic of this paper. Based on the one-dimensional results of Tang (2004), Chen et al. (2011) derived uniform asymptotics of $\psi_a(\vec{x},t)$ and $\psi_b(\vec{x},t)$ under the condition that the claim distributions belonged to the class $\mathcal{C}$. Chen et al. (2013) generalized this model by enlarging the class of the claim distributions from the class $\mathcal{C}$ to the class $\mathcal{L}\cap\mathcal{D}$. Besides, they allowed the claim inter-arrival times to be dependent according to certain dependence structure. More recently, Lu and Zhang (2016) obtained uniform asymptotics for $\psi_a(\vec{x},t)$, $\psi_b(\vec{x},t)$ and ruin probabilities of some other forms by assuming that the claim distributions belonged to the class $\mathcal{S}_*$. However, they needed some extra conditions.

This paper aims to remove the limitations in Lu and Zhang (2016). More concretely, we will establish the following uniform asymptotics for $\psi_a(\vec{x},t)$, $\psi_b(\vec{x},t)$ for general strongly subexponential claims. For notational convenience, for two functions $a(x,t)$ and $b(x,t)$, we write $a(x,t) \lesssim b(x,t)$, if $\limsup_{(x,t)\to(\infty,\infty)} a^{-1}(x,t)b(x,t) \le 1$; $a(x,t) \gtrsim b(x,t)$, if $\liminf_{(x,t)\to(\infty,\infty)} a^{-1}(x,t)b(x,t) \ge 1$ and $a(x,t) \sim b(x,t)$, if both $a(x,t) \lesssim b(x,t)$ and $a(x,t) \gtrsim b(x,t)$. Hereafter, unless otherwise stated, the limit procedure is to let $(x_1\wedge x_2, t) \to (\infty,\infty)$ but with $\kappa x_1 \le x_2 \le \kappa^{-1}x_1$ for some positive constant $\kappa \in (0,1)$, where $x_1 \wedge x_2 = \min\{x_1, x_2\}$.

With the above set-up, we are now ready to state our main result which gives asymptotic estimates of the finite-time ruin probabilities $\psi_a(\vec{x},t)$ and $\psi_b(\vec{x},t)$.

Theorem 1.4. *Consider the two-dimensional risk model whose aggregate loss process is defined by* (1.2). *Suppose that Assumptions* 1.1–1.3 *hold. If $F_1, F_2 \in \mathcal{S}_*$, then*

$$\psi_a(\vec{x},t) \sim \prod_{i=1}^{2}\mu_i^{-1}\int_{x_i}^{x_i+\mu_i\lambda t}\overline{F_i}(y)dy \tag{1.3}$$

and

$$\psi_b(\vec{x},t) \sim \sum_{i=1}^{2} \mu_i^{-1} \int_{x_i}^{x_i+\mu_i\lambda t} \overline{F_i}(y)dy. \tag{1.4}$$

By Theorem 1.4, we immediately obtain the following result, which generalizes Theorem 2 of Chen et al. (2011) and Theorem 3.1 of Lu and Zhang (2016).

Corollary 1.5. *Let the conditions of Theorem* 1.4 *be valid. Then for any function* $f(\cdot) : [0,\infty) \mapsto [0,\infty)$ *such that* $f(x) \to \infty$ *as* $x \to \infty$,

$$\limsup_{x_1\wedge x_2\to\infty} \sup_{t\geq f(x_1\wedge x_2)} \left| \frac{\psi_a(\vec{x},t)}{\prod_{i=1}^{2} \mu_i^{-1} \int_{x_i}^{x_i+\mu_i\lambda t} \overline{F_i}(y)dy} - 1 \right| = 0$$

and

$$\limsup_{x_1\wedge x_2\to\infty} \sup_{t\geq f(x_1\wedge x_2)} \left| \frac{\psi_b(\vec{x},t)}{\sum_{i=1}^{2} \mu_i^{-1} \int_{x_i}^{x_i+\mu_i\lambda t} \overline{F_i}(y)dy} - 1 \right| = 0.$$

The rest of this paper is organized as follows. In Section 2 we first develop some lemmas, then prove the main result given in Section 1. In Section 3, we extend the two-dimensional risk model to one with Brownian motion diffusions.

2. **Proof of Theorem 1.4**

2.1. **Some Lemmas.**

In order to prove Theorem 1.4, we need some lemmas. The first lemma investigates some properties of long-tailed distributions, which has its own independent interest.

Lemma 2.1. *Let* $F \in \mathcal{L}$. *Then for any* $\varepsilon > 0$, *there exists some* $x_0 > 0$ *such that for all* $x > x_0$ *and all* $0 \leq u < x - x_0$,

$$e^{\varepsilon(u-1)}\overline{F}(x) \leq \overline{F}(x-u) \leq e^{\varepsilon(u+1)}\overline{F}(x).$$

Proof. We only prove the inequality on the right-hand side, and the one on the left-hand side can be proved similarly.

For any $\varepsilon > 0$, by $F \in \mathcal{L}$, we may choose some $x_0 > 0$ such that for all $x > x_0$,

$$\overline{F}(x-1) \leq e^{\varepsilon}\overline{F}(x).$$

Then for all $x>x_0$ and all positive integer n such that $x - n + 1 > x_0$, we have

$$\frac{\overline{F}(x-n)}{\overline{F}(x)} = \frac{\overline{F}(x-n)}{\overline{F}(x-n+1)} \cdot \frac{\overline{F}(x-n+1)}{\overline{F}(x-n+2)} \cdots \frac{\overline{F}(x-1)}{\overline{F}(x)} \leq e^{\varepsilon n}. \tag{2.1}$$

For any non-negative number u such that $0 \leq u < x-x_0$, let n_u be the largest integer which is no larger than u, then $u-1 < n_u \leq u < n_u+1$. Therefore, for all $x > x_0$ and all $0 \leq u < x - x_0$, we have $x-(n_u+1)+1 = x-n_u > x_0$, thus by (2.1),

$$\frac{\overline{F}(x-u)}{\overline{F}(x)} \leq \frac{\overline{F}(x-(n_u+1))}{\overline{F}(x)} \leq e^{\varepsilon(n_u+1)} \leq e^{\varepsilon(u+1)}.$$

This completes the proof. $\square$

Lemma 1 and Theorem 1 of Korshunov (2002) stated the following uniform asymptotic result for the maximum of a random walk with a negative mean, which plays an important role in this paper. Here we make the convention that a summation over an empty set of indices is zero.

Lemma 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. r.v.s with a common distribution F and a common mean $-\mu < 0$. Let $\{S_n = \sum_{k=1}^n X_k, n \geq 0\}$ be the random walk generated by $\{X_n, n \geq 1\}$.*

(i) *If $F \in \mathcal{L}$, then*

$$\liminf_{x\to\infty} \inf_{n\geq 1} \frac{P(\max_{1\leq k\leq n} S_k > x)}{\mu^{-1}\int_x^{x+n\mu} \overline{F}(y)dy} \geq 1.$$

(ii) *If $F \in \mathcal{S}_*$, then*

$$\limsup_{x\to\infty} \sup_{n\geq 1} \left| \frac{P(\max_{1\leq k\leq n} S_k > x)}{\mu^{-1}\int_x^{x+n\mu} \overline{F}(y)dy} - 1 \right| = 0.$$

The following Lemma is due to Lemma 3.3 of Leipus and Šiaulys (2007). The readers are referred to Lemma 4.4 of Wang et al. (2012) for a more general form in the dependent case.

