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# LAW OF LARGE NUMBERS AND CENTRAL LIMIT THEOREM FOR INDEPENDENT AND NON-IDENTICAL DISTRIBUTED RANDOM VARIABLES UNDER SUBLINEAR EXPECTATIONS

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**ABSTRACT:** In this paper, we study some limit theorems for random variables under sublinear expectations. First, a law of large numbers is proved for independent and non-identical distributed random variables with only finite first order moments. Second, a central limit theorem is proved for independent and non-identical distributed random variables with only finite second order moments. These results include and extend some existing results.

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## 1. INTRODUCTION

The law of large numbers and central limit theorem as fundamental limit theorems in probability theory play a fruitful role in the development of probability theory and its applications. However, these kinds of limit theorems have always considered additive probabilities and additive expectations. In fact, the additivity of probabilities and expectations has been abandoned in some areas because many uncertain phenomena cannot be well modeled by using additive probabilities and additive expectations.

Since the paper (Artzner et al. [1]) on coherent risk measures, people are more and more interested in sublinear expectations (or more generally, convex expectations, see [4, 6, 7, 8]). By Peng [16], we know that a sublinear expectation $\hat{E}$ can be represented as the upper expectation of a subset of linear expectations $\{E_\theta : \theta \in \Theta\}$, i.e., $\hat{E}[\cdot] = \sup_{\theta\in\Theta} E_\theta[\cdot]$. In most cases, this subset is often treated as an uncertain model of probabilities $\{P_\theta : \theta \in \Theta\}$ and the notion of sublinear expectation provides a robust way to measure a risk loss $X$. In fact, the nonlinear expectation theory provides many rich, flexible and elegant tools.

Since the notion of independent identically distributed (IID) random variables under sublinear expectations initiated by Peng, many limit results such as strong (weak) law of large numbers, central limit theorem and law of iterated logarithm under sublinear expectations have been studied. For more details, we can see [2, 3, 9, 10, 11, 12, 13, 14, 15, 18, 19].

In this paper, we study some limit theorems for random variables under sublinear expectations. First, a law of large numbers is proved for independent and non-identical distributed random variables with only finite first order moments. Second, a central limit theorem is proved for independent and non-identical distributed random variables with only finite second order moments. These results generalize the known results in [10, 12, 13, 14, 15, 19].

## 2. PRELIMINARIES

In this section, we present some preliminaries in the theory of sublinear expectations. For more detail, we can see [5, 16, 17, 18, 19].

Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$. We suppose that $\mathcal{H}$ satisfies $c \in \mathcal{H}$ for each constant $c$ and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. The space $\mathcal{H}$ can be considered as the space of random variables.

**Definition 2.1.** (see [16, 17, 18, 19]). A sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E}$: $\mathcal{H} \to \overline{\mathbb{R}}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a) Monotonicity: If $X \geq Y$, then $\hat{E}[X] \geq \hat{E}[Y]$;

(b) Constant preserving: $\hat{E}[c] = c, \forall c \in \mathbb{R}$;

(c) Sub-additivity: $\hat{E}[X+Y] \leq \hat{E}[X] + \hat{E}[Y]$ whenever $\hat{E}[X] + \hat{E}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;

(d) Positive homogeneity: $\hat{E}[\lambda X] = \lambda \hat{E}[X], \forall \lambda \geq 0$.

Here $\overline{\mathbb{R}} = [-\infty, +\infty]$. The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

Give a sublinear expectation $\hat{E}$, let us denote the conjugate expectation $\hat{\varepsilon}$ of $\hat{E}$ by

$$\hat{\varepsilon}[X] := -\hat{E}[-X], \quad \forall X \in \mathcal{H}.$$

It is obvious that $\hat{\varepsilon}[X] \leq \hat{E}[X]$, for all $X \in \mathcal{H}$.

In this paper, we consider the following sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$: if $X_1, \cdots, X_n \in \mathcal{H}$, then $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l.Lip}(\mathbb{R}^n)$, where $C_{l.Lip}(\mathbb{R}^n)$ denotes the linear space of functions $\varphi$ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y| \quad \forall x, y \in \mathbb{R}^n,$$

for some $C > 0,\ m \in \mathbb{N}$ depending on $\varphi$. Let $C_{b.Lip}(\mathbb{R}^n)$ denote the linear space of bounded functions $\varphi$ satisfying

$$|\varphi(x) - \varphi(y)| \leq C|x - y| \quad \forall x, y \in \mathbb{R}^n,$$

for some $C > 0$ depending on $\varphi$.

**Definition 2.2.** (lSee [16, 17, 18, 19]). *Identical distribution:* Let $X_1$ and $X_2$ be two n-dimensional random vectors defined in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$, respectively. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if

$$\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)], \quad \forall \varphi \in C_{l.Lip}(\mathbb{R}^n),$$

whenever the sublinear expectations are finite.

*Independence:* In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, a random vector $Y = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$ is called independent to another random vector $X := (X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$ under $\hat{E}$, if for each test function $\varphi \in C_{l.Lip}(\mathbb{R}^m \times \mathbb{R}^n)$, we have

$$\hat{E}[\varphi(X, Y)] = \hat{E}[\hat{E}[\varphi(x, Y)]_{x=X}],$$

whenever $\overline{\varphi}(x) := \hat{E}[|\varphi(x, Y)|] < \infty$ for all $x$ and $\hat{E}[\overline{\varphi}(X)] < \infty$.

*Sequence of IID random variables:* A sequence of IID random sequence $\{X_i\}_{i=1}^{\infty}$ is called IID random variables, if $X_i \overset{d}{=} X_1$ and $X_{i+1}$ is independent to $Y := (X_1, \cdots, X_i)$ for each $i \geq 1$.

**Definition 2.3.** *(Maximal distribution)* (see [16, 17]). A random variable $\eta$ in a sublinear expectation space $(\Omega, \mathcal{H}, \tilde{E})$ is called maximal distributed if

$$\tilde{E}[\varphi(\eta)] = \sup_{\underline{\mu} \leq y \leq \overline{\mu}} \varphi(y), \quad \forall \varphi \in C_{l.Lip}(\mathbb{R}),$$

where $\overline{\mu} := \tilde{E}[\eta]$ and $\underline{\mu} := \tilde{\varepsilon}[\eta]$.

**Remark 2.4.** Let $\eta$ be maximal distributed with $\overline{\mu} := \tilde{E}[\eta]$, $\underline{\mu} := \tilde{\varepsilon}[\eta]$, the distribution of $\eta$ is characterized by the following parabolic partial differential equation (PDE):

$$\partial_t u - g(\partial_x u) = 0, \quad u(0, x) = \varphi(x),$$

where $u(t, x) := \tilde{E}[\varphi(x + t\eta)],\ (t, x) \in [0, \infty) \times \mathbb{R},\ g(x) := \overline{\mu}x^+ - \underline{\mu}x^-$.

**Definition 2.5.** *(G-normal distribution)* (see [16, 17]). A random variable $X$ in a sublinear expectation space $(\Omega, \mathcal{H}, \tilde{E})$ with $\overline{\sigma}^2 := \tilde{E}[X^2]$, $\underline{\sigma}^2 := \tilde{\varepsilon}[X^2]$ is called G-normal distributed, denoted by $X \sim \mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$, if for each $Y \in \mathcal{H}$ which is independent to $X$ such that $Y \overset{d}{=} X$, it holds that $aX + bY \overset{d}{=} \sqrt{a^2+b^2}X$, $\forall a, b \geq 0$.

**Remark 2.6.** Let $X \sim \mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$ under $\tilde{E}$. For each $\varphi \in C_{l.Lip}(\mathbb{R})$, we define a function

$$v(t,x) := \tilde{E}[\varphi(x + \sqrt{t}X)], \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Then $v$ is the unique viscosity solution of the following parabolic PDE:

$$\partial_t v - G(\partial^2_{xx} v) = 0, \quad v(0,x) = \varphi(x),$$

where $G(\alpha) := \frac{1}{2}\tilde{E}[\alpha X^2] = \frac{1}{2}(\overline{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-)$.

