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SOME REMARKS ON THE SOLUTIONS OF A SECOND-ORDER EVOLUTION INCLUSION

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ABSTRACT: We prove the Lipschitz dependence on the initial data of the solution set of a Cauchy problem associated to a second-order evolution inclusion by using the contraction principle in the space of selections of the multifunction instead of the space of solutions. A Filippov type existence theorem for this problem is also provided.

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1. INTRODUCTION

This paper is concerned with the following problem

$$x''(t) \in A(t)x(t) + F(t, x(t)), \quad x(0) = x_0, \quad x'(0) = y_0, \tag{1.1}$$

where $F : [0, T] \times X \to \mathcal{P}(X)$ is a set-valued map, X is a separable Banach space, $x_0, y_0 \in X$ and $\{A(t)\}_{t\geq 0}$ is a family of linear closed operators from X into X that genearates an evolution system of operators $\{\mathcal{U}(t, s)\}_{t,s\in[0,T]}$. The general framework of evolution operators $\{A(t)\}_{t\geq 0}$ that define problem (1.1) has been developed by Kozak ([13]) and improved by Henriquez ([10]).

The present paper is motivated by several recent papers ([1, 2, 3, 4, 10, 11]) where existence results and qualitative properties of solutions for problem (1.1) have been obtained by using fixed point techniques.

In the present paper we study the properties of the map that associates to given initial conditions the set of mild solutions of problem (1.1) and the main purpose is to prove that this solution map depends Lipschitz-continuously on the initial conditions. Our approach is based on an idea of Tallos ([12, 15]) applying the set-valued contraction principle in the space of selections of the multifunction instead of the space of solutions as usual. This approach allows us to obtain also a Filippov type existence result for mild solutions of problem (1.1). Recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem consists in proving the existence of a solution starting from a given "quasi" solution. Moreover, the result provides an estimate between the "quasi" solution and the solution obtained.

The results in this paper may be interpreted as extensions of similar results obtained for other classes of second order differential inclusions ([6, 7]) to the more general problem (1.1).

The paper is organized as follows: in Section 2 we recall some preliminary results that we use in the sequel and in Section 3 we prove our main results.

2. PRELIMINARIES

Let denote by I the interval $[0,T]$, $T>0$ and let X be a real separable Banach space with the norm $|.|$ and with the corresponding metric $d(.,.)$. As usual, we denote by $C(I,X)$ the Banach space of all continuous functions $x(.):I\to X$ endowed with the norm $|x(.)|_C=\sup_{t\in I}|x(t)|$ and by $L^1(I,X)$ the Banach space of all (Bochner) integrable functions $x(.):I\to X$ endowed with the norm $|x(.)|_1=\int_0^T|x(t)|\mathrm{d}t$. With $B(X)$ we denote the Banach space of linear bounded operators on X.

In what follows $\{A(t)\}_{t\geq 0}$ is a family of linear closed operators from X into X that genearates an evolution system of operators $\{\mathcal{U}(t,s)\}_{t,s\in I}$. By hypothesis the domain of $A(t)$, $D(A(t))$ is dense in X and is independent of t.

Definition 1. ([10, 13]) A family of bounded linear operators $\mathcal{U}(t,s):X\to X$, $(t,s)\in\Delta:=\{(t,s)\in I\times I;s\leq t\}$ is called an evolution operator of the equation

$$x''(t)=A(t)x(t) \tag{2.1}$$

if

i) For any $x\in X$, the map $(t,s)\to\mathcal{U}(t,s)x$ is continuously differentiable and:

a) $\mathcal{U}(t,t)=0$, $t\in I$.

b) If $t\in I, x\in X$ then $\frac{\partial}{\partial t}\mathcal{U}(t,s)x|_{t=s}=x$ and $\frac{\partial}{\partial s}\mathcal{U}(t,s)x|_{t=s}=-x$.

ii) If $(t,s)\in\Delta$, then $\frac{\partial}{\partial s}\mathcal{U}(t,s)x\in D(A(t))$, the map $(t,s)\to\mathcal{U}(t,s)x$ is of class C^2 and:

a) $\frac{\partial^2}{\partial t^2}\mathcal{U}(t,s)x \equiv A(t)\mathcal{U}(t,s)x$.

