PROBABILISTIC ROBUSTNESS FOR
DISPERSIVE-DISSIPATIVE WAVE EQUATIONS
DRIVEN BY SMALL LAPLACE-MULTIPLIER NOISE

RENHAI WANG[1], YANGRONG LI[2], AND FUZHI LI[3]

[1,2,3]School of Mathematics and Statistics
Southwest University, Chongqing, 400715, P.R. China

ABSTRACT: This paper is devoted to limit-dynamics for dispersive-dissipative wave equations on an unbounded domain. An interesting feature is that the stochastic term is multiplied by an unbounded Laplace operator. A random attractor in the Sobolev space is obtained when the density of noise is small and the growth rate of nonlinearity is subcritical. The random attractor is upper semicontinuous to the global attractor when the density of noise tends to zero. Both methods of spectrum and tail-estimate are combined to prove the collective limit-set compactness. Furthermore, a probabilistic method is used to show that the robustness of attractors is basically uniform in probability.

1. INTRODUCTION

This paper investigates **probabilistic robustness** of random attractors for dispersive-dissipative wave equations driven by a Laplace-type noise:

$$d(u_t + \alpha u - \Delta u - \beta \Delta u_t) + (\lambda u - \Delta u + f(x, u))dt$$
$$= g(x)dt + \varepsilon \mathcal{S} u \circ dW, \quad x \in \mathbb{R}^3. \tag{1}$$

with the initial conditions: $u(0) = u_0$ and $u_t(0) = u_1$, where $\alpha, \beta, \lambda > 0$, $g \in L^2(\mathbb{R}^3)$,

W is a real-valued Wiener process and $\mathcal{S} = I - \beta\Delta$. The term $\varepsilon\mathcal{S}u \circ dW$ means a **Laplace-multiplier noise** with a density $\varepsilon > 0$, see [15].

The deterministic equation ($\varepsilon = 0$) is used to mathematically describe the spread of longitudinal strain waves in nonlinear elastic road and weakly nonlinear ion-acoustic weaves (see [2, 6, 16]). The terms $-\Delta u_t$ and $-\Delta u_{tt}$ in Eq.(1) are called the viscosity dissipative and the dispersive terms respectively (see [12]). The well-posedness and dynamics for deterministic equation were widely investigated in [19, 27] and [3, 4, 23] respectively.

Recently, Jones and Wang [12] studied the random attractor for dispersive-dissipative wave equation perturbed by additive noise, i.e. $\mathcal{S}u = h$ where h is a known function. The wave equation without the dispersive term was also discussed in Wang [21] and Yang, Duan and Kloeden [24] for such additive noise and in Wang, Zhou and Gu [22] for usual multiplicative noise, i.e. $\mathcal{S}u = u$, also see [8, 9, 11, 13, 17, 18, 20, 25, 26, 32].

However, one hardly convert Eq.(1) for $\mathcal{S} = I$ into a random equation. If the noise is multiplied by a Laplace operator, then, it is possible to convert it into a coupled first-order system without stochastic differential.

In this paper, the first goal is to prove the existence of a random attractor on $E = H^1(\mathbb{R}^3)^2$. We need two assumptions: the nonlinearity f has a subcritical growth, and the density ε of noise is small. Note that the second assumption is special for the equation with a Laplace-multiplier noise and different from the usual assumptions in literatures.

The second goal is to prove **convergence** (or **robustness**) of the random attractors to the global attractor as the density ε tends to zero. By applying the abstract result given in Li et al.[14], we need to verify the uniform absorption (Section 3), the collectively limit-set compactness (Section 4) and the convergence (Section 5) of the random system. In particular, the novelties and difficulties come from verifying the limit-set compactness.

The third goal is to prove that the robustness of attractors is **basically uniform in probability**, that is, the random attractor converges to the global attractor, uniformly in a probabilistic subspace of probability $1 - \eta$ for any small $\eta > 0$. This topic of probabilistic robustness seems to be new in literatures.

2. SMALL LAPLACE-MULTIPLIER NOISE

2.1. TRANSLATION OF VARIABLES

Let $z := u_t + \delta u$ for a suitable $\delta > 0$. We have

$$u_t = -\delta u + z, \tag{2}$$

$$d(z - \beta \Delta z) + ((\alpha - \delta)z - (1 - \beta\delta)\Delta z + \delta_1 u - \delta_2 \Delta u)dt$$
$$+ f(x, u) - g)dt = \varepsilon \mathcal{S} u \circ dW, \tag{3}$$

$$u(x, 0) = u_0(x), \quad z(x, 0) = u_1(x) + \delta u_0(x), \tag{4}$$

where, $\delta_1 := \lambda - \alpha\delta + \delta^2$ and $\delta_2 := 1 - \delta + \beta\delta^2$.

We then identify the Winner process $W(\cdot, \omega)$ with the standard process $\omega(\cdot)$ on a metric dynamical system $(\Omega, \mathcal{F}, P, \theta_t)$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) | \ \omega(0) = 0\}$ with the Frechet topology, $\mathcal{F}$ is the corresponding Borel σ-algebra, P is the Wiener measure and θ_t is a group defined by $\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t)$ for $(\omega, t) \in \Omega \times \mathbb{R}$.

By [5], there is a solution $y(\theta_t \omega) = -\delta \int_{-\infty}^{0} e^{\delta\tau}(\theta_t \omega)(\tau)d\tau$ for the Ornstein-Uhlenbeck equation: $dy + \delta y dt = dW(t)$.

Lemma 1. *[1, 10]. The mapping $t \to y(\theta_t \omega)$ is continuous and tempered on Ω_0 with $P(\Omega_0) = 1$,*

$$\lim_{t \to \pm\infty} \frac{y(\theta_t \omega)}{t} = \lim_{t \to \pm\infty} \frac{1}{t} \int_{-t}^{0} y(\theta_s \omega)ds = 0, \tag{5}$$

$$\lim_{t \to \pm\infty} \frac{1}{t} \int_{-t}^{0} |y(\theta_s \omega)|^m ds = \frac{\Gamma(\frac{1+m}{2})}{\sqrt{\pi\delta^m}}, \quad \forall m > 0, \tag{6}$$

for all $\omega \in \Omega_0$, where Γ is the Gamma function.

Let $v(t, \omega) := z(t, \omega) - \varepsilon y(\theta_t \omega)u(t, \omega)$. By (2), (3), we have

$$u_t = v - \delta u + \varepsilon y u, \tag{7}$$

$$v_t - \beta\Delta v_t + (\alpha - \delta)v - (1 - \beta\delta)\Delta v + \delta_1 u - \delta_2 \Delta u + f(x, u)$$
$$= g - \varepsilon y v + \varepsilon\beta y \Delta v - (\varepsilon\delta_3 y + \varepsilon^2 y^2)u + (\varepsilon\delta_4 y + \varepsilon^2 \beta y^2)\Delta u, \tag{8}$$

$$u(x, 0) = u_0(x), \quad v(x, 0) = u_1(x) + \delta u_0(x) - \varepsilon y(\omega)u_0(x), \tag{9}$$

where $\delta_3 := \alpha - 3\delta$, $\delta_4 := 1 - 3\beta\delta$.

2.2. HYPOTHESES AND CONTINUOUS RDS

Hypothesis F. $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is continuous with r-th growth:

$$|f(x, s)| \leq \gamma_1 |s|^r + \phi_1(x), \qquad \phi_1 \in L^2(\mathbb{R}^3), \tag{10}$$

$$f(x,s)s \geq \gamma_2 F(x,s) + \phi_2(x), \qquad \phi_2 \in L^1(\mathbb{R}^3), \qquad (11)$$

$$F(x,s) \geq \gamma_3 |s|^{r+1} - \phi_3(x), \qquad \phi_3 \in L^1(\mathbb{R}^3), \qquad (12)$$

$$|\frac{\partial f}{\partial s}(x,s)| \leq \gamma_4 |s|^{r-1} + \phi_4(x), \qquad \phi_4 \in H^1(\mathbb{R}^3). \qquad (13)$$

where $r \in [1,4]$, $\gamma_i > 0$ and $F(x,s) := \int_0^s f(x,\tau)d\tau$.