**Lemma 2.7.** *(see [3, 5]). Suppose $X \sim \mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$ under $\tilde{E}$. Let $P$ be a probability measure and $\varphi$ be a bounded continuous function on $\mathbb{R}$. If $\{B_t\}_{t\geq 0}$ is a 1-dimensional Brownian motion under $P$, then*

$$\tilde{E}[\varphi(X)] = \sup_{\theta\in\Theta} E_P\left[\varphi\left(\int_0^1 \theta_s dB_s\right)\right], \quad \tilde{\varepsilon}[\varphi(X)] = \inf_{\theta\in\Theta} E_P\left[\varphi\left(\int_0^1 \theta_s dB_s\right)\right],$$

*where*

$$\Theta := \left\{\{\theta_t\}_{t\geq 0} : \theta_t \text{ is } \mathcal{F}_t\text{-adapted process such that } \underline{\sigma} \leq \theta_t \leq \overline{\sigma}\right\},$$
$$\mathcal{F}_t := \sigma\{B_s : 0 \leq s \leq t\} \vee \mathcal{N}, \quad \mathcal{N} \text{ is the collection of } P\text{-null subsets.}$$

**Lemma 2.8.** *(Hölder's inequality) (see [17]). Let $X, Y$ be two random variables in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, then for $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\hat{E}[|XY|] \leq (\hat{E}[|X|^p])^{\frac{1}{p}} \cdot (\hat{E}[|Y|^q])^{\frac{1}{q}}.$$

**Lemma 2.9.** *(Rosenthal's inequality) (see [18]). Let $(X_1, \cdots, X_n)$ be a sequence of random variables in $(\Omega, \mathcal{H}, \hat{E})$. Suppose that $X_{k+1}$ is independent to $(X_1, \cdots, X_k)$ for each $k = 1, \cdots, n-1$. Denote $S_n := \sum_{k=1}^n X_k$.*

*(a) Then*

$$\hat{E}\left[\max_{k\leq n}|S_k|^p\right] \leq C_p\left\{\sum_{k=1}^n \hat{E}[|X_k|^p] + \left(\sum_{k=1}^n \hat{E}[X_k^2]\right)^{\frac{p}{2}}\right\}$$
$$+ C_p\left\{\left(\sum_{k=1}^n \left[(\hat{\varepsilon}[X_k])^- + (\hat{E}[X_k])^+\right]\right)^p\right\}, \quad \textit{for } p \geq 2. \quad (2.1)$$

*(b) Furthermore, we assume that* $\hat{E}[X_k] \le 0, \quad k = 1, \cdots, n$. *Then*

$$\hat{E}\left[(S_n^+)^p\right] \le \begin{cases} 2^{2-p} \sum_{k=1}^{n} \hat{E}\left[|X_k|^p\right], & for\ 1 \le p \le 2, \\ C_p n^{\frac{p}{2}-1} \sum_{k=1}^{n} \hat{E}[|X_k|^p], & for\ p \ge 2. \end{cases} \tag{2.2}$$

**Definition 2.10.** A set function $V : \mathcal{F} \to [0,1]$ is called a capacity if it satisfies the following:

(1) $V(\emptyset) = 0, V(\Omega) = 1$;

(2) $V(A) \le V(B)$, whenever $A \subset B$ and $A, B \in \mathcal{F}$.

It is called a sub-additive capacity if it further satisfies $V(A \cup B) \le V(A) + V(B)$ for all $A, B \in \mathcal{F}$ with $A \bigcup B \in \mathcal{F}$.

Let $(\Omega, \mathcal{H}, \hat{E})$ be a sublinear expectation space, and $\hat{\varepsilon}$ be the conjugate expectation of $\hat{E}$. We denote a pair $(\mathbb{V}, v)$ of capacities by

$$\mathbb{V}(A) := \inf\{\hat{E}[\xi] : I_A \le \xi, \xi \in \mathcal{H}\}, \quad v(A) := 1 - \mathbb{V}(A^c), \quad \forall A \in \mathcal{F},$$

where $A^c$ is the complement set of $A$. Then

$$\begin{aligned} &\mathbb{V}(A) := \hat{E}[I_A], \quad v(A) := \hat{\varepsilon}[I_A], \quad \text{if } I_A \in \mathcal{H}, \\ &\hat{E}[f] \le \mathbb{V}(A) \le \hat{E}[g], \quad \hat{\varepsilon}[f] \le v(A) \le \hat{\varepsilon}[g], \quad \text{if } f \le I_A \le g, \quad f, g \in \mathcal{H}. \end{aligned} \tag{2.3}$$

Obviously, $\mathbb{V}$ is sub-additive. But $v$ is not. However, we have $v(A \bigcup B) \le v(A) + \mathbb{V}(B)$.

## 3. MAIN RESULTS

**Theorem 3.1.** *Let a sequence* $\{X_i\}_{i=1}^{\infty}$*, which is in a sublinear expectation space* $(\Omega, \mathcal{H}, \hat{E})$*, satisfy the following conditions:*

(i) *each* $X_{i+1}$ *is independent to* $(X_1, \cdots, X_i)$*, for* $i = 1, 2, \cdots$;

(ii) $\hat{E}[X_i] = \overline{\mu_i}$, $\hat{\varepsilon}[X_i] = \underline{\mu_i}$, *where* $-\infty < \underline{\mu_i} \le \overline{\mu_i} < \infty$;

(iii) *there are two constants* $\overline{\mu}$ *and* $\underline{\mu}$ *such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |\underline{\mu_i} - \underline{\mu}| = 0, \quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |\overline{\mu_i} - \overline{\mu}| = 0;$$

(iv) $\lim_{d\to\infty} \sup_{i\ge1} \hat{E}\left[(|X_i| - d)^+\right] = 0$;

(v) $\sup_{i\ge1} \hat{E}[|X_i|] < \infty$. *Then for any continuous function* $\varphi$ *satisfying* $|\varphi(x)| \le C(1 + |x|)$*, we have*

$$\lim_{n\to\infty} \hat{E}\left[\varphi\left(\frac{S_n}{n}\right)\right] = \widetilde{E}[\varphi(\eta)], \tag{3.1}$$

*where* $S_n = \sum_{i=1}^{n} X_i$, $\eta$ *is maximal distributed under* $\widetilde{E}$ *with* $\overline{\mu} := \widetilde{E}[\eta]$, $\underline{\mu} := \widetilde{\varepsilon}[\eta]$. *Furthermore, if* $p > 1$ *and* $\sup_{i\geq 1}\hat{E}[|X_i|^p] < \infty$, *then* (3.1) *holds for any continuous function* $\varphi$ *satisfying* $|\varphi(x)| \leq C(1+|x|^p)$.

**Proof.** Let $Y_i = (-i)\vee(X_i\wedge i)$, $T_n = \sum_{i=1}^{n} Y_i$. In order to prove Theorem 3.1, we need the following facts:

(A1) Suppose that the condition (iv) is satisfied, then

$$\frac{1}{n}\sum_{i=1}^{n}\hat{E}[|X_i - Y_i|] \to 0 \quad \text{as} \quad n\to\infty.$$

(A2) Suppose that the conditions (iv) and (v) are satisfied, then

$$\frac{\sum_{i=1}^{n}\hat{E}[|Y_i|^{\alpha+1}]}{n^{\alpha+1}} \to 0 \quad \text{as} \quad n\to\infty, \quad \forall\ 0<\alpha<1.$$

(A3) Suppose that the conditions (i) and (v) are satisfied, then

$$\hat{E}\left[|T_n|^{2p}\right] \leq C_{2p} n^{2p}, \quad \forall\ p\geq 1.$$

For (A1), by Stolz theorem, it is sufficient to show that

$$\hat{E}[|X_n - Y_n|] \to 0 \quad \text{as} \quad n\to\infty.$$

Note that

$$\hat{E}[|X_n - Y_n|] = \hat{E}[(|X_n|-n)^+] \leq \sup_{i\geq 1}\hat{E}[(|X_i|-n)^+] \to 0 \quad \text{as} \quad n\to\infty.$$

So (A1) holds.