b) $\frac{\partial^2}{\partial s^2}\mathcal{U}(t,s)x \equiv \mathcal{U}(t,s)A(t)x$.

c) $\frac{\partial^2}{\partial s\partial t}\mathcal{U}(t,s)x|_{t=s} = 0$.

iii) If $(t,s) \in \Delta$, then there exist $\frac{\partial^3}{\partial t^2\partial s}\mathcal{U}(t,s)x$, $\frac{\partial^3}{\partial s^2\partial t}\mathcal{U}(t,s)x$ and:

a) $\frac{\partial^3}{\partial t^2\partial s}\mathcal{U}(t,s)x \equiv A(t)\frac{\partial}{\partial s}\mathcal{U}(t,s)x$ and the map $(t,s) \to A(t)\cdot\frac{\partial}{\partial s}\mathcal{U}(t,s)x$ is continuous.

b) $\frac{\partial^3}{\partial s^2\partial t}\mathcal{U}(t,s)x \equiv \frac{\partial}{\partial t}\mathcal{U}(t,s)A(s)x$.

As an example for equation (2.1) one may consider the problem (e.g., [11])

$$\frac{\partial^2 z}{\partial t^2}(t,\tau) = \frac{\partial^2 z}{\partial \tau^2}(t,\tau) + a(t)\frac{\partial z}{\partial t}(t,\tau), \quad t \in [0,T], \tau \in [0,2\pi],$$

$$z(t,0) = z(t,\pi) = 0, \quad \frac{\partial z}{\partial \tau}(t,0) = \frac{\partial z}{\partial \tau}(t,2\pi),\ t \in [0,T],$$

where $a(.) : I \to \mathbf{R}$ is a continuous function. This problem is modeled in the space $X = L^2(\mathbf{R},\mathbf{C})$ of 2π-periodic 2-integrable functions from $\mathbf{R}$ to $\mathbf{C}$, $A_1 z = \frac{d^2 z(\tau)}{d\tau^2}$ with domain $H^2(\mathbf{R},\mathbf{C})$ the Sobolev space of 2π-periodic functions whose derivatives belong to $L^2(\mathbf{R},\mathbf{C})$. It is well known that$A_1$ is the infinitesimal generator of strongly continuous cosine functions $C(t)$ on X. Moreover, A_1 has discrete spectrum; namely the spectrum of A_1 consists of eigenvalues $-n^2$, $n \in \mathbf{Z}$ with associated eigenvectors $z_n(\tau) = \frac{1}{\sqrt{2\pi}}e^{in\tau}$, $n \in \mathbf{N}$. The set z_n, $n \in \mathbf{N}$ is an orthonormal basis of X. In particular, $A_1 z = \sum_{n\in\mathbf{Z}} -n^2 < z, z_n > z_n$, $z \in D(A_1)$. The cosine function is given by $C(t)z = \sum_{n\in\mathbf{Z}} \cos(nt) < z, z_n > z_n$ with the associated sine function $S(t)z = t < z, z_0 > z_0 + \sum_{n\in\mathbf{Z}^*} \frac{\sin(nt)}{n} < z, z_n > z_n$.

For $t \in I$ define the operator $A_2(t)z = a(t)\frac{dz(\tau)}{d\tau}$ with domain $D(A_2(t)) = H^1(\mathbf{R},\mathbf{C})$. Set $A(t) = A_1 + A_2(t)$. It has been proved in [10] that this family generates an evolution operator as in Definition 1.

Definition 2. A continuous mapping $x(.) \in C(I,X)$ is called a mild solution of problem (1.1) if there exists a (Bochner) integrable function $f(.) \in L^1(I,X)$ such that

$$f(t) \in F(t,x(t)) \quad a.e.\,(I), \tag{2.2}$$

$$x(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)y_0 + \int_0^t \mathcal{U}(t,s)f(s)ds,\ t \in I. \tag{2.3}$$

We shall call $(x(.), f(.))$ a *trajectory-selection pair* of (1.1) if $f(.)$ verifies (2.2) and $x(.)$ is defined by (2.3).