We then choose $\delta > 0$ such that $\delta_i > 0$ $(i = 1, 2, 3, 4)$ and set

$$\kappa_1 := \min\{\alpha - \delta, \frac{1 - \beta\delta}{\beta}, \delta, \delta\gamma_2\}, \qquad (14)$$

$$\kappa_2 := \max\{2(\delta_3 + 1), \frac{2(\delta_3 + 1)}{\delta_1}, \frac{2(\delta_4 + \beta)}{\beta}, \frac{2(\delta_4 + \beta)}{\delta_2}, \frac{\gamma_1}{\gamma_3}, 4\}. \qquad (15)$$

Hypothesis S. The density of noise is small: $\varepsilon \in (0, \varepsilon_0]$, where

$$\varepsilon_0 = \min\{1, \frac{\kappa_1}{30\kappa_2(\frac{2}{\sqrt{\pi\delta}} + \frac{1}{\delta})}\}. \qquad (16)$$

Let $E = H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ with the norm:

$$\|\varphi\|_E = (\|v\|^2 + \beta\|\nabla v\|^2 + \delta_1\|u\|^2 + \delta_2\|\nabla u\|^2)^{\frac{1}{2}} \qquad (17)$$

for $\varphi = (u, v) \in E$. $\| \cdot \|_E$ is equivalent to the usual Sobolev norm.

By following the argument of [19], one can prove that for each $\omega \in \Omega_0$ and $\varphi_0^\varepsilon = (u_0^\varepsilon, v_0^\varepsilon) \in E$, the problem (7)-(9) has a unique solution $\varphi^\varepsilon(\cdot, \omega, \varphi_0^\varepsilon) \in C([0, \infty), E)$. By [5], $\Phi^\varepsilon : \mathbb{R}^+ \times \Omega_0 \times E \to E$ is a continuous *random dynamic system* (RDS) on E, where, for each $\varepsilon \leq \varepsilon_0$,

$$\Phi^\varepsilon(t, \omega, \varphi_0^\varepsilon) = \varphi^\varepsilon(t, \omega, \varphi_0^\varepsilon) = (u^\varepsilon(\cdot, \omega, u_0^\varepsilon), v^\varepsilon(\cdot, \omega, v_0^\varepsilon)). \qquad (18)$$

Let $\mathfrak{D}$ be a universe of all tempered random sets $\mathcal{D}$ such that

$$\lim_{t \to +\infty} e^{-\frac{1}{15}\kappa_1 t}\|\mathcal{D}(\theta_{-t}\omega)\|_E^2 = 0. \qquad (19)$$

3. COLLECTIVELY UNIFORM ESTIMATES

Lemma 2. *For each $\mathcal{D} \in \mathfrak{D}$ and $\omega \in \Omega_0$ there is a $T_1 = T_1(\mathcal{D}, \omega) > 0$ such that for all $t \geq T_1$ and $\varphi_0 \in \mathcal{D}(\theta_{-t}\omega)$,*

$$\|\varphi(t, \theta_{-t}\omega, \varphi_0)\|_E^2 \leq c + cR^\varepsilon(\omega), \qquad (20)$$

where,

$$R^\varepsilon(\omega) := \int_{-\infty}^0 e^{\kappa_1 s + \sqrt{\varepsilon}|y(\theta_s\omega)| + \varepsilon\kappa_2 \int_s^0 |y(\theta_\tau\omega)| + |y(\theta_\tau\omega)|^2 d\tau} ds < +\infty. \qquad (21)$$

Moreover, for all $s, t \geq 0$, we have,

$$\|\varphi(s, \theta_{-t}\omega, \varphi_0)\|_E^2$$

$$\leq ce^{-\kappa_1 s + \varepsilon\kappa_2 \int_{-t}^{s-t} |y(\theta_\tau\omega)| + |y(\theta_\tau\omega)|^2 d\tau}(\|\varphi_0\|_E^2 + \int_{\mathbb{R}^3} F(x, u_0)dx)$$

$$+ c\int_{-t}^{s-t} e^{\kappa_1(\sigma - s + t) + \sqrt{\varepsilon}|y(\theta_\sigma\omega)| + \varepsilon\kappa_2 \int_\sigma^{s-t} |y(\theta_\tau\omega)| + |y(\theta_\tau\omega)|^2 d\tau} d\sigma + c. \qquad (22)$$

Proof. Taking the inner product of Eq.(8) with v in L^2, we have

$$\frac{d}{dt}(\|\varphi\|_E^2 + 2\int_{\mathbb{R}^3} F(x, u)dx) + 2\kappa_1\|\varphi\|_E^2 + 2\delta(f(x, u), u)$$

$$\leq 2\varepsilon y(f(x, u), u) + 2(g, v) + I_1 + I_2. \qquad (23)$$

where I_1, I_2 are defined and estimated as follows.

$$I_1 := -2\varepsilon y\|v\|^2 - 2\varepsilon\beta y\|\nabla v\|^2 + 2\varepsilon\delta_1 y\|u\|^2 + 2\varepsilon\delta_2 y\|\nabla u\|^2$$

$$\leq 2\varepsilon|y|\|\varphi\|_E^2 \leq \frac{1}{2}\varepsilon\kappa_2|y|\|\varphi\|_E^2 \leq \frac{1}{2}\varepsilon\kappa_2(|y| + |y|^2)\|\varphi\|_E^2. \qquad (24)$$

Since $\varepsilon \leq \varepsilon_0 \leq 1$, it follows from the definition of κ_2 that

$$I_2 := -2(\varepsilon\delta_3 y + \varepsilon^2 y^2)(u, v) + 2(\varepsilon\delta_4 y + \varepsilon^2\beta y^2)(\Delta u, v)$$

$$\leq \frac{1}{2}\varepsilon\kappa_2(|y| + |y|^2)\|\varphi\|_E^2. \qquad (25)$$

By the Young inequality, we see that

$$2(g, v) \leq 2\|v\|\|g\| \leq c\|\varphi\|_E\|g\| \leq \frac{1}{2}\kappa_1\|\varphi\|_E^2 + c. \qquad (26)$$

By (10), we see $\delta\gamma_2 \geq \kappa_1$ and $F + \phi_3 \geq 0$. By (11),

$$2\delta(f(x, u), u) \geq 2\delta\gamma_2 \int F(x, u) + 2\delta \int \phi_2(x)$$

$$\geq 2\kappa_1 \int F(x, u) + 2(\kappa_1 - \delta\gamma_2) \int \phi_3(x) + 2\delta \int \phi_2(x). \qquad (27)$$

By $\gamma_1 \leq \kappa_2\gamma_3$ and (10),

$$2\varepsilon y(f(x, u), u) \leq 2\varepsilon\gamma_1|y| \int_{\mathbb{R}^3} |u|^{r+1}dx + 2\varepsilon|y|\|\phi_1\|\|u\|$$

$$\leq 2\varepsilon\kappa_2(|y| + |y|^2) \int_{\mathbb{R}^3} \gamma_3|u|^{r+1}dx + 2\varepsilon|y|\|\phi_1\|\|u\| \quad \text{(by (12))}$$

$$\leq 2\varepsilon\kappa_2(|y| + |y|^2) \int_{\mathbb{R}^3} F(x, u)dx + \frac{1}{2}\kappa_1\|\varphi\|_E^2 + \varepsilon c(|y| + |y|^2). \qquad (28)$$

Substituting (24)-(28) into (23), we find

$$\frac{d}{dt}(\|\varphi\|_E^2 + 2\int F(x, u)dx)$$

$$+ (\kappa_1 - \varepsilon\kappa_2(|y| + |y|^2))(\|\varphi\|_E^2 + 2\int F(x,u)dx) \le ce^{\sqrt{\varepsilon}|y|}. \qquad (29)$$

Applying the Gronwall lemma over $[0, s]$ for any $s \ge 0$, we find

$$\|\varphi(s,\omega,\varphi_0)\|_E^2 + 2\int_{\mathbb{R}^3} F(x, u(s,\omega,u_0))dx$$

$$\le e^{-\kappa_1 s + \varepsilon\kappa_2 \int_0^s |y(\theta_\tau\omega)| + |y(\theta_\tau\omega)|^2 d\tau}(\|\varphi_0\|_E^2 + 2\int_{\mathbb{R}^3} F(x, u_0)dx)$$

$$+ c\int_0^s e^{\kappa_1(\sigma - s) + \sqrt{\varepsilon}|y(\theta_\sigma\omega)| + \varepsilon\kappa_2 \int_\sigma^s |y(\theta_\tau\omega)| + |y(\theta_\tau\omega)|^2 d\tau}d\sigma. \qquad (30)$$