For (A2), by Stolz theorem, it is sufficient to show that

$$\frac{\hat{E}[|Y_n|^{\alpha+1}]}{n^{\alpha+1}-(n-1)^{\alpha+1}} \to 0 \quad \text{as} \quad n\to\infty.$$

Note that

$$\frac{\hat{E}[|Y_n|^{\alpha+1}]}{n^{\alpha+1}-(n-1)^{\alpha+1}} \leq \frac{\hat{E}[|Y_n|^{\alpha+1}]}{(n-1)^{\alpha}},$$

$$\begin{aligned}\hat{E}[|Y_n|^{\alpha+1}] &\leq \hat{E}[|X_n||Y_n|^{\alpha}] \leq \hat{E}[(|X_n|-d+d)|Y_n|^{\alpha}] \\ &\leq n^{\alpha}\hat{E}[(|X_n|-d)^+] + d\hat{E}[|Y_n|^{\alpha}] \\ &\leq n^{\alpha}\hat{E}[(|X_n|-d)^+] + d\left(\hat{E}[|X_n|]\right)^{\alpha} \\ &\leq n^{\alpha}\cdot\sup_{i\geq 1}\hat{E}[(|X_i|-d)^+] + d\left(\sup_{i\geq 1}\hat{E}[|X_i|]\right)^{\alpha}.\end{aligned}$$

So

$$\frac{\hat{E}[|Y_n|^{\alpha+1}]}{(n-1)^\alpha} \le \frac{n^\alpha \cdot \sup_{i\ge1} \hat{E}[(|X_i|-d)^+]}{(n-1)^\alpha} + \frac{d\left(\sup_{i\ge1} \hat{E}[|X_i|]\right)^\alpha}{(n-1)^\alpha} \to 0$$

$$\text{as } n\to\infty \text{ and } d\to\infty.$$

Thus, (A.2) holds.

For (A3), by the Rosenthal's inequality (2.1), we have

$$\begin{aligned}
\hat{E}[|T_n|^{2p}] &\le C_{2p}\sum_{i=1}^n \hat{E}[|Y_i|^{2p}] + C_{2p}\left(\sum_{i=1}^n \hat{E}[|Y_i|^2]\right)^p \\
&+ C_{2p}\left(\sum_{i=1}^n \left[(\hat{E}[Y_i])^+ + (\hat{E}[-Y_i])^+\right]\right)^{2p} \\
&\le C_{2p}\sum_{i=1}^n \hat{E}[|X_i||Y_i|^{2p-1}] + C_{2p}\left(\sum_{i=1}^n \hat{E}[|X_i||Y_i|]\right)^p \\
&+ C_{2p}\left(\sum_{i=1}^n 2\hat{E}[|X_i|]\right)^{2p} \\
&\le C_{2p}n^{2p-1}n\cdot\sup_{i\ge1}\hat{E}[|X_i|] + C_{2p}\left(n\cdot\ \ n\cdot\sup_{i\ge1}\hat{E}[|X_i|]\right)^p \\
&+ C_{2p}2^{2p}\left(n\cdot\sup_{i\ge1}\hat{E}[|X_i|]\right)^{2p} \\
&\le C_{2p}n^{2p}.
\end{aligned}$$

We first prove that

$$\lim_{n\to\infty} \hat{E}\left[\varphi\left(\frac{T_n}{n}\right)\right] = \widetilde{E}[\varphi(\eta)], \qquad \forall\varphi\in C_{b.Lip}(\mathbb{R}). \tag{3.2}$$

Now, for a small but fixed $h>0$, let $V$ be the unique viscosity solution of the following equation:

$$\partial_t V + g(\partial_x V) = 0, \quad (t,x)\in[0,1+h]\times\mathbb{R}, \qquad V|_{t=1+h} = \varphi(x), \tag{3.3}$$

where $g(x) := \overline{\mu}x^+ - \underline{\mu}x^-$. According to the definition of maximal distribution, we have

$$V(t,x) = \widetilde{E}[\varphi(x+(1+h-t)\eta)], \quad V(h,0) = \widetilde{E}[\varphi(\eta)], \quad V(1+h,x) = \varphi(x). \tag{3.4}$$

Since (3.3) is a uniformly parabolic PDE, by the interior regularity of $V$, we have

$$\|V\|_{C^{1+\frac{\alpha}{2},1+\alpha}([0,1]\times\mathbb{R})} < \infty, \quad \text{for some } \alpha\in(0,1).$$

Let $\delta=\frac{1}{n},\ T_0=0$, then

$$\begin{aligned} V(1,\delta T_n)-V(0,0) &= \sum_{i=0}^{n-1}\{V((i+1)\delta,\delta T_{i+1})-V(i\delta,\delta T_i)\} \\ &= \sum_{i=0}^{n-1}\{[V((i+1)\delta,\delta T_{i+1})-V(i\delta,\delta T_{i+1})] \\ &\quad +[V(i\delta,\delta T_{i+1})-V(i\delta,\delta T_i)]\} \\ &= \sum_{i=0}^{n-1}\{I_\delta^i+J_\delta^i\}, \end{aligned}$$

with, by Taylor's expansion,

$$\begin{aligned} J_\delta^i &= \partial_t V(i\delta,\delta T_i)\delta+\partial_x V(i\delta,\delta T_i)Y_{i+1}\delta \\ &= (\partial_t V(i\delta,\delta T_i)\delta+\partial_x V(i\delta,\delta T_i)X_{i+1}\delta)+(\partial_x V(i\delta,\delta T_i)(Y_{i+1}-X_{i+1})\delta) \\ &= J_{\delta,1}^i+J_{\delta,2}^i, \end{aligned}$$

$$\begin{aligned} I_\delta^i = &\int_0^1 [\partial_t V((i+\beta)\delta,\delta T_{i+1})-\partial_t V(i\delta,\delta T_{i+1})]\,d\beta\delta+[\partial_t V(i\delta,\delta T_{i+1})-\partial_t V(i\delta,\delta T_i)]\,\delta \\ &+\int_0^1 [\partial_x V(i\delta,\delta T_i+\beta\delta Y_{i+1})-\partial_x V(i\delta,\delta T_i)]\,d\beta Y_{i+1}\delta. \end{aligned}$$

Therefore,

$$\begin{aligned} &\hat{E}\left[\sum_{i=0}^{n-1}J_{\delta,1}^i\right]-\sum_{i=0}^{n-1}\left(\hat{E}[|J_{\delta,2}^i|]+\hat{E}[|I_\delta^i|]\right) \\ &\le \hat{E}[V(1,\delta T_n)]-V(0,0)\le \hat{E}\left[\sum_{i=0}^{n-1}J_{\delta,1}^i\right]+\sum_{i=0}^{n-1}\left(\hat{E}[|J_{\delta,2}^i|]+\hat{E}[|I_\delta^i|]\right). \end{aligned} \tag{3.5}$$