We shall use the following notations for the solution sets of (1.1).

$$\mathcal{S}(x_0,y_0) = \{(x(.),f(.));\ (x(.),f(.)) \text{ is a trajectory-selection pair of } (1.1)\}, \tag{2.4}$$

$$\mathcal{S}_1(x_0, y_0) = \{x(.); \quad x(.) \text{ is a mild solution of } (1.1)\}. \tag{2.5}$$

In the sequel the following conditions are satisfied.

Hypothesis H1. i) There exists an evolution operator $\{\mathcal{U}(t,s)\}_{t,s\in I}$ associated to the family $\{A(t)\}_{t\geq 0}$.

ii) There exist $M, M_0 \geq 0$ such that $|\mathcal{U}(t,s)|_{B(X)} \leq M$ and $|\frac{\partial}{\partial s}\mathcal{U}(t,s)| \leq M_0$, for all $(t,s) \in \Delta$.

iii) $F(.,.) : I \times X \to \mathcal{P}(X)$ has nonempty closed values and for every $x \in X$, $F(.,x)$ is measurable.

iv) There exists $L(.) \in L^1(I, \mathbf{R}_+)$ such that for almost all $t \in I, F(t,.)$ is $L(t)$-Lipschitz in the sense that

$$d_H(F(t,x), F(t,y)) \leq L(t)|x-y| \quad \forall\ x,y \in X,$$

here $d_H(A,B)$ is the Hausdorff distance between $A, B \subset X$

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \quad d^*(A,B) = \sup\{d(a,B);\\ a \in A\}.$$

v) $d(0, F(t,0)) \leq L(t)$ $\quad$ *a.e.* (I)

Let $m(t) = \int_0^t L(u)du$ and for given $\alpha \in \mathbf{R}$ we consider on $L^1(I,X)$ the following norm

$$|f|_1 = \int_0^T e^{-\alpha m(t)}|f(t)|dt, \quad f \in L^1(I,X),$$

which is equivalent with the usual norm on $L^1(I,X)$.

Consider the following norm on $C(I,X) \times L^1(I,X)$

$$|(x,f)|_{C\times L} = |x|_C + |f|_1 \quad \forall\ (x,f) \in C(I,X) \times L^1(I,X).$$

Finally we recall some basic results concerning set valued contractions that we shall use in the sequel.

Let (Z,d) be a metric space and consider a set valued map T on Z with nonempty closed values in Z. T is said to be a λ-contraction if there exists $0 < \lambda < 1$ such that:

$$d_H(T(x), T(y)) \leq \lambda d(x,y) \quad \forall x,y \in Z$$

If Z is complete, then every set valued contraction has a fixed point, i.e. a point $z \in Z$ such that $z \in T(z)$ ([8]).

We denote by $Fix(T)$ the set of all fixed point of the multifunction T. Obviously, $Fix(T)$ is closed.

Theorem 3. *([14]) Let Z be a complete metric space and suppose that T_1, T_2 are λ-contractions with closed values in Z. Then*

$$d_H(Fix(T_1), Fix(T_2)) \leq \frac{1}{1-\lambda} \sup_{z\in Z} d_H(T_1(z), T_2(z)).$$

3. THE MAIN RESULTS

We are ready now to show that the set of all trajectory-selection pairs of (1.1) depends Lipschitz-continuously on the initial condition.

Theorem 4. *Let Hypothesis H1 be satisfied and let $\alpha > M$.*

Then the map $(x_0, y_0) \to \mathcal{S}(x_0, y_0)$ is Lipschitz-continuous on $X \times X$ with nonempty closed values in $C(I, X) \times L^1(I, X)$.