Lemma 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. r.v.s with a common mean $-\mu < 0$ and $\{S_n, n \geq 0\}$ be the random walk generated by $\{X_n, n \geq 1\}$. If X_1 is bounded above, then there exist two constants $r, M > 0$ such that for all $x > 0$,*

$$P\left(\max_{n\geq 1} S_n > x\right) \leq Me^{-rx}.$$

The following Lemma is due to Theorem 1(i) of Kočetova et al. (2009). See Lemma 4.6 of Wang et al. (2012) for its generalization to the dependent case.

Lemma 2.4. *Consider the renewal counting process $\{N(t), t \geq 0\}$ introduced in (1.1). For any $a > \lambda$, there exists some $b > 1$ such that*

$$\lim_{t\to\infty} \sum_{k>at} P(N(t) \geq k)b^k = 0.$$

The last lemma is due to (3.2) of Chen et al. (2011).

Lemma 2.5. *Let $f(x)$ be a non-increasing function defined on $[a, c]$, $a < c$. If $a < b < c$, then*

$$\int_a^c f(x)dx \leq (b-a)^{-1}(c-a)\int_a^b f(x)dx.$$

2.2. **Proof of (1.3).** In this subsection, we prove (1.3) in Theorem 1.4. It's sufficient to prove the following result:

$$\prod_{i=1}^{2}\mu_i^{-1}\int_{x_i}^{x_i+\mu_i\lambda t}\overline{F_i}(y)dy \lesssim \psi_a(\vec{x},t) \lesssim \prod_{i=1}^{2}\mu_i^{-1}\int_{x_i}^{x_i+\mu_i\lambda t}\overline{F_i}(y)dy. \tag{2.2}$$

2.2.1. *Proof of the upper limit part in* (2.2). For any $\varepsilon > 0$, let

$$\begin{aligned}\psi_a(\vec{x},t) = &\Bigg(\sup_{0\le k\le N(t)}\sum_{j=1}^{k}(X_{1j}-c_1\theta_j) > x_1,\\ &\sup_{0\le k\le N(t)}\sum_{j=1}^{k}(X_{2j}-c_2\theta_j) > x_2, N(t)\le \lambda t(1+\varepsilon)\Bigg)\\ &+P\Bigg(\sup_{0\le k\le N(t)}\sum_{j=1}^{k}(X_{1j}-c_1\theta_j) > x_1,\\ &\sup_{0\le k\le N(t)}\sum_{j=1}^{k}(X_{2j}-c_2\theta_j) > x_2, N(t) > \lambda t(1+\varepsilon)\Bigg)\\ :=\;& \psi_{a1}(\vec{x},t)+\psi_{a2}(\vec{x},t).\end{aligned} \tag{2.3}$$

We first deal with $\psi_{a2}(\vec{x},t)$. By Lemma 2.4, there exists some $\beta > 0$ such that

$$\limsup_{t\to\infty}\sum_{n>\lambda t(1+\varepsilon)} P(N(t)\ge n)e^{2\beta n} = 0. \tag{2.4}$$

Furthermore, by $F_i \in \mathcal{S}_*$ and Kesten's inequality, there exists some $K := K(\beta) > 0$ such that for all $x \ge 0$ and all $n \ge 1$,

$$\overline{F_i^{*n}}(x) \le K e^{\beta n}\overline{F_i}(x), \quad i = 1, 2. \tag{2.5}$$

We know by (2.4) that there exists some $t_1' > 0$ such that for all $t \ge t_1'$,

$$\mu_i\lambda t > 1, \quad i = 1, 2 \tag{2.6}$$

and

$$K^2\sum_{n>\lambda t(1+\varepsilon)} P(N(t)\ge n)e^{2\beta n} < \varepsilon. \tag{2.7}$$

By (2.5) and (2.7), it holds for all $t \ge t_1'$ and $x_1, x_2 \ge 0$ that

$$\begin{aligned}\psi_{a2}(\vec{x},t) &\le \sum_{n>\lambda t(1+\varepsilon)} P\left(\sum_{j=1}^{n}X_{1j} > x_1, \sum_{j=1}^{n}X_{2j} > x_2, N(t) = n\right)\\ &\le K^2\overline{F_1}(x_1)\overline{F_2}(x_2)\sum_{n>\lambda t(1+\varepsilon)} e^{2\beta n}P(N(t)=n)\\ &\le \varepsilon\overline{F_1}(x_1)\overline{F_2}(x_2).\end{aligned} \tag{2.8}$$

Still by $F \in \mathcal{S}_*$, there exists some x_0' sufficiently large such that for all $x > x_0'$,

$$\overline{F_i}(x+1) \geq \frac{1}{2}\overline{F_i}(x), \quad i = 1, 2. \tag{2.9}$$

By (2.6), (2.8) and (2.9), it holds for $x_1 \wedge x_2 > x_0'$ and $t > t_1'$ that

$$\begin{aligned} \psi_{a2}(\vec{x}, t) &\leq \frac{\varepsilon \overline{F_1}(x_1)\overline{F_2}(x_2)}{\prod_{i=1}^{2} \mu_i^{-1} \int_{x_i}^{x_i+1} \overline{F_i}(y)dy} \prod_{i=1}^{2} \mu_i^{-1} \int_{x_i}^{x_i+\mu_i\lambda t} \overline{F_i}(y)dy \\ &\leq 4\varepsilon\mu_1\mu_2 \prod_{i=1}^{2} \mu_i^{-1} \int_{x_i}^{x_i+\mu_i\lambda t} \overline{F_i}(y)dy. \end{aligned} \tag{2.10}$$

Next, we deal with $\psi_{a1}(\vec{x}, t)$. Let $c = c_1 \vee c_2 = \max\{c_1, c_2\}$. For any $0 < \delta < \min\{1, \lambda c^{-1}(\mu_1 \wedge \mu_2)\}$, we write $\xi_{ij}(\delta) = X_{ij} - \lambda^{-1}c_i(1-\delta)$, $\mu_{i\delta} = -E\xi_{ij}(\delta) = \mu_i - \lambda^{-1}c_i\delta$, $S_i(t, \delta) = \sup_{0\leq k\leq \lambda t(1+\varepsilon)} \sum_{j=1}^{k} \xi_{ij}(\delta)$, $\eta_j(\delta) = \lambda^{-1}(1-\delta) - \theta_j$, $j \geq 1$, $i = 1, 2$ and $\eta(\delta) = c \sup_{k\geq 0} \sum_{j=1}^{k} \eta_j(\delta)$. For any fixed $\delta > 0$, since $\{\eta_j(\delta), j \geq 1\}$ is bounded above, it follows from Lemma 2.3 that there exists some $0 < \gamma_1 := \gamma_1(\delta) < \varepsilon$ such that

$$Ee^{2\gamma_1\eta(\delta)} < \infty.$$

Hence there exists some $l_1 > 0$ such that

$$Ee^{2\gamma_1\eta(\delta)}\mathbf{1}(\eta(\delta) > l_1) < \varepsilon, \tag{2.11}$$

where $\mathbf{1}(A)$ is the indicator function of the event A.