We first consider $\hat{E}\left[\sum_{i=0}^{n-1}J_{\delta,1}^i\right]$. From (3.3) and the condition (i), it follows that

$$\begin{aligned} \hat{E}[J_{\delta,1}^i] &= \hat{E}\left[\partial_t V(i\delta,\delta T_i)\delta+\partial_x V(i\delta,\delta T_i)X_{i+1}\delta\right] \\ &= \hat{E}\Big\{\partial_t V(i\delta,\delta T_i)\delta+\delta\left[(\partial_x V(i\delta,\delta T_i))^+\overline{\mu_{i+1}}-(\partial_x V(i\delta,\delta T_i))^-\underline{\mu_{i+1}}\right]\Big\} \\ &\le \hat{E}\Big\{\partial_t V(i\delta,\delta T_i)\delta+\delta\left[(\partial_x V(i\delta,\delta T_i))^+\overline{\mu}-(\partial_x V(i\delta,\delta T_i))^-\underline{\mu}\right]\Big\} \\ &\quad +\delta\hat{E}\left[(\partial_x V(i\delta,\delta T_i))^+(\overline{\mu_{i+1}}-\overline{\mu})-(\partial_x V(i\delta,\delta T_i))^-(\underline{\mu_{i+1}}-\underline{\mu})\right] \\ &= \delta\hat{E}\left[(\partial_x V(i\delta,\delta T_i))^+(\overline{\mu_{i+1}}-\overline{\mu})-(\partial_x V(i\delta,\delta T_i))^-(\underline{\mu_{i+1}}-\underline{\mu})\right] \\ &\le \delta\hat{E}\left[|\partial_x V(i\delta,\delta T_i)|\right]\left(|\overline{\mu_{i+1}}-\overline{\mu}|+|\underline{\mu_{i+1}}-\underline{\mu}|\right). \end{aligned}$$

Since $\partial_x V$ is uniformly $\frac{\alpha}{2}$-hölder continuous in $t$ and $\alpha$-hölder continuous in $x$ on $[0,1]\times\mathbb{R}$, we have

$$\begin{aligned}\hat{E}[|\partial_x V(i\delta,\delta T_i)|] &\le \hat{E}[|\partial_x V(i\delta,\delta T_i)-\partial_x V(0,0)|]+|\partial_x V(0,0)|\\ &\le C(1+|i\delta|^{\frac{\alpha}{2}}+\hat{E}[|\delta T_i|^{\alpha}]).\end{aligned}$$

Since

$$\hat{E}[|\delta T_i|^{\alpha}] \le \hat{E}[|\delta T_i|]+1 \le \sup_{i\ge 1}\hat{E}[|X_i|]+1,$$

we claim that there is a constant $C_1>0$, such that

$$\hat{E}[|\partial_x V(i\delta,\delta T_i)|] \le C_1. \tag{3.6}$$

Then we obtain

$$\hat{E}\left[\sum_{i=0}^{n-1} J_{\delta,1}^i\right] \le \sum_{i=0}^{n-1}\hat{E}[J_{\delta,1}^i] \le C_1\frac{1}{n}\sum_{i=0}^{n-1}\left(|\overline{\mu_{i+1}}-\overline{\mu}|+|\underline{\mu_{i+1}}-\underline{\mu}|\right).$$

In a similar manner as above, we also have

$$\hat{E}\left[\sum_{i=0}^{n-1} J_{\delta,1}^i\right] \ge -C_1\frac{1}{n}\sum_{i=0}^{n-1}\left(|\overline{\mu_{i+1}}-\bar{\mu}|+|\underline{\mu_{i+1}}-\underline{\mu}|\right).$$

Thus, from the condition (iii), we can obtain

$$\lim_{n\to\infty}\hat{E}\left[\sum_{i=0}^{n-1} J_{\delta,1}^i\right]=0. \tag{3.7}$$

Next, we consider $\sum_{i=0}^{n-1}\hat{E}\left[|J_{\delta,2}^i|\right]$. According to the condition (i) and (3.6), we have

$$\begin{aligned}\hat{E}\left[|J_{\delta,2}^i|\right] &= \hat{E}[|\partial_x V(i\delta,\delta T_i)(Y_{i+1}-X_{i+1})\delta|]\\ &\le \delta\hat{E}[|\partial_x V(i\delta,\delta T_i)|]\hat{E}[|X_{i+1}-Y_{i+1}|]\\ &\le \delta C_1\hat{E}[|X_{i+1}-Y_{i+1}|].\end{aligned}$$

Then, we obtain

$$\sum_{i=0}^{n-1}\hat{E}\left[|J_{\delta,2}^i|\right] \le C_1\frac{1}{n}\sum_{i=0}^{n-1}\hat{E}[|X_{i+1}-Y_{i+1}|].$$

By (A1), it follows that

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\hat{E}\left[|J_{\delta,2}^i|\right]=0. \tag{3.8}$$

Finally, we consider $\sum_{i=0}^{n-1}\hat{E}\left[|I_\delta^i|\right]$. For $I_\delta^i$, since both $\partial_t V$ and $\partial_x V$ are uniformly $\frac{\alpha}{2}$-hölder continuous in $t$ and $\alpha$-hölder continuous in $x$ on $[0,1]\times\mathbb{R}$, then we have

$$\hat{E}[|I_\delta^i|] \le C\delta^{\frac{\alpha}{2}+1}+C\delta^{\alpha+1}\left(\hat{E}[|Y_{i+1}|^{\alpha}]+\hat{E}[|Y_{i+1}|^{\alpha+1}]\right).$$

It follows that

$$\sum_{i=0}^{n-1} \hat{E}\left[|I_\delta^i|\right] \le C\left(\frac{1}{n}\right)^{\frac{\alpha}{2}} + C\left(\frac{1}{n}\right)^{\alpha+1} \sum_{i=0}^{n-1}\left(\hat{E}[|Y_{i+1}|^\alpha] + \hat{E}[|Y_{i+1}|^{\alpha+1}]\right).$$

For $\sum_{i=0}^{n-1} \hat{E}[|Y_{i+1}|^\alpha] \Big/ n^{\alpha+1}$, we have

$$\frac{\sum_{i=0}^{n-1} \hat{E}[|Y_{i+1}|^\alpha]}{n^{\alpha+1}} \le \frac{\sum_{i=0}^{n-1} \hat{E}[|X_{i+1}|^\alpha]}{n^{\alpha+1}} \le \frac{\left(\sup_{i\ge 1} \hat{E}[|X_i|]\right)^\alpha}{n^\alpha} \to 0 \quad \text{as} \quad n \to \infty.$$

By (A2), we can obtain

$$\sum_{i=0}^{n-1} \hat{E}[|Y_{i+1}|^{\alpha+1}] \Big/ n^{\alpha+1} \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore,

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \hat{E}\left[|I_\delta^i|\right] = 0. \tag{3.9}$$

From (3.5), (3.7), (3.8) and (3.9), we have

$$\lim_{n\to\infty} \hat{E}[V(1, \delta T_n)] = V(0,0). \tag{3.10}$$

Additionally, it is obvious that if $\varphi \in C_{b.Lip}(\mathbb{R})$, i.e., $|\varphi(x) - \varphi(y)| \le C|x-y|$, then for each $t, s \in [0, 1+h]$ and $x \in \mathbb{R}$,

$$|V(t,x) - V(s,x)| \le C\widetilde{E}[|\eta|]|t-s| \le C|t-s|. \tag{3.11}$$

In particular,

$$|V(0,0) - V(h,0)| \le Ch. \tag{3.12}$$

Combining (3.4), (3.11), with (3.12), we have

$$\begin{aligned}
&\left|\hat{E}[\varphi(\delta T_n)] - \widetilde{E}[\varphi(\eta)]\right| = \left|\hat{E}[V(1+h, \delta T_n)] - V(h,0)\right| \\
&\le \left|\hat{E}[V(1+h, \delta T_n)] - \hat{E}[V(1, \delta T_n)]\right| + \left|\hat{E}[V(1, \delta T_n)] - V(0,0)\right| + |V(0,0) - V(h,0)| \\
&\le 2Ch + \left|\hat{E}[V(1, \delta T_n)] - V(0,0)\right|.
\end{aligned}$$

From (3.10), we obtain

$$\limsup_{n\to\infty} \left|\hat{E}[\varphi(\delta T_n)] - \widetilde{E}[\varphi(\eta)]\right| \le 2Ch.$$

So (3.2) is proved.