Replacing ω by $\theta_{-t}\omega$ in (30), we find, for all $s, t \ge 0$,

$$\|\varphi(s,\theta_{-t}\omega,\varphi_0)\|_E^2 + 2\int_{\mathbb{R}^3} F(x, u(s,\theta_{-t}\omega,u_0))dx$$

$$\le ce^{-\kappa_1 s + \varepsilon\kappa_2 \int_{-t}^{s-t} |y(\theta_\tau\omega)| + |y(\theta_\tau\omega)|^2 d\tau}(\|\varphi_0\|_E^2 + \int_{\mathbb{R}^3} F(x, u_0)dx)$$

$$+ c\int_{-t}^{s-t} e^{\kappa_1(\sigma - s + t) + \sqrt{\varepsilon}|y(\theta_\sigma\omega)| + \varepsilon\kappa_2 \int_\sigma^{s-t} |y(\theta_\tau\omega)| + |y(\theta_\tau\omega)|^2 d\tau}d\sigma, \qquad (31)$$

which implies (22) in view of (12). In particular, we have

$$\|\varphi(t,\theta_{-t}\omega,\varphi_0)\|_E^2$$

$$\le ce^{-\kappa_1 t + \varepsilon\kappa_2 \int_{-t}^0 |y(\theta_\tau\omega)| + |y(\theta_\tau\omega)|^2 d\tau}(\|\varphi_0\|_E^2 + \int_{\mathbb{R}^3} F(x, u_0)dx)$$

$$+ c\int_{-t}^0 e^{\kappa_1 s + \sqrt{\varepsilon}|y(\theta_s\omega)| + \varepsilon\kappa_2 \int_s^0 |y(\theta_\tau\omega)| + |y(\theta_\tau\omega)|^2 d\tau}ds + c. \qquad (32)$$

By Hypothesis **S**, we know that there is a $T_0 = T_0(\omega) > 0$ such that

$$\varepsilon\kappa_2 \int_{-t}^0 |y(\theta_\tau\omega)| + |y(\theta_\tau\omega)|^2 d\tau \le \varepsilon_0\kappa_2(\frac{2\Gamma(1)}{\sqrt{\pi\delta}} + \frac{2\Gamma(3/2)}{\sqrt{\pi\delta}})t$$

$$= \frac{\kappa_1\kappa_2}{30\kappa_2(\frac{2}{\sqrt{\pi\delta}} + \frac{1}{\delta})}(\frac{2}{\sqrt{\pi\delta}} + \frac{1}{\delta})t = \frac{1}{30}\kappa_1 t, \quad \text{for all } t \ge T_0. \qquad (33)$$

By (5), $\sqrt{\varepsilon}|y(\theta_s\omega)| \le -\frac{1}{30}\kappa_1 s$ for all $s \le -T_0$. Therefore, we have

$$\int_{-\infty}^{-T_0} e^{\kappa_1 s + \sqrt{\varepsilon}|y(\theta_s\omega)| + \varepsilon\kappa_2 \int_s^0 |y(\theta_\tau\omega)| + |y(\theta_\tau\omega)|^2 d\tau}ds \le \int_{-\infty}^0 e^{\frac{14}{15}\kappa_1 s}ds < +\infty,$$

which implies $R^\varepsilon(\omega)$ (given in (21)) is finite. On the other hand, by (19) and (33), we see that for $\varphi_0 \in \mathcal{D}(\theta_{-t}\omega)$, when $t \to +\infty$,

$$ce^{-\kappa_1 t + \varepsilon\kappa_2 \int_{-t}^0 |y(\theta_\tau\omega)| + |y(\theta_\tau\omega)|^2 d\tau}\|\varphi_0\|_E^2 \le ce^{-\frac{29}{30}\kappa_1 t}\|\mathcal{D}(\theta_{-t}\omega)\|_E^2 \to 0.$$

By the Sobolev embedding $H^1 \hookrightarrow L^p$ for $p \in [2, 6]$,

$$ce^{-\kappa_1 t + \varepsilon\kappa_2 \int_{-t}^0 |y(\theta_\tau\omega)| + |y(\theta_\tau\omega)|^2 d\tau} \int F(x, u_0)$$

$$\leq c e^{-\frac{29}{30}\kappa_1 t}(1 + \|u_0\|_{r+1}^{r+1} + \|\phi_1\|\|u_0\|)$$

$$\leq c e^{-\frac{29}{30}\kappa_1 t} + c(e^{-\frac{29\times 2}{30(r+1)}\kappa_1 t}\|u_0\|_{H^1}^2)^{\frac{r+1}{2}} + c e^{-\frac{29}{30}\kappa_1 t}\|u_0\|^2$$

$$\leq c e^{-\frac{29}{30}\kappa_1 t} + c(e^{-\frac{1}{15}\kappa_1 t}\|\mathcal{D}(\theta_{-t}\omega)\|_E^2)^{\frac{r+1}{2}} + c e^{-\frac{1}{15}\kappa_1 t}\|\mathcal{D}(\theta_{-t}\omega)\|_E^2$$

which tends to zero as $t \to +\infty$ in view of (19). $\qquad\square$

4. TAIL-ESTIMATE AND SPECTRUM

We need an auxiliary estimate.

Lemma 3. *Let Hypotheses* **F** *be satisfied. We have*

$$\|v_t\|_{H^1}^2 + \|u_t\|_{H^1}^2 \leq c e^{|y(\theta_t\omega)|}(1 + \|\varphi\|_E^2 + \|\varphi\|_E^{2r}). \tag{34}$$

Proof. We multiply (8) with v_t to obtain

$$\|v_t\|^2 + \beta\|\nabla v_t\|^2 = I_1 + I_2 + I_3, \tag{35}$$

where we estimate I_1, I_2, I_3 as follows.

$$I_1 := -(\alpha - \delta)(v, v_t) + (1 - \beta\delta)(\Delta v, v_t) - \delta_1(u, v_t) + \delta_2(\Delta u, v_t)$$

$$\leq \frac{1}{4}\|v_t\|^2 + \frac{1}{4}\beta\|\nabla v_t\|^2 + c\|\varphi\|_E^2.$$

By (10) and $H^1 \hookrightarrow L^p$ for $p \in [2,6]$,

$$I_2 := (g, v_t) - (f(x,u), v_t) \leq \gamma_1 \int_{\mathbb{R}^3} |u|^r|v_t|dx + \|v_t\|\|\phi_1\|$$

$$\leq \frac{1}{8}\|v_t\|^2 + c + c\|u\|_{H^1}^r\|v_t\|_{H^1} + \|v_t\|\|\phi_1\|$$

$$\leq \frac{1}{4}\|v_t\|^2 + \frac{1}{4}\beta\|\nabla v_t\|^2 + c(1 + \|\varphi\|_E^{2r}),$$

similarly, the rest terms on the right-hand side of (35) are bounded by

$$I_3 := \varepsilon\beta y(\Delta v, v_t) + 2(\varepsilon\delta_4 y + \varepsilon^2\beta y^2)(\Delta u, v_t) - 2(\varepsilon\delta_3 y + \varepsilon^2 y^2)(u, v_t)$$

$$- \varepsilon y(v, v_t) \leq \frac{1}{4}\|v_t\|^2 + \frac{1}{4}\beta\|\nabla v_t\|^2 + c + c(|y| + |y|^2)\|\varphi\|_E^2.$$

By $1 + |y| + |y|^2 \leq 2e^{|y|}$, we obtain (34). $\qquad\square$

4.1. COLLECTIVE TAIL-ESTIMATES

Lemma 4. *Let $\mathcal{D} \in \mathfrak{D}$ and $\omega \in \Omega_0$. We have*

$$\lim_{t,k\to+\infty} \sup_{\varphi_0\in\mathcal{D}(\theta_{-t}\omega)} \sup_{\varepsilon\leq\varepsilon_0} \|\varphi^\varepsilon(t,\theta_{-t}\omega,\varphi_0)\|_{E(\mathcal{O}_k^c)} = 0, \tag{36}$$

where $\mathcal{O}_k = \{x : |x| < k\}$, $\mathcal{O}_k^c = \mathbb{R}^3\backslash\mathcal{O}_k$ and $E(\mathcal{O}_k^c) = H^1(\mathcal{O}_k^c)^2$.