For some fixed $0 < \sigma < 2^{-1}$, we have

$$\begin{aligned} \psi_{a1}(\vec{x}, t) &\leq P\left(S_1(t, \delta) + \eta(\delta) > x_1, S_2(t, \delta) + \eta(\delta) > x_2\right) \\ &\leq P\left(S_1(t, \delta) > x_1 - l_1\right)P(S_2(t, \delta) > x_2 - l_1) \\ &\quad + P(S_1(t, \delta) + \eta(\delta) > x_1, S_2(t, \delta) + \eta(\delta) > x_2, l_1 < \eta(\delta) \leq \sigma(x_1 \wedge x_2)) \\ &\quad + P(\eta(\delta) > \sigma(x_1 \wedge x_2)) \\ &:= \psi_{a11}(\vec{x}, t) + \psi_{a12}(\vec{x}, t) + \psi_{a13}(\vec{x}, t). \end{aligned} \tag{2.12}$$

Firstly, we deal with $\psi_{a11}(\vec{x}, t)$. By $F \in \mathcal{S}_*$, there exists some $x_1' \geq x_0'$ such that for all $x > x_1'$,

$$(1-\varepsilon)\overline{F_i}(x) \leq \overline{F_i}(x \pm l_1) \leq (1+\varepsilon)\overline{F_i}(x), \quad i = 1, 2. \tag{2.13}$$

Moreover, by Lemma 2.2(ii), we may assume that x_1' is sufficiently large such that for all $x \geq x_1'$ and all $t > 0$,

$$\begin{aligned} P(S_i(t, \delta) > x) &\leq (1+\varepsilon)\mu_{i\delta}^{-1} \int_{x}^{x+\mu_{i\delta}\lambda t(1+\varepsilon)} \overline{F_i}(u + \lambda^{-1}c_i(1-\delta))du \\ &\leq (1+\varepsilon)\mu_{i\delta}^{-1} \int_{x}^{x+\mu_{i\delta}\lambda t(1+\varepsilon)} \overline{F_i}(u)du, \quad i = 1, 2. \end{aligned} \tag{2.14}$$

We know by (2.14), (2.13) and Lemma 2.5 that when $x_1 \wedge x_2 \geq x_2' := x_1' + l_1$ and $t > 0$,

$$P(S_i(t,\delta) > x_i - l_1) \leq (1+\varepsilon)\mu_{i\delta}^{-1} \int_{x_i - l_1}^{x_i - l_1 + \mu_{i\delta}\lambda t(1+\varepsilon)} \overline{F_i}(y)dy$$
$$= (1+\varepsilon)\mu_{i\delta}^{-1} \int_{x_i}^{x_i + \mu_{i\delta}\lambda t(1+\varepsilon)} \overline{F_i}(z - l_1)dz$$
$$\leq (1+\varepsilon)^3\mu_{i\delta}^{-1} \int_{x_i}^{x_i + \mu_{i\delta}\lambda t} \overline{F_i}(y)dy, \quad i = 1, 2. \tag{2.15}$$

Plugging (2.15) into $\psi_{a11}(\vec{x}, t)$, we obtain that when $x_1 \wedge x_2 \geq x_2'$ and $t > 0$,

$$\psi_{a11}(\vec{x}, t) \leq (1+\varepsilon)^6 \prod_{i=1}^{2} \mu_{i\delta}^{-1} \int_{x_i}^{x_i + \mu_{i\delta}\lambda t} \overline{F_i}(y)dy$$
$$\leq (1+\varepsilon)^6 \prod_{i=1}^{2} \mu_{i\delta}^{-1} \int_{x_i}^{x_i + \mu_i\lambda t} \overline{F_i}(y)dy. \tag{2.16}$$

Secondly, we deal with $\psi_{a12}(\vec{x}, t)$. In view of Lemma 2.1, we know that for sufficiently large x_2', the following relations hold for all $x \geq x_2'$ and any $0 \leq u < x - x_2'$:

$$\overline{F_i}(x - u) \leq e^{\gamma_1(u+1)}\overline{F_i}(x), \quad i = 1, 2. \tag{2.17}$$

It's clear that

$$\psi_{a12}(\vec{x}, t) = \int_{l_1}^{\sigma(x_1 \wedge x_2)} P\left(\bigcap_{i=1}^{2}\{S_i(t,\delta) > x_i - u\}\right) dP(\eta(\delta) \leq u). \tag{2.18}$$

Take x_3' sufficiently large such that $(1-\sigma)x_3' \geq x_2'$, then we have $x_i - u \geq (1-\sigma)x_3' \geq x_2'$, $i = 1, 2$ for all $l_1 \leq u \leq \sigma(x_1 \wedge x_2)$ if $x_1 \wedge x_2 \geq x_3'$. Therefore, it follows from (2.14) that when $x_1 \wedge x_2 \geq x_3'$ and $t > 0$, it holds uniformly for all $l_1 \leq u \leq \sigma(x_1 \wedge x_2)$ that

$$P\left(\bigcap_{i=1}^{2}\{S_i(t,\delta) > x_i - u\}\right) \leq (1+\varepsilon)^2 \prod_{i=1}^{2} \mu_{i\delta}^{-1} \int_{x_i - u}^{x_i - u + \mu_{i\delta}\lambda t(1+\varepsilon)} \overline{F_i}(y)dy$$
$$= 1+\varepsilon)^2 \prod_{i=1}^{2} \mu_{i\delta}^{-1} \int_{x_i}^{x_i + \mu_{i\delta}\lambda t(1+\varepsilon)} \overline{F_i}(\omega - u)d\omega. \tag{2.19}$$

Combining (2.18)–(2.19) and Fubini's theorem, when $x_1 \wedge x_2 \geq x_3'$ and $t > 0$, we have

$$\psi_{a12}(\vec{x}, t) \leq (1+\varepsilon)^2 \int_{l_1}^{\sigma(x_1 \wedge x_2)} \iint_{(\omega_1,\omega_2)\in D(t,\varepsilon,\delta)} \prod_{i=1}^{2} \mu_{i\delta}^{-1}\overline{F_i}(\omega_i - u)d\omega_1 d\omega_2 dP(\eta(\delta) \leq u)$$
$$\leq (1+\varepsilon)^2 \iint_{(\omega_1,\omega_2)\in D(t,\varepsilon,\delta)} \int_{l_1}^{\sigma(x_1 \wedge x_2)} \prod_{i=1}^{2} \mu_{i\delta}^{-1}\overline{F_i}(\omega_i - u)dP(\eta(\delta) \leq u)d\omega_1 d\omega_2, \tag{2.20}$$

where $D(t, \varepsilon, \delta) := \{(\omega_1, \omega_2) : x_1 \leq \omega_1 \leq x_1 + \mu_{1\delta}\lambda t(1+\varepsilon), x_2 \leq \omega_2 \leq x_2 + \mu_{2\delta}\lambda t(1+\varepsilon)\}$. Noting that when $x_1 \wedge x_2 \geq x_3'$, we have $\omega_i - u \geq (1-\sigma)x_i \geq x_2'$ for all $x_i \leq \omega_i \leq$