By the Lipschitz continuity of $\varphi$ and (A1), we have

$$\left|\hat{E}\left[\varphi\left(\frac{S_n}{n}\right)\right]-\hat{E}\left[\varphi\left(\frac{T_n}{n}\right)\right]\right|\le C\frac{1}{n}\sum_{i=1}^{n}\hat{E}\left[|X_i-Y_i|\right]\to 0\quad\text{as}\quad n\to\infty.$$

Thus

$$\lim_{n\to\infty}\hat{E}\left[\varphi\left(\frac{S_n}{n}\right)\right]=\widetilde{E}[\varphi(\eta)],\qquad\forall\varphi\in C_{b.Lip}(\mathbb{R}).$$

If $\varphi$ is a bounded and uniformly continuous function, we can find a sequence $\{\varphi_k\}_{k=1}^{\infty}\in C_{b.Lip}(\mathbb{R})$ such that $\varphi_k\to\varphi$ uniformly on $\mathbb{R}$. By

$$\begin{aligned}\left|\hat{E}\left[\varphi\left(\frac{S_n}{n}\right)\right]-\widetilde{E}\left[\varphi(\eta)\right]\right|&\le\left|\hat{E}\left[\varphi\left(\frac{S_n}{n}\right)\right]-\hat{E}\left[\varphi_k\left(\frac{S_n}{n}\right)\right]\right|\\&+\left|\widetilde{E}[\varphi(\eta)]-\widetilde{E}[\varphi_k(\eta)]\right|+\left|\hat{E}\left[\varphi_k\left(\frac{S_n}{n}\right)\right]-\widetilde{E}\left[\varphi_k(\eta)\right]\right|,\end{aligned}$$

we can easily check that (3.1) holds.

Finally, suppose that $p\ge 1$, $\sup_{i\ge1}\hat{E}[|X_i|^p]<\infty$, the conditions (i)-(iv) are satisfied, and $\varphi$ is a continuous function satisfying $|\varphi(x)|\le C(1+|x|^p)$. Give a number $N>1$. Define $\varphi_1(x)=\varphi((-N)\vee(x\wedge N))$ and $\varphi_2(x)=\varphi(x)-\varphi_1(x)$. Then $\varphi_1$ is a bounded and uniformly continuous function and

$$|\varphi_2(x)|\le 4C|x|^pI_{\{|x|>N\}}\le 8C(|x|^p-N/2)^+.$$

Define $M:=N/2$, then $|\varphi_2(x)|\le 8C(|x|^p-M)^+$. So

$$\begin{aligned}\left|\hat{E}\left[\varphi\left(\frac{S_n}{n}\right)\right]-\widetilde{E}[\varphi(\eta)]\right|&\le\left|\hat{E}\left[\varphi_1\left(\frac{S_n}{n}\right)\right]-\widetilde{E}[\varphi_1(\eta)]\right|\\&+8C\widetilde{E}[(|\eta|^p-M)^+]+8C\hat{E}\left[\left(\left|\frac{S_n}{n}\right|^p-M\right)^+\right].\end{aligned}$$

Since

$$8C\widetilde{E}[(|\eta|^p-M)^+]\le 8C\frac{\widetilde{E}[|\eta|^{2p}]}{M}\to 0\quad\text{as}\quad M\to\infty,$$

then it is sufficient to show that

$$\lim_{M\to\infty}\limsup_{n\to\infty}\hat{E}\left[\left(\left|\frac{S_n}{n}\right|^p-M\right)^+\right]=0. \tag{3.13}$$

Let $\hat{Y}_i=X_i-Y_i$, $\hat{S}_n=\sum_{i=1}^{n}(\hat{Y}_i-\hat{E}[\hat{Y}_i])$, then

$$S_n^+\le T_n^++\hat{S}_n^++\sum_{i=1}^{n}\hat{E}[|\hat{Y}_i|],$$

$$\left(\left|\frac{S_n^+}{n}\right|^p - M\right)^+ \le \left(3^{p-1}\left|\frac{T_n^+}{n}\right|^p - M\right)^+ + 3^{p-1}\left|\frac{\hat{S}_n^+}{n}\right|^p + 3^{p-1}\left(\sum_{i=1}^n \frac{\hat{E}[|\hat{Y}_i|]}{n}\right)^p.$$

By (A1), we have

$$\sum_{i=1}^n \frac{\hat{E}[|\hat{Y}_i|]}{n} = \frac{1}{n}\sum_{i=1}^n \hat{E}[|X_i - Y_i|] \to 0 \quad \text{as} \quad n \to \infty.$$

And by (A3), we have

$$\hat{E}\left[\left(3^{p-1}\left|\frac{T_n^+}{n}\right|^p - M\right)^+\right] \le M^{-1}3^{2p-2}\hat{E}\left[\left|\frac{T_n}{n}\right|^{2p}\right] \le M^{-1}3^{2p-2}C_{2p} \to 0 \quad \text{as} \quad M \to \infty.$$

For $\hat{E}[|\hat{S}_n^+/n|^p]$, applying the Rosenthal's inequality (2.2), we can obtain the following result: When $p = 1$,

$$\hat{E}\left[\left|\frac{\hat{S}_n^+}{n}\right|\right] \le 4\frac{1}{n}\sum_{i=1}^n \hat{E}[|\hat{Y}_i|] \le 4\frac{1}{n}\sum_{i=1}^n \hat{E}[(|X_i| - i)^+] \le 4\frac{1}{n}\sum_{i=1}^n \sup_{k\ge 1}\hat{E}[(|X_k| - i)^+],$$

so

$$\hat{E}\left[\left|\frac{\hat{S}_n^+}{n}\right|\right] \to 0 \quad \text{as} \quad n \to \infty.$$

When $1 < p \le 2$,

$$\begin{aligned}\hat{E}[|\hat{S}_n^+|^p] &\le 2^{2-p}\sum_{i=1}^n \hat{E}[|\hat{Y}_i - \hat{E}[\hat{Y}_i]|^p] \le 2^{2-p}\sum_{i=1}^n \left(2^p\hat{E}[|\hat{Y}_i|^p]\right) \\ &\le 4\sum_{i=1}^n \hat{E}[|X_i|^p] \le 4n \cdot \sup_{i\ge 1}\hat{E}[|X_i|^p],\end{aligned}$$

so

$$\hat{E}\left[\left|\frac{\hat{S}_n^+}{n}\right|^p\right] \to 0 \quad \text{as} \quad n \to \infty.$$

When $p > 2$,

$$\begin{aligned}\hat{E}[|\hat{S}_n^+|^p] &\le C_p n^{\frac{p}{2}-1}\sum_{i=1}^n \hat{E}[|\hat{Y}_i - \hat{E}[\hat{Y}_i]|^p] \le C_p n^{\frac{p}{2}-1}\sum_{i=1}^n \left(2^p\hat{E}[|\hat{Y}_i|^p]\right) \\ &\le 2^p C_p n^{\frac{p}{2}-1}\sum_{i=1}^n \hat{E}[|X_i|^p] \le 2^p C_p \left(\sup_{i\ge 1}\hat{E}[|X_i|^p]\right) n^{\frac{p}{2}},\end{aligned}$$

so

$$\hat{E}\left[\left|\frac{\hat{S}_n^+}{n}\right|^p\right] \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, it follows that

$$\lim_{M\to\infty}\limsup_{n\to\infty}\hat{E}\left[\left(\left|\frac{S_n^+}{n}\right|^p - M\right)^+\right] = 0.$$

Similarly,

$$\lim_{M\to\infty}\limsup_{n\to\infty}\hat{E}\left[\left(\left|\frac{S_n^-}{n}\right|^p - M\right)^+\right] = 0.$$

So (3.13) is proved and the proof is completed. □

**Theorem 3.2.** *Under the conditions of Theorem* 3.1*, then for any* $\varepsilon > 0$*,*

$$\lim_{n\to\infty} v\left(\frac{S_n}{n} \in \left(\underline{\mu} - \varepsilon, \overline{\mu} + \varepsilon\right)\right) = 1, \tag{3.14}$$

The proof is similar to the proof of Theorem 3.3 in [10] and so it is omitted.