Proof. For $k \geq 1$, we let $\rho_k(x) := \rho(\frac{|x|^2}{k^2})$ for $x \in \mathbb{R}^3$, where $\rho : \mathbb{R} \mapsto [0,1]$ is a smooth function such that $\rho \equiv 0$ on $[0,1]$ and $\rho \equiv 1$ on $[2,\infty)$.

We take the inner product of (8) with $\rho_k v$ in L^2, after some calculations, we obtain

$$\frac{d}{dt}\int \rho_k(|\varphi|^2 + 2F(x,u)) + 2\kappa_1\int \rho_k|\varphi|^2 + 2\delta\int \rho_k f(x,u)u$$

$$\leq 2\varepsilon y\int \rho_k f(x,u)u + 2\int \rho_k vg + H_1 + H_2 + H_3, \tag{37}$$

where $|\varphi| := (|v|^2 + \beta|\nabla v|^2 + \delta_1|u|^2 + \delta_2|\nabla u|^2)^{\frac{1}{2}}$ and

$$H_1 := -2\varepsilon y\int_{\mathbb{R}^3} \rho_k|v|^2 - 2\varepsilon\beta y\int_{\mathbb{R}^3} \rho_k|\nabla v|^2 dx + 2\varepsilon\delta_1 y\int_{\mathbb{R}^3} \rho_k|u|^2 dx$$

$$\leq 2\varepsilon|y|\int_{\mathbb{R}^3} \rho_k|\varphi|^2 dx \leq \frac{1}{2}\varepsilon\kappa_2(|y| + |y|^2)\int_{\mathbb{R}^3} \rho_k|\varphi|^2 dx,$$

$$H_2 := 2\varepsilon\delta_2 y\int_{\mathbb{R}^3} \rho_k|\nabla u|^2 dx - 2(\varepsilon\delta_3 y + \varepsilon^2 y^2)\int \rho_k vu$$

$$+ 2(\varepsilon\delta_4 y + \varepsilon^2\beta y^2)\int \rho_k v\Delta u \leq \frac{1}{2}\varepsilon\kappa_2(|y| + |y|^2)\int_{\mathbb{R}^3} \rho_k|\varphi|^2 dx.$$

Similarly,

$$H_3 := -2\delta_2\int u_t(\nabla u \cdot \nabla\rho_k) - 2\delta_2(\delta - \varepsilon y)\int u(\nabla u \cdot \nabla\rho_k)$$

$$- 2\beta\int v(\nabla v_t \cdot \nabla\rho_k) - (2(1 - \beta\delta) + 2\varepsilon\beta y)\int v(\nabla v \cdot \nabla\rho_k)$$

$$\leq \frac{c}{k}e^{2|y|}(1 + \|\varphi\|_E^2 + \|\varphi\|_E^{2r}).$$

The Young inequality implies that

$$2\int \rho_k vg \leq c\int \rho_k|\varphi||g| \leq \frac{1}{2}\kappa_1\int \rho_k|\varphi|^2 + c\int \rho_k|g|^2. \tag{38}$$

By (14), we see $\delta\gamma_2 - \kappa_1 \geq 0$, then

$$2\delta\int \rho_k f(x,u)u \geq 2\delta\gamma_2\int \rho_k F(x,u) + 2\delta\int \rho_k\phi_2(x)$$

$$\geq 2\kappa_1\int \rho_k F(x,u) + 2(\kappa_1 - \delta\gamma_2)\int \rho_k\phi_3 + 2\delta\int \rho_k\phi_2. \tag{39}$$

By (15), $\gamma_1 \leq \kappa_2 \gamma_3$, then the Young inequality implies that

$$2\varepsilon y \int \rho_k f(x, u) u \leq 2\varepsilon \gamma_1 |y| \int \rho_k |u|^{r+1} + 2\varepsilon |y| \int \rho_k |u| |\phi_1|$$

$$\leq 2\varepsilon \kappa_2 (|y| + |y|^2) \int \gamma_3 \rho_k |u|^{r+1} + 2\varepsilon |y| \int \rho_k |u| |\phi_1|$$

$$\leq 2\varepsilon \kappa_2 (|y| + |y|^2) \int \rho_k F(x, u) + \frac{1}{2} \kappa_1 \int \rho_k |\varphi|^2$$

$$+ c(|y| + |y|^2) \int \rho_k (|\phi_3| + |\phi_1|^2). \tag{40}$$

Since $\lim_{k \to +\infty} \int_{\mathbb{R}^3} \rho_k (|g|^2 + |\phi_2| + |\phi_3| + |\phi_1|^2) dx = 0$, it follows from (37)-(40) that for every $\eta > 0$, there is a $K_0 > 1$ such that

$$\frac{d}{dt} \int \rho_k (|\varphi|^2 + 2F(x, u))$$

$$+ (\kappa_1 - \varepsilon \kappa_2 (|y| + |y|^2)) \int \rho_k (|\varphi|^2 + 2F(x, u))$$

$$\leq \eta e^{2|y|} (1 + \|\varphi\|_E^2 + \|\varphi\|_E^{2r}), \quad \text{for } k \geq K_0. \tag{41}$$

We use the Gronwall lemma to find that for all $k \geq K_0$,

$$\int \rho_k |\varphi(t, \theta_{-t}\omega, \varphi_0)|^2 + 2 \int \rho_k F(x, u) \leq cQ_1 + c\eta (Q_2 + Q_3), \tag{42}$$

where, as $t \to +\infty$

$$Q_1 := ce^{-\kappa_1 t + \varepsilon \kappa_2 \int_{-t}^0 |y(\theta_\tau \omega)| + |y(\theta_\tau \omega)|^2 d\tau} \int \rho_k (|\varphi_0|^2 + F(x, u_0))$$

tends to zero. It is easy to see that

$$Q_2 := \int_0^t e^{\kappa_1 (s-t) + 2|y(\theta_{s-t}\omega)| + \varepsilon \kappa_2 \int_{s-t}^0 |y(\theta_\tau \omega)| + |y(\theta_\tau \omega)|^2 d\tau} ds$$

is finite. It suffices to prove the finiteness of the following term:

$$Q_3 := \int_0^t e^{\kappa_1 (s-t) + 2|y(\theta_{s-t}\omega)| + \varepsilon \kappa_2 \int_{s-t}^0 |y(\theta_\tau \omega)| + |y(\theta_\tau \omega)|^2 d\tau} \|\varphi(s)\|_E^{2r} ds,$$

where $\varphi(s) = \varphi(s, \theta_{-t}\omega, \varphi_0)$. For this end, we use (22) to obtain that

$$\|\varphi(s, \theta_{-t}\omega, \varphi_0)\|_E^2 \leq \sum_{j=1}^3 q(s, t),$$

where

$$q_1(s, t) := ce^{-\kappa_1 s + \varepsilon \kappa_2 \int_{-t}^{s-t} |y(\theta_\tau \omega)| + |y(\theta_\tau \omega)|^2 d\tau} \|\varphi_0\|_E^2,$$

$$q_2(s,t) := ce^{-\kappa_1 s + \varepsilon \kappa_2 \int_{-t}^{s-t} |y(\theta_\tau \omega)| + |y(\theta_\tau \omega)|^2 d\tau} \int_{\mathbb{R}^3} F(x, u_0) dx,$$

$$q_3(s,t) := c \int_{-t}^{s-t} e^{\kappa_1(\sigma - s + t) + |y(\theta_\sigma \omega)| + \varepsilon \kappa_2 \int_{\sigma}^{s-t} |y(\theta_\tau \omega)| + |y(\theta_\tau \omega)|^2 d\tau} d\sigma + c.$$