$x_i + \mu_{i\delta}\lambda t(1+\varepsilon)$, $i = 1, 2$ and $l_1 \le u \le \sigma(x_1 \wedge x_2)$. Therefore, by (2.17) and (2.11), when $x_1 \wedge x_2 \ge x_3'$ and $t > 0$, we have

$$\begin{aligned}\int_{l_1}^{\sigma(x_1\wedge x_2)} \prod_{i=1}^{2} \overline{F_i}(\omega_i - u) dP(\eta(\delta) \le u) &\le \int_{l_1}^{\sigma(x_1\wedge x_2)} e^{2\gamma_1(u+1)} dP(\eta(\delta) \le u) \prod_{i=1}^{2} \overline{F_i}(\omega_i) \\ &\le e^{2\gamma_1} E e^{2\gamma_1 \eta(\delta)} \mathbf{1}(\eta(\delta) > l_1) \prod_{i=1}^{2} \overline{F_i}(\omega_i) \\ &\le \varepsilon e^{2\varepsilon} \overline{F_1}(\omega_1) \overline{F_2}(\omega_2). \end{aligned} \tag{2.21}$$

Plugging (2.21) into (2.20) and using Lemma 2.5, we see that when $x_1 \wedge x_2 \ge x_3'$ and $t > 0$,

$$\begin{aligned}\psi_{a12}(\vec{x}, t) &\le \varepsilon(1+\varepsilon)^2 e^{2\varepsilon} \int_{x_1}^{x_1+\mu_{1\delta}\lambda t(1+\varepsilon)} \int_{x_2}^{x_2+\mu_{2\delta}\lambda t(1+\varepsilon)} \prod_{i=1}^{2} \mu_{i\delta}^{-1} \overline{F_i}(\omega_i) d\omega_1 d\omega_2 \\ &\le \varepsilon(1+\varepsilon)^4 e^{2\varepsilon} \mu_{1\delta}^{-1} \mu_{2\delta}^{-1} \int_{x_1}^{x_1+\mu_{1\delta}\lambda t} \overline{F_1}(y) dy \int_{x_2}^{x_2+\mu_{2\delta}\lambda t} \overline{F_2}(y) dy \\ &\le \varepsilon(1+\varepsilon)^4 e^{2\varepsilon} \prod_{i=1}^{2} \mu_{i\delta}^{-1} \int_{x_i}^{x_i+\mu_i\lambda t} \overline{F_i}(y) dy. \end{aligned} \tag{2.22}$$

At last, we deal with $\psi_{a13}(\vec{x}, t)$. By Lemma 2.3, there exist two constants $M := M(\delta)$ and $r := r(\delta)$ such that for all $x_1, x_2 > 0$ and $t > 0$,

$$\psi_{a13}(\vec{x}, t) \le M e^{-r\sigma(x_1\wedge x_2)}.$$

Recall that there is some $0 < \kappa < 1$ such that $\kappa x_1 \le x_2 \le \kappa^{-1} x_1$, so we have $x_1 \wedge x_2 \ge 2^{-1}\kappa(x_1 + x_2)$. Thus when $x_1 \wedge x_2 > 0$ and $t > 0$,

$$\psi_{a13}(\vec{x}, t) \le M e^{-2^{-1} r\sigma\kappa x_1} \cdot e^{-2^{-1} r\sigma\kappa x_2}. \tag{2.23}$$

Since $F_i \in \mathcal{S}_*$, we conclude by the proof of Lemma 1 of Embrechts et al. (1979) that any exponential function with negative parameters is a higher-order infinitesimal of $\overline{F_i}(x)$, $i = 1, 2$. Hence it is implied by (2.23) that there exists a constant $x_4' \ge x_3'$ such that when $x_1 \wedge x_2 \ge x_4'$ and $t > 0$,

$$\psi_{a13}(\vec{x}, t) \le \varepsilon \overline{F_1}(x_1) \overline{F_2}(x_2).$$

Using the same idea in the argument (2.8)–(2.10), we know that when $x_1 \wedge x_2 \ge x_4'$ and $t \ge t_1'$,

$$\psi_{a13}(\vec{x}, t) \le 4\varepsilon\mu_1\mu_2 \prod_{i=1}^{2} \mu_i^{-1} \int_{x_i}^{x_i+\mu_i\lambda t} \overline{F_i}(y) dy. \tag{2.24}$$

Plugging (2.16), (2.22) and (2.24) into (2.12), we see that when $x_1 \wedge x_2 \ge x_4'$ and $t \ge t_1'$,

$$\psi_{a1}(\vec{x}, t) \le \left((1+\varepsilon)^6 + \varepsilon(1+\varepsilon)^4 e^{2\varepsilon} + 4\varepsilon\mu_1\mu_2\right) \prod_{i=1}^{2} \mu_{i\delta}^{-1} \int_{x_i}^{x_i+\mu_i\lambda t} \overline{F_i}(y) dy. \tag{2.25}$$

Therefore, we conclude from (2.3), (2.10) and (2.25) that when $x_1 \wedge x_2 \geq x_4'$ and $t \geq t_1'$,

$$\psi_a(\vec{x},t) \leq \left((1+\varepsilon)^6 + \varepsilon(1+\varepsilon)^4 e^{2\varepsilon} + 8\varepsilon\mu_1\mu_2\right) \prod_{i=1}^{2} \mu_{i\delta}^{-1} \int_{x_i}^{x_i+\mu_i\lambda t} \overline{F_i}(y)dy.$$

Thus we prove the upper limit in (2.2) due to the arbitrariness of ε and δ. □

2.2.2. *Proof of the lower limit part in* (2.2). Recall that $c = c_1 \vee c_2$. For any $\delta > 0$, let $\widetilde{\xi}_{ij}(\delta) = X_{ij} - \lambda^{-1}c_i(1+\delta)$, $\widetilde{\mu}_{i\delta} = -E\widetilde{\xi}_{ij}(\delta) = \mu_i + \lambda^{-1}c_i\delta$, $\widetilde{S}_i(n,\delta) = \max_{0\leq k\leq n}\sum_{j=1}^{k}\widetilde{\xi}_{ij}(\delta)$, $\widetilde{\eta}_j(\delta) = \lambda^{-1}(1+\delta) - \theta_j$, $j \geq 1$, $i = 1,2$, and $\widetilde{\eta}(\delta) = c\inf_{k\geq 0}\sum_{j=1}^{k}\widetilde{\eta}_j(\delta)$.