**Theorem 3.3.** *Let* $\{X_i\}_{i=1}^{\infty}$ *be a sequence of random variables in a sublinear expectation space* $(\Omega, \mathcal{H}, \hat{E})$*, satisfy the following conditions:*

(i) *each* $X_{i+1}$ *is independent to* $(X_1, \cdots, X_i)$*, for* $i = 1, 2, \cdots$*;*

(ii) $\hat{E}[X_i] = \hat{\varepsilon}[X_i] = 0$*,* $\hat{E}[X_i^2] = \overline{\sigma}_i^2$*,* $\hat{\varepsilon}[X_i^2] = \underline{\sigma}_i^2$*, where* $0 \le \underline{\sigma}_i \le \overline{\sigma}_i < \infty$*;*

(iii) *there are two positive constants* $\underline{\sigma}$ *and* $\overline{\sigma}$ *such that*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|\underline{\sigma}_i^2 - \underline{\sigma}^2| = 0, \quad \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|\overline{\sigma}_i^2 - \overline{\sigma}^2| = 0;$$

(iv) $\lim_{c\to\infty}\sup_{i\ge 1}\hat{E}[(X_i^2 - c)^+] = 0$*;*

(v) $\sup_{i\ge 1}\hat{E}[X_i^2] < \infty$*. Then for any continuous function* $\varphi$ *satisfying* $|\varphi(x)| \le C(1 + x^2)$*, we have*

$$\lim_{n\to\infty}\hat{E}\left[\varphi\left(\frac{S_n}{\sqrt{n}}\right)\right] = \widetilde{E}[\varphi(\xi)], \tag{3.15}$$

*where* $S_n = \sum_{i=1}^{n} X_i$*,* $\xi \sim \mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$ *under* $\widetilde{E}$*. Furthermore, if* $p > 2$ *and* $\sup_{i\ge 1}\hat{E}[|X_i|^p] < \infty$*, then* (3.15) *holds for any continuous function* $\varphi$ *satisfying* $|\varphi(x)| \le C(1 + |x|^p)$*.*

**Proof.** Let $Y_i = (-\sqrt{i}) \vee (X_i \wedge \sqrt{i})$, $T_n = \sum_{i=1}^{n} Y_i$. In order to prove Theorem 3.3, we need the following facts:

(B1) Suppose that the condition (iv) is satisfied, then

$$\frac{\sum_{i=1}^{n}\hat{E}[|X_i - Y_i|]}{\sqrt{n}} \to 0 \quad \text{as} \quad n \to \infty.$$

(B2) Suppose that the conditions (iv) and (v) are satisfied, then

$$\frac{\sum\limits_{i=1}^{n} \hat{E}[|Y_i|^{\alpha+2}]}{n^{\frac{\alpha}{2}+1}} \to 0 \quad \text{as} \quad n \to \infty, \quad \forall 0 < \alpha < 1.$$

(B3) Suppose that the conditions (i), (ii), (iv) and (v) are satisfied, then

$$\hat{E}[|T_n|^p] \le C_p n^{\frac{p}{2}}, \quad \forall p \ge 2.$$

For (B1), note that

$$\sqrt{n}\hat{E}[|X_n - Y_n|] \le \hat{E}[(X_n^2 - n)^+] \le \sup_{i \ge 1} \hat{E}[(X_i^2 - n)^+].$$

So (B1) holds.

For (B2), note that

$$\begin{aligned}
\hat{E}[|Y_n|^{\alpha+2}] &\le \hat{E}[X_n^2 |Y_n|^{\alpha}] \le \hat{E}[(X_n^2 - c + c)|Y_n|^{\alpha}] \\
&\le n^{\frac{\alpha}{2}} \hat{E}[(X_n^2 - c)^+] + c\hat{E}[|Y_n|^{\alpha}] \\
&\le n^{\frac{\alpha}{2}} \hat{E}[(X_n^2 - c)^+] + c\left(\hat{E}[X_n^2]\right)^{\frac{\alpha}{2}} \\
&\le n^{\frac{\alpha}{2}} \cdot \sup_{i \ge 1} \hat{E}[(X_i^2 - c)^+] + c\left(\sup_{i \ge 1} \hat{E}[X_i^2]\right)^{\frac{\alpha}{2}}
\end{aligned}$$

for any $c > 1$. By Stolz theorem, (B2) is true.

For (B3), by the Rosenthal's inequality (2.1) and (B1), we have

$$\begin{aligned}
\hat{E}[|T_n|^p] &\le C_p \sum_{i=1}^{n} \hat{E}[|Y_i|^p] + C_p \left(\sum_{i=1}^{n} \hat{E}[Y_i^2]\right)^{\frac{p}{2}} + C_p \left(\sum_{i=1}^{n} \left[(\hat{E}[Y_i])^+ + (\hat{E}[-Y_i])^+\right]\right)^p \\
&\le C_p n^{\frac{p}{2}-1} \sum_{i=1}^{n} \hat{E}[X_i^2] + C_p \left(\sum_{i=1}^{n} \hat{E}[X_i^2]\right)^{\frac{p}{2}} + C_p \left(\sum_{i=1}^{n} 2\hat{E}[|X_i - Y_i|]\right)^p \\
&\le C_p n^{\frac{p}{2}-1} \cdot n \cdot \sup_{i \ge 1} \hat{E}[X_i^2] + C_p \left(n \cdot \sup_{i \ge 1} \hat{E}[X_i^2]\right)^{\frac{p}{2}} + C_p \left(\sum_{i=1}^{n} 2\hat{E}[|X_i - Y_i|]\right)^p \\
&\le C_p n^{\frac{p}{2}}.
\end{aligned}$$

So (B3) is true.

First, we show that

$$\lim_{n \to \infty} \hat{E}\left[\varphi\left(\frac{T_n}{\sqrt{n}}\right)\right] = \widetilde{E}[\varphi(\xi)], \qquad \forall \varphi \in C_{b.Lip}(\mathbb{R}). \tag{3.16}$$

Now, for a small but fixed $h > 0$, let $V$ be the unique viscosity solution of the following equation:

$$\partial_t V + G(\partial^2_{xx} V) = 0, \quad (t, x) \in [0, 1+h] \times \mathbb{R}, \quad V|_{t=1+h} = \varphi(x), \tag{3.17}$$

where $G(\alpha) := \frac{1}{2}(\overline{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-)$. According to the definition of G-normal distribution, we have

$$\begin{aligned} V(t,x) =& \widetilde{E}\left[\varphi\left(x + \sqrt{1+h-t}\xi\right)\right], \\ V(h,0) =& \widetilde{E}[\varphi(\xi)], \\ V(1+h,x) =& \varphi(x). \end{aligned} \tag{3.18}$$

Since (3.17) is a uniformly parabolic PDE, by the interior regularity of $V$, we have

$$\|V\|_{C^{1+\frac{\alpha}{2},2+\alpha}([0,1]\times\mathbb{R})} < \infty, \quad \text{for some } \alpha \in (0,1).$$