Then, we have $Q_3 \leq c \sum_{j=1}^3 Q_{3,j}$, where

$$Q_{3,j} := \int_0^t e^{\kappa_1(s-t) + 2|y(\theta_{s-t}\omega)| + \varepsilon \kappa_2 \int_{s-t}^0 |y(\theta_\tau \omega)| + |y(\theta_\tau \omega)|^2 d\tau} q_j^r(s,t) ds,$$

for $j = 1, 2, 3$. After some calculations, we have

$$Q_{3,1} \leq C e^{-\frac{r}{5}\kappa_1 t + \varepsilon r \kappa_2 \int_{-t}^0 |y(\theta_\tau \omega)| + |y(\theta_\tau \omega)|^2 d\tau} \|\varphi_0\|_E^{2r}$$
$$\leq c(e^{-\frac{1}{15}\kappa_1 t} \|\mathcal{D}(\theta_{-t}\omega)\|_E^2)^r \to 0$$

as $t \to +\infty$. By the same argument, we have $Q_{3,2}(t, \omega) \leq C\tilde{Q}_{3,2}$, where,

$$\tilde{Q}_{3,2} := e^{-\frac{r}{5}\kappa_1 t + \varepsilon r \kappa_2 \int_{-t}^0 |y(\theta_\tau \omega)| + |y(\theta_\tau \omega)|^2 d\tau} \left(\int_{\mathbb{R}^3} F(x, u_0(\theta_{-t}\omega)) dx \right)^r$$

$$\leq c e^{-\frac{r}{6}\kappa_1 t} \left(\int_{\mathbb{R}^3} |f(x, u_0)| |u_0| + |\phi_2| dx \right)^r \quad \text{(by(10))}$$

$$\leq c e^{-\frac{r}{6}\kappa_1 t} + c e^{-\frac{r}{6}\kappa_1 t} \|u_0\|_{r+1}^{r^2+r} + c e^{-\frac{r}{6}\kappa_1 t} \|\phi_1\|^r \|u_0\|^r$$

$$\leq c e^{-\frac{r}{6}\kappa_1 t} + c(e^{-\frac{1}{3(r+1)}\kappa_1 t} \|u_0\|_{H^1}^2)^{\frac{r^2+r}{2}} + c(e^{-\frac{1}{3}\kappa_1 t} \|u_0\|^2)^{\frac{r}{2}}$$

$$\leq c e^{-\frac{r}{6}\kappa_1 t} + c(e^{-\frac{1}{15}\kappa_1 t} \|\mathcal{D}(\theta_{-t}\omega)\|_E^2)^{\frac{r^2+r}{2}} + c(e^{-\frac{1}{3}\kappa_1 t} \|\mathcal{D}\|_E^2)^{\frac{r}{2}},$$

where, we take the minimal coefficient $1/15$ in (19). Then $\tilde{Q}_{3,2} \to 0$ as $t \to +\infty$ and thus $Q_{3,2}$ is bounded. By the tempered property of y, one can verify that $Q_{3,3}$ is finite and thus Q_3 is finite.

Finally, by using $F + \phi_3 \geq 0$ (see (12)), it follows from (42) that

$$\int_{\mathbb{R}^3} \rho_k |\varphi|^2 dx \leq \sum_{i=1}^4 Q_i(t, k, \omega) + c \int_{\mathcal{O}_k^c} |\phi_3| dx \to 0$$

as $t, k \to +\infty$. The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.2. ORTHOGONAL DECOMPOSITION

Let $\xi_k(x) := 1 - \rho_k(x)$ for $k \geq 1$, and

$$\bar{\varphi} = (\bar{u}, \bar{v}) := \xi_k \varphi = (\xi_k u, \xi_k v), \tag{43}$$

for each solution $\varphi = (u, v)$ of system (7)-(9). Then, $\bar{\varphi} \in H^1(\mathcal{O}_{2k}) \times H^1(\mathcal{O}_{2k})$ has the orthogonal decomposition:

$$\bar{\varphi} = P_i \bar{\varphi} \oplus (I - P_i) \bar{\varphi} := \bar{\varphi}_{i,1} + \bar{\varphi}_{i,2} = (\bar{u}_{i,1}, \bar{v}_{i,1}) + (\bar{u}_{i,2}, \bar{v}_{i,2}), \tag{44}$$

where $P_i : L^2(\mathcal{O}_{2k}) \times L^2(\mathcal{O}_{2k}) \mapsto Y_i \times Y_i := \text{span}\{e_1, e_2 \cdots, e_i\} \times \text{span}\{e_1, e_2 \cdots, e_i\}$ is a canonical projection and $\{e_j\}_{j=1}^{\infty}$ is the family of eigenfunctions for $-\Delta$ in $L^2(\mathcal{O}_{2k})$ with corresponding positive eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to \infty$ as $j \to \infty$. We also see $Y_i \times Y_i \subset H^1(\mathcal{O}_{2k}) \times H^1(\mathcal{O}_{2k})$, thus it easily follows from (43) that

$$\xi_k \Delta v = \Delta \bar{v} - v \Delta \xi_k - 2\nabla \xi_k \cdot \nabla v, \tag{45}$$

$$\xi_k \Delta v_t = \Delta \bar{v}_t - v_t \Delta \xi_k - 2\nabla \xi_k \cdot \nabla v_t, \tag{46}$$

$$\xi_k \Delta u = \Delta \bar{u} - u \Delta \xi_k - 2\nabla \xi_k \cdot \nabla u. \tag{47}$$

Multiplying (7)-(8) by ξ_k and substituting (45)-(47) into the obtained results, we have

$$\bar{u}_t = \bar{v} - \delta \bar{u} + \varepsilon y \bar{u} \tag{48}$$

$$\bar{v}_t - \beta \Delta \bar{v}_t + (\alpha - \delta)\bar{v} - (1 - \beta\delta)\Delta \bar{v} + \delta_1 \bar{u} - \delta_2 \Delta \bar{u}$$
$$= -\xi_k f(x, u) + \xi_k g - \varepsilon y \bar{v} + \varepsilon \beta y \Delta \bar{v} - (\varepsilon \delta_3 y$$
$$+ \varepsilon^2 y^2)\bar{u} + (\varepsilon \delta_4 y + \varepsilon^2 \beta y^2)\Delta \bar{u} - J \tag{49}$$

where

$$J := \beta v_t \Delta \xi_k + 2\beta \nabla \xi_k \cdot \nabla v_t + (1 - \beta\delta)v\Delta \xi_k + \delta_2 u \Delta \xi_k + \varepsilon \beta y v \Delta \xi_k$$
$$+ 2(1 - \beta\delta)\nabla \xi_k \cdot \nabla v + 2\delta_2 \nabla \xi_k \cdot \nabla u + 2\varepsilon \beta y \nabla \xi_k \cdot \nabla v$$
$$+ (\varepsilon \delta_4 y + \varepsilon^2 \beta y^2)u\Delta \xi_k + 2(\varepsilon \delta_4 y + \varepsilon^2 \beta y^2)\nabla \xi_k \cdot \nabla u.$$

Lemma 5. *Let* $\mathcal{D} \in \mathfrak{D}$, $\omega \in \Omega_0$ *and* $k \geq 1$. *We have*

$$\lim_{t, i \to +\infty} \sup_{\varphi_0 \in \mathcal{D}(\theta_{-t}\omega)} \sup_{\varepsilon \leq \varepsilon_0} \|(I - P_i)\xi_k \varphi^{\varepsilon}(t, \theta_{-t}\omega, \varphi_0)\|_{E(\mathcal{O}_{2k})} = 0. \tag{50}$$

Proof. Applying $I - P_i$ to (49) and taking the inner product of the result equation with $\bar{v}_{i,2}$, we have

$$\frac{d}{dt}(\|\bar{v}_{i,2}\|^2 + \beta\|\nabla \bar{v}_{i,2}\|^2) + 2(\alpha - \delta)\|\bar{v}_{i,2}\|^2 + 2(1 - \beta\delta)\|\nabla \bar{v}_{i,2}\|^2$$
$$+ 2\delta_1(\bar{u}_{i,2}, \bar{v}_{i,2}) - 2\delta_2(\Delta \bar{u}_{i,2}, \bar{v}_{i,2}) = -2(\xi_k f(x, u), \bar{v}_{i,2}) + 2(\xi_k g, \bar{v}_{i,2})$$
$$- 2\varepsilon y\|\bar{v}_{i,2}\|^2 - 2\varepsilon \beta y\|\nabla \bar{v}_{i,2}\|^2 - 2(\varepsilon \delta_3 y + \varepsilon^2 y^2)(\bar{u}, \bar{v}_{i,2})$$
$$+ 2(\varepsilon \delta_4 y + \varepsilon^2 \beta y^2)(\Delta \bar{u}, \bar{v}_{i,2}) - 2(J, \bar{v}_{i,2}). \tag{51}$$