Noting that $E\widetilde{\eta}_1(\delta) > 0$, so $\widetilde{\eta}(\delta)$ is proper. Thus for any $0 < \varepsilon < 2^{-1}$, there exists some $l_2 > 0$ large enough such that

$$P(\widetilde{\eta}(\delta) > -l_2) > 1-\varepsilon. \tag{2.26}$$

By Lemma 2.2(i) and $F_1, F_2 \in \mathcal{S}_*$, there exists some $x_5' > 0$ such that when $x_1 \wedge x_2 \geq x_5'$ and $t > 0$, it holds uniformly for all $n \geq \lambda t(1-\delta)$ that

$$\begin{aligned} P\left(\widetilde{S}_i(n,\delta) > x_i + l_2\right) &\geq (1-\varepsilon)(\widetilde{\mu}_{i\delta})^{-1} \int_{x_i+l_2}^{x_i+l_2+n\widetilde{\mu}_{i\delta}} \overline{F_i}(y+\lambda^{-1}c_i(1+\delta))dy \\ &\geq (1-\varepsilon)(\widetilde{\mu}_{i\delta})^{-1} \int_{x_i}^{x_i+\lambda t(1-\delta)\widetilde{\mu}_{i\delta}} \overline{F_i}\left(u + l_2 + \lambda^{-1}c_i(1+\delta)\right)du \\ &\geq (1-\varepsilon)^2(1-\delta)(\widetilde{\mu}_{i\delta})^{-1} \int_{x_i}^{x_i+\lambda t\widetilde{\mu}_{i\delta}} \overline{F_i}(u)du, \quad i = 1,2, \end{aligned}$$

where in the last step we used the long-tailed property of F_1, F_2 and Lemma 2.5. Thus by the independence of $\{X_{1n}, n \geq 1\}$ and $\{X_{2n}, n \geq 1\}$, it holds uniformly for all $x_1 \wedge x_2 \geq x_5'$, $t > 0$ and $n \geq \lambda t(1-\delta)$ that

$$\begin{aligned} &P\left(\tilde{S}_1(n,\delta) > x_1 + l_2, \tilde{S}_2(n,\delta) > x_2 + l_2\right) \\ &\geq (1-\varepsilon)^4(1-\delta)^2(\widetilde{\mu}_{1\delta}\widetilde{\mu}_{2\delta})^{-1} \int_{x_1}^{x_1+\widetilde{\mu}_{1\delta}\lambda t} \overline{F_1}(u)du \int_{x_2}^{x_2+\widetilde{\mu}_{2\delta}\lambda t} \overline{F_2}(u)du. \end{aligned} \tag{2.27}$$

Since $\{N(t), t \geq 0\}$ is a renewal counting process, we know that there exists some $t_2' > 0$ sufficiently large such that for all $t \geq t_2'$,

$$P(N(t) > (1-\delta)\lambda t) > 1-\varepsilon. \tag{2.28}$$

Thus by (2.26) and (2.28), for all $t \geq t_2'$,

$$\begin{aligned} \sum_{n>(1-\delta)\lambda t} P(N(t) = n, \widetilde{\eta}(\delta) > -l_2) &\geq P(N(t) > (1-\delta)\lambda t) - P(\widetilde{\eta}(\delta) \leq -l_2) \\ &\geq 1-2\varepsilon. \end{aligned} \tag{2.29}$$

Therefore, by (2.27), (2.29) and Lemma 2.5, when $x_1 \wedge x_2 \geq x'_5$ and $t \geq t'_2$,

$$\begin{aligned}
&\psi_a(\vec{x}, t) \\
&\quad = P\left(\sup_{0\leq k\leq N(t)} \sum_{j=1}^{k}\left(\widetilde{\xi}_{1j}(\delta) + c_1\widetilde{\eta}_j(\delta)\right) > x_1, \sup_{0\leq k\leq N(t)} \sum_{j=1}^{k}\left(\widetilde{\xi}_{2j}(\delta) + c_2\widetilde{\eta}_j(\delta)\right) > x_2\right) \\
&\quad \geq P\left(\widetilde{S}_1(N(t),\delta) > x_1 + l_2, \widetilde{S}_2(N(t),\delta) > x_2 + l_2, \widetilde{\eta}(\delta) > -l_2, N(t) \geq \lambda t(1-\delta)\right) \\
&\quad \geq (1-\varepsilon)^4(1-2\varepsilon)(1-\delta)^2(\widetilde{\mu}_{1\delta}\widetilde{\mu}_{2\delta})^{-1}\int_{x_1}^{x_1+\widetilde{\mu}_{1\delta}\lambda t}\overline{F_1}(u)du\int_{x_2}^{x_2+\widetilde{\mu}_{2\delta}\lambda t}\overline{F_2}(u)du \\
&\quad \geq (1-\varepsilon)^4(1-2\varepsilon)(1-\delta)^2\prod_{i=1}^{2}(\widetilde{\mu}_{i\delta})^{-1}\int_{x_i}^{x_i+\mu_i\lambda t}\overline{F_i}(y)dy.
\end{aligned}$$

Since ε and δ are arbitrarily fixed, we prove the lower limit in (2.2). □

2.3. **Proof of (1.4).** Obviously, we have

$$\psi_b(\vec{x}, t) = P\left(\sup_{0\leq s\leq t} L_1(s) > x_1\right) + P\left(\sup_{0\leq s\leq t} L_2(s) > x_2\right) - \psi_a(\vec{x}, t). \tag{2.30}$$

For any $0 < \varepsilon < 3^{-1}$, by (1.3) and the finiteness of EX_{11}, there exists some $x'_6 > 0$ and $t'_3 > 0$ large enough such that when $x_1 \wedge x_2 > x'_6$ and $t \geq t'_3$,

$$\begin{aligned}
\psi_a(\vec{x}, t) &\leq (1+\varepsilon)\prod_{i=1}^{2}\mu_i^{-1}\int_{x_i}^{x_i+\mu_i\lambda t}\overline{F_i}(y)dy \\
&\leq \varepsilon(1+\varepsilon)\mu_2^{-1}\int_{x_2}^{x_2+\mu_2\lambda t}\overline{F_2}(y)dy.
\end{aligned} \tag{2.31}$$

By Corollary 2.1 of Wang et al. (2012), for sufficiently large x'_6 and t'_3, when $x_1 \wedge x_2 > x'_6$ and $t \geq t'_3$,

$$\begin{aligned}
(1-\varepsilon)\mu_i^{-1}\int_{x_i}^{x_i+\mu_i EN(t)}\overline{F_i}(y)dy &\leq P\left(\sup_{0\leq s\leq t} L_i(s) > x_i\right) \\
&\leq (1+\varepsilon)\mu_i^{-1}\int_{x_i}^{x_i+\mu_i EN(t)}\overline{F_i}(y)dy, i = 1, 2.
\end{aligned} \tag{2.32}$$

Since $\{N(t), t \geq 0\}$ is a renewal counting process, for sufficiently large t'_3, we know that

$$(1-\varepsilon)\lambda t \leq EN(t) \leq (1+\varepsilon)\lambda t \quad \text{for all } t \geq t'_3. \tag{2.33}$$

Combining (2.32), (2.33) and Lemma 2.5, we know that when $x_1 \wedge x_2 > x'_6$ and $t > t'_3$,

$$\begin{aligned}
(1-\varepsilon)^2\mu_i^{-1}\int_{x_i}^{x_i+\mu_i\lambda t}\overline{F_i}(y)dy &\leq P\left(\sup_{0\leq s\leq t} L_i(s) > x_i\right) \\
&\leq (1+\varepsilon)^2\mu_i^{-1}\int_{x_i}^{x_i+\mu_i\lambda t}\overline{F_i}(y)dy.
\end{aligned} \tag{2.34}$$

By (2.30), (2.31) and (2.34), when $x_1 \wedge x_2 > x_6'$ and $t > t_3'$,

$$\psi_b(\vec{x},t) \le (1+\varepsilon)^2 \left(\mu_1^{-1} \int_{x_1}^{x_1+\mu_1\lambda t} \overline{F_1}(y)dy + \mu_2^{-1} \int_{x_2}^{x_2+\mu_2\lambda t} \overline{F_2}(y)dy \right)$$

and

$$\psi_b(\vec{x},t) \ge (1-\varepsilon)^2 \mu_1^{-1} \int_{x_1}^{x_1+\mu_1\lambda t} \overline{F_1}(y)dy + (1-3\varepsilon)\mu_2^{-1} \int_{x_2}^{x_2+\mu_2\lambda t} \overline{F_2}(y)dy.$$