Let $\delta = \frac{1}{n}$, $T_0 = 0$, then

$$\begin{aligned} V(1, \sqrt{\delta}T_n) - V(0,0) &= \sum_{i=0}^{n-1} \left\{ V((i+1)\delta, \sqrt{\delta}T_{i+1}) - V(i\delta, \sqrt{\delta}T_i) \right\} \\ &= \sum_{i=0}^{n-1} \left\{ \left[ V((i+1)\delta, \sqrt{\delta}T_{i+1}) - V(i\delta, \sqrt{\delta}T_{i+1}) \right] \right. \\ &\quad \left. + \left[ V(i\delta, \sqrt{\delta}T_{i+1}) - V(i\delta, \sqrt{\delta}T_i) \right] \right\} \\ &= \sum_{i=0}^{n-1} \left\{ I^i_\delta + J^i_\delta \right\}, \end{aligned}$$

with, by Taylor's expansion,

$$\begin{aligned} J^i_\delta &= \partial_t V(i\delta, \sqrt{\delta}T_i)\delta + \frac{1}{2}\partial^2_{xx} V(i\delta, \sqrt{\delta}T_i) Y^2_{i+1}\delta + \partial_x V(i\delta, \sqrt{\delta}T_i) Y_{i+1}\sqrt{\delta} \\ &= \left( \partial_t V(i\delta, \sqrt{\delta}T_i)\delta + \frac{1}{2}\partial^2_{xx} V(i\delta, \sqrt{\delta}T_i) X^2_{i+1}\delta + \partial_x V(i\delta, \sqrt{\delta}T_i) X_{i+1}\sqrt{\delta} \right) \\ &\quad + \left( \frac{1}{2}\partial^2_{xx} V(i\delta, \sqrt{\delta}T_i)(Y^2_{i+1} - X^2_{i+1})\delta + \partial_x V(i\delta, \sqrt{\delta}T_i)(Y_{i+1} - X_{i+1})\sqrt{\delta} \right) \\ &= J^i_{\delta,1} + J^i_{\delta,2}, \end{aligned}$$

$$\begin{aligned} I^i_\delta =& \int_0^1 \left[ \partial_t V((i+\beta)\delta, \sqrt{\delta}T_{i+1}) - \partial_t V(i\delta, \sqrt{\delta}T_{i+1}) \right] d\beta\delta \\ &+ \left[ \partial_t V(i\delta, \sqrt{\delta}T_{i+1}) - \partial_t V(i\delta, \sqrt{\delta}T_i) \right] \delta \\ &+ \int_0^1 \int_0^1 \left[ \partial^2_{xx} V(i\delta, \sqrt{\delta}T_i + \gamma\beta Y_{i+1}\sqrt{\delta}) - \partial^2_{xx} V(i\delta, \sqrt{\delta}T_i) \right] \gamma d\beta d\gamma Y^2_{i+1}\delta. \end{aligned}$$

Therefore,

$$\hat{E}\left[\sum_{i=0}^{n-1} J_{\delta,1}^i\right] - \sum_{i=0}^{n-1}\left(\hat{E}[|J_{\delta,2}^i|] + \hat{E}[|I_\delta^i|]\right) \le \hat{E}[V(1,\sqrt{\delta}T_n)] - V(0,0)$$
$$\le \hat{E}\left[\sum_{i=0}^{n-1} J_{\delta,1}^i\right] + \sum_{i=0}^{n-1}\left(\hat{E}[|J_{\delta,2}^i|] + \hat{E}[|I_\delta^i|]\right). \tag{3.19}$$

For $I_\delta^i$, since both $\partial_t V$ and $\partial_{xx}^2 V$ are uniformly $\frac{\alpha}{2}$-hölder continuous in $t$ and $\alpha$-hölder continuous in $x$ on $[0,1]\times\mathbb{R}$, then we have

$$|I_\delta^i| \le C\delta^{1+\frac{\alpha}{2}}(1+|Y_{i+1}|^\alpha + |Y_{i+1}|^{2+\alpha}).$$

From (B2), we have

$$\sum_{i=0}^{n-1}\hat{E}\left[|I_\delta^i|\right] \le C\left(\frac{1}{n}\right)^{1+\frac{\alpha}{2}}\sum_{i=0}^{n-1}\left(1+\hat{E}[|Y_{i+1}|^\alpha] + \hat{E}[|Y_{i+1}|^{2+\alpha}]\right) \to 0 \quad \text{as} \quad n\to\infty.$$

Thus

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\hat{E}\left[|I_\delta^i|\right] = 0. \tag{3.20}$$

For $J_{\delta,1}^i$, from the conditions (i) and (ii) we have

$$\hat{E}\left[\partial_x V(i\delta,\sqrt{\delta}T_i)X_{i+1}\sqrt{\delta}\right] = \hat{E}\left[-\partial_x V(i\delta,\sqrt{\delta}T_i)X_{i+1}\sqrt{\delta}\right] = 0.$$

We then combine the above equality with (3.17) as well as the condition (i), it follows that

$$\begin{aligned}
\hat{E}[J_{\delta,1}^i] &= \hat{E}\left[\partial_t V(i\delta,\sqrt{\delta}T_i)\delta + \frac{1}{2}\partial_{xx}^2 V(i\delta,\sqrt{\delta}T_i)X_{i+1}^2\delta\right] \\
&= \hat{E}\Big\{\partial_t V(i\delta,\sqrt{\delta}T_i)\delta + \frac{\delta}{2}\left[(\partial_{xx}^2 V(i\delta,\sqrt{\delta}T_i))^+\overline{\sigma}_{i+1}^2 - (\partial_{xx}^2 V(i\delta,\sqrt{\delta}T_i))^-\underline{\sigma}_{i+1}^2\right]\Big\} \\
&\le \hat{E}\Big\{\partial_t V(i\delta,\sqrt{\delta}T_i)\delta + \frac{\delta}{2}\left[(\partial_{xx}^2 V(i\delta,\sqrt{\delta}T_i))^+\overline{\sigma}^2 - (\partial_{xx}^2 V(i\delta,\sqrt{\delta}T_i))^-\underline{\sigma}^2\right]\Big\} \\
&\quad + \frac{\delta}{2}\hat{E}\left[(\partial_{xx}^2 V(i\delta,\sqrt{\delta}T_i))^+(\overline{\sigma}_{i+1}^2 - \overline{\sigma}^2) - (\partial_{xx}^2 V(i\delta,\sqrt{\delta}T_i))^-(\underline{\sigma}_{i+1}^2 - \underline{\sigma}^2)\right] \\
&= \frac{\delta}{2}\hat{E}\left[(\partial_{xx}^2 V(i\delta,\sqrt{\delta}T_i))^+(\overline{\sigma}_{i+1}^2 - \overline{\sigma}^2) - (\partial_{xx}^2 V(i\delta,\sqrt{\delta}T_i))^-(\underline{\sigma}_{i+1}^2 - \underline{\sigma}^2)\right] \\
&\le \frac{\delta}{2}\hat{E}\left[\left|\partial_{xx}^2 V(i\delta,\sqrt{\delta}T_i)\right|\right]\left(\left|\overline{\sigma}_{i+1}^2 - \overline{\sigma}^2\right| + \left|\underline{\sigma}_{i+1}^2 - \underline{\sigma}^2\right|\right).
\end{aligned}$$