Applying $I - P_i$ to (48), we have

$$2(\bar{u}_{i,2}, \bar{v}_{i,2}) = \frac{d}{dt}\|\bar{u}_{i,2}\|^2 + 2\delta\|\bar{u}_{i,2}\|^2 - 2\varepsilon y\|\bar{u}_{i,2}\|^2, \tag{52}$$

$$-2(\Delta \bar{u}_{i,2}, \bar{v}_{i,2}) = \frac{d}{dt}\|\nabla \bar{u}_{i,2}\|^2 + 2\delta\|\nabla \bar{u}_{i,2}\|^2 - 2\varepsilon y\|\nabla \bar{u}_{i,2}\|^2. \tag{53}$$

Then, it follows from (51)-(53) that

$$
\frac{d}{dt}\|\bar{\varphi}_{i,2}\|_E^2 + 2\kappa_1\|\bar{\varphi}_{i,2}\|_E^2 \leq -2(\xi_k f(x,u), \bar{v}_{i,2}) + 2(\xi_k g, \bar{v}_{i,2})
$$
$$
+ J_1 + J_2 - 2(J, \bar{v}_{i,2}), \tag{54}
$$

where J_1, J_2 are given by

$$
J_1 := -2\varepsilon y\|\bar{v}_{i,2}\|^2 - 2\varepsilon\beta y\|\nabla\bar{v}_{i,2}\|^2 + 2\varepsilon\delta_1 y\|\bar{u}_{i,2}\|^2 + 2\varepsilon\delta_2 y\|\nabla\bar{u}_{i,2}\|^2
$$
$$
\leq 2\varepsilon|y|\|\bar{\varphi}_{i,2}\|_E^2 \leq \frac{1}{2}\varepsilon\kappa_2(|y| + |y|^2)\|\bar{\varphi}_{i,2}\|_E^2.
$$
$$
J_2 := -2(\varepsilon\delta_3 y + \varepsilon^2 y^2)(\bar{u}, \bar{v}_{i,2}) + 2(\varepsilon\delta_4 y + \varepsilon^2\beta y^2)(\Delta\bar{u}, \bar{v}_{i,2})
$$
$$
\leq \frac{1}{2}\varepsilon\kappa_2(|y| + |y|^2)\|\bar{\varphi}_{i,2}\|_E^2.
$$

Let $\mu = \frac{r-1}{2r+2} \in [0,1)$ since $r \in [1,4]$. Then, by the interpolation inequality and the Young inequality, we see

$$
-2(\xi_k f(x,u), \bar{v}_{i,2}) \leq c\int_{\mathbb{R}^3} \xi_k|u|^r|\bar{v}_{i,2}|dx + c\int_{\mathbb{R}^3} \xi_k|\phi_1||\bar{v}_{i,2}|dx
$$
$$
\leq c\|u\|_{r+1}^r\|\nabla\bar{v}_{i,2}\|^\mu\|\bar{v}_{i,2}\|^{1-\mu} + c\|\phi_1\|\|\bar{v}_{i,2}\|
$$
$$
\leq c\lambda_{i+1}^{\frac{\mu-1}{2}}\|u\|_{H^1}^r\|\nabla\bar{v}_{i,2}\| + c\lambda_{i+1}^{-\frac{1}{2}}\|\phi_1\|\|\nabla\bar{v}_{i,2}\|
$$
$$
\leq \frac{1}{2}\kappa_1\|\bar{\varphi}_{i,2}\|_E^2 + c\lambda_{i+1}^{\mu-1}\|\varphi\|_E^{2r} + c\lambda_{i+1}^{-1}. \tag{55}
$$

By the Young inequality and $g \in L^2(\mathbb{R}^3)$, we have

$$
2(\xi_k g, \bar{v}_{i,2}) \leq c\lambda_{i+1}^{-\frac{1}{2}}\|\nabla\bar{v}_{i,2}\| \leq \frac{1}{4}\kappa_1\|\bar{\varphi}_{i,2}\|_E^2 + c\lambda_{i+1}^{-1}. \tag{56}
$$

By (34), we can similarly obtain that

$$
(J, \bar{v}_{i,2}) \leq \|J\|\|\bar{v}_{i,2}\| \leq c\lambda_{i+1}^{-\frac{1}{2}}\|J\|\|\nabla\bar{v}_{i,2}\| \leq c\lambda_{i+1}^{-\frac{1}{2}}\|J\|\|\bar{\varphi}_{i,2}\|
$$
$$
\leq \frac{1}{4}\kappa_1\|\bar{\varphi}_{i,2}\|_E^2 + c\lambda_{i+1}^{-1}e^{|y|}(\|\varphi\|_E^2 + \|u_t\|_{H^1}^2 + \|v_t\|_{H^1}^2)
$$
$$
\leq \frac{1}{4}\kappa_1\|\bar{\varphi}_{i,2}\|_E^2 + c\lambda_{i+1}^{-1}e^{2|y|}(1 + \|\varphi\|_E^2 + \|\varphi\|_E^{2r}). \tag{57}
$$

Substituting (55)-(57) into (54) and noting $\lambda_{i+1}^{\mu-1} + \lambda_{i+1}^{-1} \to 0$ as $i \to +\infty$, we obtain that for $\eta > 0$, there is an $i_1 \in \mathbb{N}$ such that for all $i \geq i_1$,

$$
\frac{d}{dt}\|\varphi_{i,2}\|_E^2 + (\kappa_1 - \varepsilon\kappa_2(|y| + |y|^2))\|\varphi_{i,2}\|_E^2
$$
$$
\leq \eta c e^{2|y|}(1 + \|\varphi\|_E^2 + \|\varphi\|_E^{2r}). \tag{58}
$$

Applying the Gronwall lemma to (58) over $[0,t]$ and replacing ω by $\theta_{-t}\omega$, we find

$$
\|\varphi_{i,2}(t, \theta_{-t}\omega, (I - P_i)(\xi_k\varphi_0))\|_{E(\mathbb{R}^3)}^2 \leq \eta c(Q_2 + Q_3)
$$

$$+ e^{-\kappa_1 t + \varepsilon \kappa_2 \int_{-t}^0 |y(\theta_\tau \omega)| + |y(\theta_\tau \omega)|^2 d\tau} \|(I - P_i)(\xi_k \varphi_0))\|_E^2, \tag{59}$$

where Q_2, Q_3 is finite as given in the proof of Lemma 4. By $\|I - P_i\| \leq 2$, $\xi_k \leq 1$ and (33), we see that for $t \geq T_0$,

$$e^{-\kappa_1 t + \varepsilon \kappa_2 \int_{-t}^0 |y(\theta_\tau \omega)| + |y(\theta_\tau \omega)|^2 d\tau} \|(I - P_i)\xi_k \varphi_0(\theta_{-t}\omega))\|_{E(\mathbb{R}^3)}^2$$

$$\leq c e^{-\frac{1}{15}\kappa_1 t} \|\mathcal{D}(\theta_{-t}\omega))\|_{E(\mathbb{R}^3)}^2 \to 0 \quad \text{as } t \to +\infty,$$

which implies (49) as required. $\qquad\qquad\square$

5. CONVERGENCE OF THE SYSTEM

Proposition 6. *Let* $\varphi^\varepsilon := (u^\varepsilon, v^\varepsilon)$ *and* $\varphi^0 := (u^0, v^0)$ *be the solutions of* (7)-(9) *for* $\varepsilon > 0$ *and* $\varepsilon = 0$ *respectively. Suppose the initial value* $\varphi_0^\varepsilon \to \varphi_0^0$ *in* E *as* $\varepsilon \to 0$, *then, for each* $T > 0$,

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|\varphi^\varepsilon(t, \omega, \varphi_0^\varepsilon) - \varphi^0(t, \varphi_0^0)\|_E = 0. \tag{60}$$