Therefore, we prove (1.4) by the arbitrariness of ε. □

Remark 2.6. Chen et al. (2013) considered a risk model with extended negatively orthant dependent (ENOD, see Liu et al. (2009) for its definition) inter-arrival times $\{\theta_n, n \ge 1\}$, and obtained a similar result to Theorem 1.4 under the condition that the claim distributions belonged to the distribution class $\mathcal{L} \cap \mathcal{D}$. We see that if the inter-arrival times are ENOD, then Lemma 2.3 still holds by Lemma 4.4 of Wang et al. (2012), which implies that (2.11) and (2.23) are true; and by Lemma 4.6 of Wang et al. (2012), Lemma 2.4 holds, which implies (2.4); and by Theorem 4.2 of Chen et al. (2010), (2.28) holds true. Thus Theorem 1.4 still holds if Assumption 1.2 is replaced by the following Assumption 1.2*.

Assumption 1.2*. The inter-arrival claim times $\{\theta_n, n \ge 1\}$ are ENOD and identically distributed r.v.s with a common finite mean λ^{-1}.

3. The case with Brownian motion diffusions

In this section, we extend the two-dimensional risk model introduced in Section 1 to one with Brownian motion diffusions. By definition, a standard Brownian motion $\{B(t), t \ge 0\}$ is a random process with independent increments and almost-surely continuous paths, and for each given t, $B(t)$ is a centered Gaussian r.v. with variance t. We consider such a risk model that its aggregate loss process has the following form:

$$\begin{pmatrix} \widetilde{L}_1(t) \\ \widetilde{L}_1(t) \end{pmatrix} = \sum_{j=1}^{N(t)} \begin{pmatrix} X_{1j} \\ X_{2j} \end{pmatrix} - \begin{pmatrix} c_1 t \\ c_2 t \end{pmatrix} - \begin{pmatrix} \sigma_1 B_1(t) \\ \sigma_2 B_2(t) \end{pmatrix}, \quad t \ge 0, \tag{3.1}$$

where the two diffusion processes $\{B_i(t), t \ge 0\}$, $i = 1, 2$ are mutually independent standard Brownian motions, σ_i, $i = 1, 2$ are positive constants and the other quantities are the same as in Section 1. The corresponding ruin probability can be defined as

$$\widetilde{\psi}_a(\vec{x},t) = P\left(\bigcap_{i=1}^{2} \left\{ \sup_{0 \le s \le t} \widetilde{L}_i(s) > x_i \right\} \right)$$

or

$$\widetilde{\psi}_b(\vec{x},t) = P\left(\bigcup_{i=1}^{2} \left\{ \sup_{0 \le s \le t} \widetilde{L}_i(s) > x_i \right\} \right).$$

Our main result is as follows.

Theorem 3.1. *Consider the risk model introduced above. Suppose that Assumptions* 1.1–1.3 *hold. Suppose that* $\{(B_1(t), B_2(t))^T, t \geq 0\}$ *is independent of* $\{(X_{1n}, X_{2n})^T, n \geq 1\}$ *and* $\{\theta_n, n \geq 1\}$. *If* $F_1, F_2 \in \mathcal{S}_*$, *then*

$$\widetilde{\psi}_a(\vec{x}, t) \sim \prod_{i=1}^{2} \mu_i^{-1} \int_{x_i}^{x_i+\mu_i\lambda t} \overline{F_i}(y)dy \tag{3.2}$$

and

$$\widetilde{\psi}_b(\vec{x}, t) \sim \sum_{i=1}^{2} \mu_i^{-1} \int_{x_i}^{x_i+\mu_i\lambda t} \overline{F_i}(y)dy. \tag{3.3}$$

By Theorem 3.1, we immediately obtain the following result.

Corollary 3.2. *Consider the two-dimensional risk model whose aggregate loss process is defined by* (3.1). *Suppose that Assumptions* 1.1–1.3 *hold. If* $F_1, F_2 \in \mathcal{S}_*$, *then for any function* $f(\cdot) : [0, \infty) \mapsto [0, \infty)$ *such that* $f(x) \to \infty$ *as* $x \to \infty$,

$$\limsup_{x_1 \wedge x_2 \to \infty} \sup_{t \geq f(x_1 \wedge x_2)} \left| \frac{\widetilde{\psi}_a(\vec{x}, t)}{\prod_{i=1}^{2} \mu_i^{-1} \int_{x_i}^{x_i+\mu_i\lambda t} \overline{F_i}(y)dy} - 1 \right| = 0$$

and

$$\limsup_{x_1 \wedge x_2 \to \infty} \sup_{t \geq f(x_1 \wedge x_2)} \left| \frac{\widetilde{\psi}_b(\vec{x}, t)}{\sum_{i=1}^{2} \mu_i^{-1} \int_{x_i}^{x_i+\mu_i\lambda t} \overline{F_i}(y)dy} - 1 \right| = 0.$$

In order to prove Theorem 3.1, we need the following lemma, which is due to Theorem 3.1 of Chapter X of Rolski et al. (1999).

Lemma 3.3. *Let* $\{B(t), t \geq 0\}$ *be a standard Brownian motion. Then for any* $\sigma, \delta > 0$ *and* $x > 0$,

$$P\left(\sup_{0 \leq s < \infty} \{-\delta s + \sigma B(s)\} > x\right) = e^{-2\delta x/\sigma^2}.$$

Recall that if $\{B(t), t \geq 0\}$ is a standard Brownian motion, then $\{-B(t), t \geq 0\}$ is also a standard Brownian motion. Hence according to Lemma 3.3, we have the following result

$$P\left(\sup_{0 \leq s < \infty} \{-\delta s - \sigma B(s)\} > x\right) = e^{-2\delta x/\sigma^2}. \tag{3.4}$$

and

$$P\left(\inf_{0 \leq s < \infty} \{\delta s - \sigma B(s)\} \leq -x\right) = e^{-2\delta x/\sigma^2} \tag{3.5}$$

Proof of Theorem 3.1. The proof of (3.3) is based on (3.2) and similar to that of (1.4), so we only prove (3.2). It suffices to prove

$$\widetilde{\psi}_a(\vec{x},t) \lesssim \prod_{i=1}^{2} \mu_i^{-1} \int_{x_i}^{x_i+\mu_i\lambda t} \overline{F_i}(y)dy \tag{3.6}$$

and

$$\widetilde{\psi}_a(\vec{x},t) \gtrsim \prod_{i=1}^{2} \mu_i^{-1} \int_{x_i}^{x_i+\mu_i\lambda t} \overline{F_i}(y)dy. \tag{3.7}$$

For any $\delta > 0$ and $i = 1, 2$, we set $a_{i\delta} = \lambda^{-1}(c_i-\delta)-EX_{i1}$ and $b_{i\delta} = \lambda^{-1}(c_i+\delta)-EX_{i1}$. Since $\lim_{\delta\to 0} a_{i\delta} = \mu_i > 0$, there exists some $\delta_0 > 0$ such that $a_{i\delta} > 0$ for all $0 < \delta < \delta_0$. For simplicity, we write $L_i^+(\delta,t) = L_i(t) + \delta t$, $L_i^-(\delta,t) = L_i(t) - \delta t$, $B_i^+(\delta,t) = -\sigma_i B_i(t) + \delta t$, $B_i^-(\delta,t) = -\sigma_i B_i(t) - \delta t$, $t \geq 0$, $i = 1, 2$.