Since $\partial_{xx}^2 V$ is uniformly $\alpha$-hölder continuous in $x$ and $\frac{\alpha}{2}$-hölder continuous in $t$ on $[0,1]\times\mathbb{R}$, it follows that

$$\hat{E}\left[\left|\partial_{xx}^2 V\left(i\delta,\sqrt{\delta}T_i\right)\right|\right] \le \left|\partial_{xx}^2 V(0,0)\right| + \hat{E}\left[\left|\partial_{xx}^2 V\left(i\delta,\sqrt{\delta}T_i\right) - \partial_{xx}^2 V(0,0)\right|\right]$$

$$\leq C\left(1+|i\delta|^{\frac{\alpha}{2}}+\hat{E}\left[\left|\sqrt{\delta}T_i\right|^{\alpha}\right]\right)$$
$$\leq C\left(1+|i\delta|^{\frac{\alpha}{2}}+\left(\hat{E}\left[\left|\sqrt{\delta}T_i\right|^{2}\right]\right)^{\frac{\alpha}{2}}\right)\leq C$$

by Hölder's inequality and (B3). So we have

$$\hat{E}\left[J^i_{\delta,1}\right]\leq C\delta\left(\left|\overline{\sigma}^2_{i+1}-\overline{\sigma}^2\right|+\left|\underline{\sigma}^2_{i+1}-\underline{\sigma}^2\right|\right).$$

By the condition (iii), it follows that

$$\hat{E}\left[\sum_{i=0}^{n-1}J^i_{\delta,1}\right]\leq\sum_{i=0}^{n-1}\hat{E}\left[J^i_{\delta,1}\right]\leq C\frac{1}{n}\sum_{i=0}^{n-1}\left(\left|\overline{\sigma}^2_{i+1}-\overline{\sigma}^2\right|+\left|\underline{\sigma}^2_{i+1}-\underline{\sigma}^2\right|\right)\to 0\quad\text{as}\quad n\to\infty.$$

Similarly,

$$\hat{E}\left[\sum_{i=0}^{n-1}J^i_{\delta,1}\right]\geq -C\frac{1}{n}\sum_{i=0}^{n-1}\left(\left|\overline{\sigma}^2_{i+1}-\overline{\sigma}^2\right|+\left|\underline{\sigma}^2_{i+1}-\underline{\sigma}^2\right|\right)\to 0\quad\text{as}\quad n\to\infty.$$

Thus

$$\lim_{n\to\infty}\hat{E}\left[\sum_{i=0}^{n-1}J^i_{\delta,1}\right]=0. \tag{3.21}$$

For $J^i_{\delta,2}$, in a similar manner as above, we have

$$\hat{E}\left[\left|\partial_x V(i\delta,\sqrt{\delta}T_i)\right|\right]\leq C.$$

By the conditions (i), (iv), (B1) and Stolz theorem,

$$\begin{aligned}\sum_{i=0}^{n-1}\hat{E}\left[|J^i_{\delta,2}|\right]\leq&\sum_{i=0}^{n-1}\left\{\frac{1}{2}\hat{E}\left[\left|\partial^2_{xx}V(i\delta,\sqrt{\delta}T_i)\right|\right]\hat{E}\left[\left|X^2_{i+1}-Y^2_{i+1}\right|\right]\delta\right.\\&\left.+\hat{E}\left[\left|\partial_x V(i\delta,\sqrt{\delta}T_i)\right|\right]\hat{E}\left[|X_{i+1}-Y_{i+1}|\right]\sqrt{\delta}\right\}\\\leq& C\frac{1}{n}\sum_{i=0}^{n-1}\hat{E}\left[\left(X^2_{i+1}-(i+1)\right)^+\right]+C\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\hat{E}\left[|X_{i+1}-Y_{i+1}|\right]\\\leq& C\frac{1}{n}\sum_{i=0}^{n-1}\sup_{k\geq 1}\hat{E}\left[\left(X^2_{k}-(i+1)\right)^+\right]+C\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\hat{E}[|X_i-Y_i|]\to 0\\&\text{as}\quad n\to\infty.\end{aligned}$$

Then combining (3.19), (3.20) and (3.21), it follows that

$$\lim_{n\to\infty}\hat{E}\left[V\left(1,\sqrt{\delta}T_n\right)\right]=V(0,0). \tag{3.22}$$

Additionally, it is obvious that if $\varphi\in C_{b.Lip}(\mathbb{R})$, i.e., $|\varphi(x)-\varphi(y)|\leq C|x-y|$, then for each $t,s\in[0,1+h]$ and $x\in\mathbb{R}$,

$$|V(t,x)-V(s,x)|\leq C\widetilde{E}[|\xi|]\sqrt{|t-s|}\leq C\sqrt{|t-s|}. \tag{3.23}$$

In particular,

$$|V(0,0) - V(h,0)| \le C\sqrt{h}. \tag{3.24}$$

Combining (3.18), (3.23), with (3.24), we have

$$\begin{aligned}
&\left|\hat{E}\left[\varphi\left(\sqrt{\delta}T_n\right)\right] - \widetilde{E}[\varphi(\xi)]\right| = \left|\hat{E}\left[V\left(1+h, \sqrt{\delta}T_n\right)\right] - V(h,0)\right| \\
&\le \left|\hat{E}\left[V\left(1+h, \sqrt{\delta}T_n\right)\right] - \hat{E}\left[V\left(1, \sqrt{\delta}T_n\right)\right]\right| \\
&+ \left|\hat{E}\left[V\left(1, \sqrt{\delta}T_n\right)\right] - V(0,0)\right| + |V(0,0) - V(h,0)| \\
&\le 2C\sqrt{h} + \left|\hat{E}\left[V\left(1, \sqrt{\delta}T_n\right)\right] - V(0,0)\right|.
\end{aligned}$$

From (3.22), we obtain

$$\limsup_{n\to\infty}\left|\hat{E}\left[\varphi\left(\sqrt{\delta}T_n\right)\right] - \widetilde{E}[\varphi(\xi)]\right| \le 2C\sqrt{h}.$$

So (3.16) is proved.

By the Lipschitz continuity of $\varphi$ and (B1), we have

$$\left|\hat{E}\left[\varphi\left(\frac{S_n}{\sqrt{n}}\right)\right] - \hat{E}\left[\varphi\left(\frac{T_n}{\sqrt{n}}\right)\right]\right| \le C\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\hat{E}\left[|X_i - Y_i|\right] \to 0 \quad \text{as} \quad n\to\infty.$$

Thus

$$\lim_{n\to\infty}\hat{E}\left[\varphi\left(\frac{S_n}{\sqrt{n}}\right)\right] = \widetilde{E}[\varphi(\xi)], \qquad \forall \varphi \in C_{b.Lip}(\mathbb{R}).$$

The rest of the proof is very similar to that of Theorem 3.5 in [19] and so it is omitted. □

**Theorem 3.4.** *Under the conditions of Theorem* 3.3*, then if $y$ is a point at which $\widetilde{v}$ is continuous, we have*

$$\lim_{n\to\infty} v\left(\frac{S_n}{\sqrt{n}} \le y\right) = \widetilde{v}(y), \tag{3.25}$$

*and if $y$ is a point at which $\widetilde{\mathbb{V}}$ is continuous, we have*

$$\lim_{n\to\infty} \mathbb{V}\left(\frac{S_n}{\sqrt{n}} \le y\right) = \widetilde{\mathbb{V}}(y), \tag{3.26}$$

*where* $\widetilde{v}(y) = \inf_{\theta\in\Theta} E_P\left[I_{\left\{\int_0^1 \theta_s dB_s \le y\right\}}\right]$, $\widetilde{\mathbb{V}}(y) = \sup_{\theta\in\Theta} E_P\left[I_{\left\{\int_0^1 \theta_s dB_s \le y\right\}}\right]$,

$\{B_t\}_{t\ge 0}$ *is a* $1-$*dimensional Brownian motion under probability measure* $P$,

$\Theta := \{\{\theta_t\}_{t\ge0} : \theta_t$ *is* $\mathcal{F}_t$*-adapted process such that* $\underline{\sigma} \le \theta_t \le \overline{\sigma}\}$,

$\mathcal{F}_t := \sigma\{B_s : 0 \le s \le t\} \vee \mathcal{N}$, $\mathcal{N}$ *is the collection of* $P$*-null subsets.*

The proof is similar to the proof of Theorem 3.1 in [12] and so it is omitted.

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