Proof. Let $\Psi^\varepsilon := (U^\varepsilon, V^\varepsilon)$ with $U^\varepsilon = u^\varepsilon - u^0$ and $V^\varepsilon = v^\varepsilon - v^0$ for $\varepsilon \in (0, \varepsilon_0)$, By (7)-(8), we obtain that

$$U_t^\varepsilon = V^\varepsilon - \delta U^\varepsilon + \varepsilon y u^\varepsilon, \tag{61}$$

$$V^\varepsilon - \beta \Delta V_t^\varepsilon + (\alpha - \delta)V^\varepsilon - (1 - \beta\delta)\Delta V^\varepsilon + \delta_1 U^\varepsilon - \delta_2 \Delta U^\varepsilon$$

$$= f(x, u^0) - f(x, u^\varepsilon) - \varepsilon y v^\varepsilon + \varepsilon \beta y \Delta v^\varepsilon$$

$$- (\varepsilon \delta_3 y + \varepsilon^2 y^2)u^\varepsilon + (\varepsilon \delta_4 y + \varepsilon^2 \beta y^2)\Delta u^\varepsilon. \tag{62}$$

Taking the inner product of the Eq.(62) with V^ε, we have

$$\frac{d}{dt}(\|V^\varepsilon\|^2 + \beta\|\nabla V^\varepsilon\|^2) + 2(\alpha - \delta)\|V^\varepsilon\|^2 + 2(1 - \beta\delta)\|\nabla V^\varepsilon\|^2$$

$$+ 2\delta_1(U^\varepsilon, V^\varepsilon) - 2\delta_2(\Delta U^\varepsilon, V^\varepsilon)$$

$$= 2(f(x, u^0) - f(x, u^\varepsilon), V^\varepsilon) - 2\varepsilon y(v^\varepsilon, V^\varepsilon) + 2\varepsilon \beta y(\Delta v^\varepsilon, V^\varepsilon)$$

$$- 2(\varepsilon \delta_3 y + \varepsilon^2 y^2)(u^\varepsilon, V^\varepsilon) + 2(\varepsilon \delta_4 y + \varepsilon^2 \beta y^2)(\Delta u^\varepsilon, V^\varepsilon). \tag{63}$$

We multiply (61) by V^ε and substitute the result into (63) to obtain

$$\frac{d}{dt}\|\Psi^\varepsilon\|_E^2 + 2\kappa_1\|\Psi^\varepsilon\|_E^2 \leq 2(f(x, u^0) - f(x, u^\varepsilon), V^\varepsilon) + 2J. \tag{64}$$

where we use the Young inequality to bound the term J:

$$J := \varepsilon\beta y(\Delta v^\varepsilon, V^\varepsilon) - \varepsilon y(v^\varepsilon, V^\varepsilon) + \varepsilon\delta_1 y(u^\varepsilon, U^\varepsilon) + \varepsilon\delta_2 y(\nabla u^\varepsilon, \nabla U^\varepsilon)$$

$$- (\varepsilon\delta_3 y + \varepsilon^2 y^2)(u^\varepsilon, V^\varepsilon) + (\varepsilon\delta_4 y + \varepsilon^2\beta y^2)(\Delta u^\varepsilon, V^\varepsilon)$$
$$\leq \kappa_1\|\Psi^\varepsilon\|_E^2/2 + \varepsilon c(|y| + |y|^2)\|\varphi^\varepsilon\|_E^2. \tag{65}$$

By the mean value theorem,

$$|f(x, u^0) - f(x, u^\varepsilon)| \leq c(|\phi_4| + |u^0|^{r-1} + |u^\varepsilon|^{r-1})|U^\varepsilon|. \tag{66}$$

So, by $H^1 \hookrightarrow L^{2r-2}$ (since $2r - 2 \leq 6$), we have

$$2(f(x, u^0) - f(x, u^\varepsilon), V^\varepsilon)$$
$$\leq c\|\phi_4\|_6\|V^\varepsilon\|\|U^\varepsilon\|_3 + c(\||u^\varepsilon|^{r-1}\| + \||u^0|^{r-1}\|)\|U^\varepsilon\|_6\|V^\varepsilon\|_3$$
$$\leq c\|\phi_4\|_{H^1}\|V^\varepsilon\|\|U^\varepsilon\|_3 + c(\|u^\varepsilon\|_{2r-2}^{r-1} + \|u^0\|_{2r-2}^{r-1})\|U^\varepsilon\|_6\|V^\varepsilon\|_3$$
$$\leq c\|V^\varepsilon\|\|U^\varepsilon\|_{H^1} + c(\|u^\varepsilon\|_{H^1}^{r-1} + \|u^0\|_{H^1}^{r-1})\|U^\varepsilon\|_{H^1}\|V^\varepsilon\|_{H^1}$$
$$\leq c\|\Psi^\varepsilon\|_E^2 + c(\|\varphi^\varepsilon\|_E^{r-1} + \|u^0\|_E^{r-1})\|\Psi^\varepsilon\|_E^2$$
$$\leq \kappa_1\|\Psi^\varepsilon\|_E^2 + c(1 + \|\varphi^\varepsilon\|_E^{r-1} + \|u^0\|_E^{r-1})\|\Psi^\varepsilon\|_E^2. \tag{67}$$

Therefore, substituting (67)-(65) into (64), we find

$$\frac{d}{dt}\|\Psi^\varepsilon\|_E^2 \leq K_1^\varepsilon(t, \omega)\|\Psi^\varepsilon\|_E^2 + \varepsilon K_2^\varepsilon(t, \omega),$$

where, by applying the Gronwall lemma to (29) over $[0, T]$, both

$$K_1^\varepsilon(t, \omega) := (1 + \|\varphi^\varepsilon\|_E^{r-1} + \|u^0\|_{H^1}^{r-1}) \text{ and}$$
$$K_2^\varepsilon(t, \omega) := c(|y(\theta_t\omega)| + |y(\theta_t\omega)|^2)\|\varphi^\varepsilon\|_E^2$$

are bounded when $t \in [0, T]$ and $\varepsilon \in (0, \varepsilon_0]$. Hence,

$$\frac{d}{dt}\|\Psi^\varepsilon\|_E^2 \leq C\|\Psi^\varepsilon\|_E^2 + \varepsilon C \tag{68}$$

Applying the Gronwall lemma to (68) over $[0, t]$ for $t \leq T$, we see

$$\sup_{t \in [0, T]}\|\Psi^\varepsilon(t)\|_E^2 \leq C\|\Psi^\varepsilon(0)\|_E^2 + \varepsilon CT. \tag{69}$$

By $\|\Psi^\varepsilon(0)\|_E^2 = \|\varphi_0^\varepsilon - \varphi_0^0\|_E \to 0$, we obtain (60) as required. □

6. ROBUSTNESS OF RANDOM ATTRACTORS

A random compact set $\mathcal{A}^\varepsilon \in \mathfrak{D}$ is said to be a $\mathfrak{D}$-**random attractor** for the RDS Φ^ε (given by (18)) if it is *invariant*, i.e. $\Phi^\varepsilon(t, \omega)\mathcal{A}^\varepsilon(\omega) = \mathcal{A}^\varepsilon(\theta_t\omega)$ for $t \geq 0$, $\omega \in \Omega_0$, and $\mathfrak{D}$-*attracting*, i.e. for each $\mathcal{D} \in \mathfrak{D}$ and $\omega \in \Omega_0$,

$$\lim_{t \to +\infty} \text{dist}_E(\Phi^\varepsilon(t, \theta_{-t}\omega)\mathcal{D}(\theta_{-t}\omega), \ \mathcal{A}^\varepsilon(\omega)) = 0.$$

For the details, see [28, 29, 30, 31]. If $\varepsilon = 0$, we obtain a semigroup Φ^0 with a global attractor A^0 on E (see, e.g. [4, 5]).

Theorem 7. *For each $\varepsilon \in (0, \varepsilon_0]$, the random dynamical system Φ^ε has a unique $\mathfrak{D}$-random attractor $\mathcal{A}^\varepsilon = \{\mathcal{A}^\varepsilon(\omega) : \omega \in \Omega\}$ on $E = H^1(\mathbb{R}^3)^2$. Moreover,*

$$\lim_{\varepsilon \to 0} dist_E(\mathcal{A}^\varepsilon(\omega), A^0) = 0, \quad \omega \in \Omega_0. \tag{70}$$

Proof. By the abstract result given by [14, Theorem 4.1], it suffices to verify the following three aspects.