By (3.4) and (3.5), for any $\varepsilon > 0$ and $0 < \delta < \delta_0$, there exists some $l_3 > 0$ and $0 < \gamma_2 < 2\delta(\sigma_1^{-2} \wedge \sigma_2^{-2})$ such that

$$P\left(\inf_{0\leq s<\infty} B_i^+(\delta,s) \geq -l_3\right) > 1-\varepsilon \tag{3.8}$$

and

$$E\exp\left\{\gamma_2 \sup_{0\leq s<\infty} B_i^-(\delta,s)\right\} \mathbf{1}\left(\sup_{0\leq s<\infty} B_i^-(\delta,s) > l_3\right) < \varepsilon, \quad i=1,2. \tag{3.9}$$

We first prove (3.6). By Theorem 1.4, there exists some $x_7' > l_3$ and $t_4' > 0$ such that for all $x_1 \wedge x_2 \geq x_7'$ and $t \geq t_4'$,

$$P\left(\bigcap_{i=1}^{2}\left\{\sup_{0\leq s\leq t} L_i^+(\delta,s) > x_i\right\}\right) < (1+\varepsilon)\prod_{i=1}^{2} a_{i\delta}^{-1} \int_{x_i}^{x_i+a_{i\delta}\lambda t} \overline{F_i}(y)dy \tag{3.10}$$

and

$$P\left(\bigcap_{i=1}^{2}\left\{\sup_{0\leq s\leq t} L_i^-(\delta,s) > x_i\right\}\right) > (1-\varepsilon)\prod_{i=1}^{2} b_{i\delta}^{-1} \int_{x_i}^{x_i+b_{i\delta}\lambda t} \overline{F_i}(y)dy. \tag{3.11}$$

We may assume that x_7' is sufficiently large such that for all $x \geq x_7'$,

$$(1-\varepsilon)\overline{F_i}(x) \leq \overline{F_i}(x \pm l_3) \leq (1+\varepsilon)\overline{F_i}(x), \quad i=1,2. \tag{3.12}$$

Hereafter, denote $E_{it} = \sup_{0\leq s\leq t} \widetilde{L}_i(s) > x_i$, $i = 1, 2$. For some fixed $0 < \sigma < 2^{-1}$. Choose some $x_8' > \sigma^{-1}x_7'$, when $x_1 \wedge x_2 > x_8'$, we decompose $\widetilde{\psi}_a(\vec{x},t)$ as follows.

$$\begin{aligned}
\widetilde{\psi}_a(\vec{x},t) \leq{}& P\left(\bigcap_{i=1}^{2} E_{it}, \bigcap_{i=1}^{2}\left\{\sup_{0\leq s\leq t} B_i^-(\delta,s) \leq l_3\right\}\right) \\
&+ P\left(\bigcap_{i=1}^{2} E_{it}, \bigcap_{i=1}^{2}\left\{l_3 < \sup_{0\leq s\leq t} B_i^-(\delta,s) \leq \sigma(x_1\wedge x_2)\right\}\right) \\
&+ P\left(\bigcap_{i=1}^{2} E_{it}, \sup_{0\leq s\leq t} B_2^-(\delta,s) \leq l_3 < \sup_{0\leq s\leq t} B_1^-(\delta,s) \leq \sigma(x_1\wedge x_2)\right)
\end{aligned}$$

$$+ P\left(\bigcap_{i=1}^{2} E_{it}, \sup_{0\le s\le t} B_1^-(\delta,s) \le l_3 < \sup_{0\le s\le t} B_2^-(\delta,s) \le \sigma(x_1\wedge x_2)\right)$$

$$+ P\left(\bigcup_{i=1}^{2}\left\{\sup_{0\le s\le t} B_i^-(\delta,s) > \sigma(x_1\wedge x_2)\right\}\right)$$

$$:= \sum_{i=1}^{5} J_i(\vec{x},t). \tag{3.13}$$

Recall that $x_7' > l_3$, so if $x_1 \wedge x_2 > x_8'$, then $x_i - l_3 > x_7'$, $i = 1,2$. Thus by (3.10) and (3.12), when $x_1\wedge x_2 > x_8'$ and $t \ge t_4'$,

$$J_1(\vec{x},t) \le P\left(\bigcap_{i=1}^{2}\left\{\sup_{0\le s\le t} L_i^+(\delta,s) > x_i - l_3\right\}\right)$$

$$\le (1+\varepsilon)\prod_{i=1}^{2} a_{i\delta}^{-1}\int_{x_i-l_3}^{x_i-l_3+a_{i\delta}\lambda t}\overline{F_i}(y)dy$$

$$= (1+\varepsilon)\prod_{i=1}^{2} a_{i\delta}^{-1}\int_{x_i}^{x_i+a_{i\delta}\lambda t}\overline{F_i}(y-l_3)dy$$

$$\le (1+\varepsilon)^2\prod_{i=1}^{2} a_{i\delta}^{-1}\int_{x_i}^{x_i+a_{i\delta}\lambda t}\overline{F_i}(y)dy. \tag{3.14}$$

Recall that $\kappa x_1 \le x_2 \le \kappa^{-1}x_1$ for some positive constant $0<\kappa<1$, so when $x_1\wedge x_2 > x_8'$ and $l_3 \le y_i \le \sigma(x_1\wedge x_2)$, $i=1,2$, we have $(x_1-y_1)\wedge(x_2-y_2) > x_7'$ and $(1-\sigma)\kappa \le (x_1-y_1)^{-1}(x_2-y_2) \le (1-\sigma)^{-1}\kappa^{-1}$. Thus by Theorem 1.4 and Lemma 2.1, there exists some $x_9' > x_8'$ and $t_5' > t_4'$ such that when $x_1\wedge x_2 > x_9'$ and $t \ge t_5'$, it holds uniformly for all $l_3 \le y_i \le \sigma(x_1\wedge x_2)$, $i=1,2$ that

$$P\left(\bigcap_{i=1}^{2}\left\{\sup_{0\le s\le t} L_i^+(\delta,s) > x_i - y_i\right\}\right) \le (1+\varepsilon)\prod_{i=1}^{2} a_{i\delta}^{-1}\int_{x_i}^{x_i+a_{i\delta}\lambda t}\overline{F_i}(u-y_i)du$$

$$\le (1+\varepsilon)\prod_{i=1}^{2} a_{i\delta}^{-1}e^{\gamma_2(y_i+1)}\int_{x_i}^{x_i+a_{i\delta}\lambda t}\overline{F_i}(u)du. \tag{3.15}$$