(i) **Covergence**. $\Phi^\varepsilon \to \Phi^0$ as $\varepsilon \to 0$, which is established by Proposition 6.

(ii) **Collective absorption**. For each $\varepsilon \in (0, \varepsilon_0]$, let

$$\mathcal{K}^\varepsilon(\omega) := \{\varphi \in E : \ \|\varphi\|_E^2 \le c(1 + R^\varepsilon(\omega))\}, \tag{71}$$

where $R^\varepsilon(\omega)$ is defined by (21). By Lemma 2, $\mathcal{K}^\varepsilon$ is a closed, bounded and random $\mathfrak{D}$-absorbing set for Φ^ε. Moreover, by (21) and (71),

$$\lim_{\varepsilon \to 0} \|\mathcal{K}^\varepsilon(\omega)\|_{E(\mathbb{R}^3)} \le c + \frac{c}{\kappa_1}, \ \omega \in \Omega_0.$$

It is easy to show $\cup_{\varepsilon \in (0, \varepsilon_0]} \mathcal{K}^\varepsilon \in \mathfrak{D}$. Then, the family $\{\mathcal{K}^\varepsilon : \varepsilon \in (0, \varepsilon_0]\}$ is collectively absorbing.

(iii) **Collective limit-set compactness**. Let $\mathcal{D} \in \mathfrak{D}$ and $\omega \in \Omega_0$. We need to show the Kuratowski measure $\chi_E M(T) \to 0$ as $T \to \infty$, where,

$$M(T) := \bigcup_{t \ge T} \bigcup_{\varepsilon \le \varepsilon_0} \Phi^\varepsilon(t, \theta_{-t}\omega)\mathcal{D}(\theta_{-t}\omega).$$

For this end, let $\eta > 0$ be small. By (36), we take $T_1 > 0$ and $k \ge 1$ such that

$$\|\varphi\|_{E(\mathcal{O}_k^c)} \le \eta, \quad \text{for all } \ \varphi \in M(T_1). \tag{72}$$

By (50), there are $i \in \mathbb{N}$ and $T_2 \ge T_1$ such that

$$\|(I - P_i)(\xi_k\varphi)\|_{E(Q_{2k})} \le \eta, \quad \text{for all } \ \varphi \in M(T_2). \tag{73}$$

By (20), there is a $T_3 \ge T_2$ such that $M(T_3)$ is bounded in $E(\mathbb{R}^3)$, which implies that the set $\{\xi_k\varphi : \ \varphi \in M(T_3)\}$ is bounded in $E(Q_{2k})$. Therefore the set $\{P_i(\xi_k\varphi) : \ \varphi \in M(T_3)\}$ is bounded in a finitely dimensional subspace and thus it is pre-compact such that

$$\chi_{E(Q_{2k})}\{P_i(\xi_k\varphi) : \ \varphi \in M(T_3)\} = 0. \tag{74}$$

By (73)-(74), we have

$$\chi_{E(Q_{2k})}\{\xi_k\varphi : \ \varphi \in M(T_3)\} \le \chi_{E(Q_{2k})}\{P_i(\xi_k\varphi : \ \varphi \in M(T_3)\}$$
$$+ \chi_{E(Q_{2k})}\{(I - P_i)(\xi_k\varphi : \ \varphi \in B(T_3)\} \le 2\eta. \tag{75}$$

Since $\xi_k\varphi = \varphi$ on Q_k, it follows from (75) that

$$\chi_{E(Q_k)}(M(T_3)) = \chi_{E(Q_k)}\{\xi_k\varphi : \ \varphi \in M(T_3)\} \leq 2\eta. \tag{76}$$

By (72) and (76), we arrive at

$$\chi_{E(\mathbb{R}^3)}(M(T_3)) \leq \chi_{E(Q_k)}(M(T_3)) + \chi_{E(Q_k^c)}(M(T_3)) \leq 4\eta,$$

which shows the needed conclusion. The measurability of attractors can be proved by the same method as given by [7]. $\qquad\square$

7. BASICALLY UNIFORM ROBUSTNESS

In this section, we will prove that the robustness (given in Theorem 7) is basically uniform in probability. The following lemma is well known.

Lemma 8. *If $\{F_n\}_{n=1}^\infty$ is an increasing family taken from $\mathcal{F}$, then $P(\cup_{n=1}^\infty F_n) = \lim_{n\to\infty} P(F_n)$. If $\{G_n\}_{n=1}^\infty$ is a decreasing family taken from $\mathcal{F}$, then $P(\cap_{n=1}^\infty G_n) = \lim_{n\to\infty} P(G_n)$.*

Theorem 9. *Let $\mathcal{A}^\varepsilon$ and A^0 be the random attractors given in Theorem 7. Then, for any $\varepsilon_n \to 0$ and $\eta > 0$, there is a $\Omega_\eta \in \mathcal{F}$ with $P(\Omega_\eta) > 1 - \eta$ such that*

$$\lim_{\varepsilon_n\to 0} \sup_{\omega\in\Omega_\eta} \ dist_E(\mathcal{A}^{\varepsilon_n}(\omega), A^0) = 0. \tag{77}$$

Proof. We set $h_n(\omega) = \text{dist}_E(\mathcal{A}^{\varepsilon_n}(\omega), A^0)$ and

$$\Omega_1 = \{\omega \in \Omega : \lim_{n\to\infty} h_n(\omega) = 0\}, \quad \hat{\Omega} = \Omega \setminus \Omega_1.$$

Then, by Theorem 7, $\Omega_1 \supset \Omega_0$ and thus $P(\Omega_1) = 1$, $P(\hat{\Omega}) = 0$. On the other hand, it is easy to prove

$$\hat{\Omega} = \bigcup_{k=1}^\infty \bigcap_{m=1}^\infty \bigcup_{n=m}^\infty \{\omega \in \Omega : h_n(\omega) \geq \frac{1}{k}\}.$$

Note that $\cup_{n=m}^\infty \{\omega \in \Omega : h_n(\omega) \geq \frac{1}{k}\}$ decreases as m increases. By (ii) of Lemma 8,

$$\lim_{m\to\infty} P(\bigcup_{n=m}^\infty \{\omega \in \Omega : h_n(\omega) \geq \frac{1}{k}\})$$

$$= P(\bigcap_{m=1}^\infty \bigcup_{n=m}^\infty \{\omega \in \Omega : h_n(\omega) \geq \frac{1}{k}\}) \leq P(\hat{\Omega}) = 0.$$

Then, for each $\eta > 0$ and $k \in \mathbb{N}$, there is an $m(k) \in \mathbb{N}$ such that

$$P(\bigcup_{n=m(k)}^{\infty} \{\omega \in \Omega : h_n(\omega) \geq \frac{1}{k}\}) < \frac{\eta}{2^k}.$$

Setting

$$\Omega_\eta := \bigcap_{k=1}^{\infty} \bigcap_{n=m(k)}^{\infty} \{\omega \in \Omega : h_n(\omega) < \frac{1}{k}\},$$

then, it is easy to find that

$$P(\Omega_0 \backslash \Omega_\eta) = P(\bigcup_{k=1}^{\infty} \bigcup_{n=N(k)}^{\infty} \{\omega \in \Omega_0 : f_{\varepsilon_n}(\omega) \geq \frac{1}{k}\})$$

$$\leq \sum_{k=1}^{\infty} P(\bigcup_{n=m(k)}^{\infty} \{\omega \in \Omega : h_{\varepsilon_n}(\omega) \geq \frac{1}{k}\}) < \sum_{k=1}^{\infty} \frac{\eta}{2^k} = \eta,$$

which proves $P(\Omega_\eta) > 1 - \eta$. On the other hand, for each $\eta' > 0$, there is a $k_0 := k_0(\eta') \in \mathbb{N}$ such that $\frac{1}{k_0} < \eta'$, in this way, we find an $m(k_0)$ such that $\Omega_\eta \subset \bigcap_{n=m(k_0)}^{\infty} \{\omega \in \Omega : h_n(\omega) < \frac{1}{k_0}\}$, then

$$\sup_{\omega \in \Omega_\eta} h_n(\omega) < \frac{1}{k_0} < \eta', \quad \text{for all } n \geq m(k_0),$$

which implies (77) as required. $\square$

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