Hereafter, denote by $G_{it}(y_i)$ the distribution of $\sup_{0\le s\le t} B_i^-(\delta,s)$, $i=1,2$ and $G_t(y_1,y_2) = G_{1t}(y_1)G_{2t}(y_2)$. Then combining (3.15) with (3.9), we get

$$J_2(\vec{x},t) \le \iint\limits_{l_3<y_1,y_2\le\sigma(x_1\wedge x_2)} P\left(\bigcap_{i=1}^{2}\left\{\sup_{0\le s\le t} L_i^+(\delta,s) > x_i - y_i\right\}\right) dG_t(y_1,y_2)$$

$$\le (1+\varepsilon)e^{2\gamma_2}\prod_{i=1}^{2} a_{i\delta}^{-1}\int_{x_i}^{x_i+a_{i\delta}\lambda t}\overline{F_i}(u)du \iint\limits_{l_3<y_1,y_2\le\sigma(x_1\wedge x_2)} e^{\gamma(y_1+y_2)}dG_t(y_1,y_2)$$

$$\le (1+\varepsilon)e^{2\gamma_2}\varepsilon^2\prod_{i=1}^{2} a_{i\delta}^{-1}\int_{x_i}^{x_i+a_{i\delta}\lambda t}\overline{F_i}(y)dy. \tag{3.16}$$

Furthermore, by replacing y_2 with l_3 in (3.15) and using Lemma 2.1 and (3.12), we know that when $x_1 \wedge x_2 > x_9'$ and $t > t_5'$, it holds uniformly for all $l_3 \le y_1 \le \sigma(x_1 \wedge x_2)$ that

$$
\begin{aligned}
& P\left(\sup_{0\le s\le t} L_1^+(\delta,s) > x_1 - y_1, \sup_{0\le s\le t} L_2^+(\delta,s) > x_2 - l_3\right) \\
&\quad \le (1+\varepsilon) a_{1\delta}^{-1} a_{2\delta}^{-1} \int_{x_1}^{x_1+a_{1\delta}\lambda t} \overline{F_i}(u-y_1)du \int_{x_2}^{x_2+a_{2\delta}\lambda t} \overline{F_2}(u-l_3)du \\
&\quad \le (1+\varepsilon)^2 e^{\gamma_2(y_1+1)} \prod_{i=1}^{2} a_{i\delta}^{-1} \int_{x_i}^{x_i+a_{i\delta}\lambda t} \overline{F_i}(u)du.
\end{aligned}
$$

Thus when $x_1 \wedge x_2 > x_9'$ and $t > t_5'$, we have

$$
\begin{aligned}
J_3(\vec{x},t) &\le P\left(E_{1t}, \sup_{0\le s\le t} L_2^+(\delta,s) > x_2 - l_3, l_3 \le \sup_{0\le s\le t} B_1^-(\delta,s) \le \sigma(x_1\wedge x_2)\right) \\
&\le \int_{l_3}^{\sigma(x_1\wedge x_2)} P\left(\sup_{0\le s\le t} L_1^+(\delta,s) > x_1 - y_1, \sup_{0\le s\le t} L_2^+(\delta,s) > x_2 - l_3\right) dG_{1t}(y_1) \\
&\le (1+\varepsilon)^2 e^{\gamma_2} \prod_{i=1}^{2} a_{i\delta}^{-1} \int_{x_i}^{x_i+a_{i\delta}\lambda t} \overline{F_i}(u)du \int_{l_3}^{\sigma(x_1\wedge x_2)} e^{\gamma_2 y_1} dG_{1t}(y_1) \\
&\le \varepsilon(1+\varepsilon)^2 e^{\gamma_2} \prod_{i=1}^{2} a_{i\delta}^{-1} \int_{x_i}^{x_i+a_{i\delta}\lambda t} \overline{F_i}(y)dy, \qquad (3.17)
\end{aligned}
$$

where (3.9) are used in the last step. Similarly, when $x_1 \wedge x_2 > x_9'$ and $t > t_5'$, we have

$$
J_4(\vec{x},t) \le \varepsilon(1+\varepsilon)^2 e^{\gamma_2} \prod_{i=1}^{2} a_{i\delta}^{-1} \int_{x_i}^{x_i+a_{i\delta}\lambda t} \overline{F_i}(y)dy. \tag{3.18}
$$

Recall that $x_1 \wedge x_2 \ge 2^{-1}\kappa(x_1+x_2)$ since $\kappa x_1 \le x_2 \le \kappa^{-1}x_1$, so by (3.4), for all $\vec{x}$ such that $x_i > 0$, $i = 1, 2$, we have

$$
\begin{aligned}
J_5(\vec{x},t) &\le \sum_{i=1}^{2} P\left(\sup_{0\le s\le t} B_i^-(\delta,s) > \sigma(x_1\wedge x_2)\right) \\
&\le \sum_{i=1}^{2} \exp\left\{-2\delta\sigma_i^{-2}\sigma(x_1\wedge x_2)\right\} \\
&\le 2\exp\left\{-\delta(\sigma_1^{-2}\wedge\sigma_2^{-2})\sigma\kappa(x_1+x_2)\right\}. \qquad (3.19)
\end{aligned}
$$

Suppose that t_5' and x_9' are sufficiently large, then by (3.19) and similar argument to (2.23)–(2.24), we know that when $x_1 \wedge x_2 > x_9'$ and $t \ge t_5'$,

$$
J_5(\vec{x},t) \le \varepsilon \prod_{i=1}^{2} a_{i\delta}^{-1} \int_{x_i}^{x_i+a_{i\delta}\lambda t} \overline{F_i}(u)du, \quad i = 1, 2. \tag{3.20}
$$

Combining (3.13), (3.14), (3.16)–(3.18) and (3.20), we pove (3.6) by the arbitrariness of ε and δ.

Next, we prove (3.7). By (3.11) and (3.12), when $x_1 \wedge x_2 > x_7'$ and $t \geq t_4'$,

$$\begin{aligned} P\left(\bigcap_{i=1}^{2}\left\{\sup_{0\le s\le t} L_i^-(\delta,s) > x_i + l_3\right\}\right) &> (1-\varepsilon)\prod_{i=1}^{2} b_{i\delta}^{-1}\int_{x_i+l_3}^{x_i+l_3+b_{i\delta}\lambda t}\overline{F_i}(y)dy \\ &= (1-\varepsilon)\prod_{i=1}^{2} b_{i\delta}^{-1}\int_{x_i}^{x_i+b_{i\delta}\lambda t}\overline{F_i}(y+l_3)dy \\ &\geq (1-\varepsilon)^3\prod_{i=1}^{2} b_{i\delta}^{-1}\int_{x_i}^{x_i+b_{i\delta}\lambda t}\overline{F_i}(y)dy. \end{aligned} \tag{3.21}$$

Combining (3.8) and (3.21), we have

$$\begin{aligned} \widetilde{\psi}_a(\vec{x},t) &\geq P\left(\bigcap_{i=1}^{2}\left\{\sup_{0\le s\le t} L_i^-(\delta,s) > x_i + l_3\right\}, \bigcap_{i=1}^{2}\left\{\inf_{0\le s\le t} B_i^+(\delta,s) \geq -l_3\right\}\right) \\ &\geq (1-\varepsilon)^5\prod_{i=1}^{2} b_{i\delta}^{-1}\int_{x_i}^{x_i+\mu_i\lambda t}\overline{F_i}(y)dy, \end{aligned}$$

which implies (3.7) by the arbitrariness of ε and δ. □

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