Proof. Let us consider $x_0, y_0 \in X$, $f(.) \in L^1(I, X)$ and define the following set valued maps

$$M_{x_0,y_0,f}(t) = F(t, -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)y_0 + \int_0^t \mathcal{U}(t,s)f(s)ds), t \in I \tag{3.1}$$

$$T_{x_0,y_0}(f) = \{\phi(.) \in L^1(I, X); \quad \phi(t) \in M_{x_0,y_0,f}(t) \quad a.e.\ (I)\}. \tag{3.2}$$

We shall prove first that $T_{x_0,y_0}(f)$ is nonempty and closed for every $f \in L^1(I, X)$. The fact that that the set valued map $M_{x_0,y_0,f}(.)$ is measurable is known. For example, the map $t \to -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)y_0 + \int_0^t \mathcal{U}(t,s)f(s)ds$ can be approximated by step functions and we can apply Theorem III. 40 in [5]. Since the values of F are closed and X is separable with the measurable selection theorem (Theorem III.6 in [5]) we infer that $M_{x_0,y_0,f}(.)$ admits a measurable selection ϕ. According to Hypothesis H1 one has

$$\begin{aligned}|\phi(t)| \leq d(0, F(t,0)) + d_H(F(t,0), F(t,x(t))) \leq L(t)(1 + |x(t)|)\\ \leq L(t)(1 + M_0|x_0| + M|y_0| + \int_0^t M|f(s)|ds).\end{aligned}$$

Thus integrating by parts we obtain

$$\begin{aligned}\int_0^T e^{-\alpha m(t)}|\phi(t)|dt \leq \int_0^T e^{-\alpha m(t)} L(t)(1 + M_0|x_0| + M|y_0| +\\ \int_0^t M|f(s)|ds)dt \leq \frac{1 + M_0|x_0|}{\alpha} + \frac{M|y_0|}{\alpha} + \frac{M|f|_1}{\alpha}.\end{aligned}$$

Hence, if $\phi(.)$ is a measurable selection of $M_{x_0,y_0,f}(.)$, then $\phi(.) \in L^1(I, X)$ and thus $T_{x_0,y_0}(f) \neq \emptyset$.

The set $T_{x_0,y_0}(f)$ is closed. Indeed, if $\phi_n \in T_{x_0,y_0}(f)$ and $|\phi_n - \phi|_1 \to 0$ then we can pass to a subsequence ϕ_{n_k} such that $\phi_{n_k}(t) \to \phi(t)$ for a.e. $t \in I$, and we find that $\phi \in T_{x_0,y_0}(f)$.

The next step of the proof will show that $T_{x_0,y_0}(.)$ is a contraction on $L^1(I, X)$.

Let $f, g \in L^1(I, X)$ be given, $\phi \in T_{x_0,y_0}(f)$ and let $\varepsilon > 0$. Consider the following set valued map

$$G(t) = M_{x_0,y_0,g}(t) \cap \{x \in X;\ |\phi(t) - x| \leq L(t)|\int_0^t \mathcal{U}(t,s)(f(s) - g(s))ds| + \varepsilon\}.$$

Since

$$\begin{aligned} d(\phi(t), M_{x_0,y_0,g}(t)) \leq & d(F(t, -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)y_0 \\ & + \int_0^t \mathcal{U}(t,s)f(s)ds), F(t, -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)y_0 \\ & + \int_0^t \mathcal{U}(t,s)g(s)ds)) \leq L(t)|\int_0^t \mathcal{U}(t,s)(f(s) - g(s))ds| \end{aligned}$$

we deduce that $G(.)$ has nonempty closed values. Moreover, according to Proposition III.4 in [5], $G(.)$ is measurable. Let $\psi(.)$ be a measurable selection of $G(.)$. It follows that $\psi \in T_{x_0,y_0}(g)$ and

$$\begin{aligned} |\phi - \psi|_1 = \int_0^T e^{-\alpha m(t)}|\phi(t) - \psi(t)|dt \leq \int_0^T e^{-\alpha m(t)} L(t)(\int_0^t M|f(s) \\ - g(s)|ds)dt + \int_0^T \varepsilon e^{-\alpha m(t)}dt \leq \frac{M}{\alpha}|f - g|_1 + \varepsilon \int_0^T e^{-\alpha m(t)}dt. \end{aligned}$$

Since ε was arbitrary, we deduce that

$$d(\phi, T_{x_0,y_0}(g)) \leq \frac{M}{\alpha}|f - g|_1.$$

Replacing f by g we obtain

$$d(T_{x_0,y_0}(f), T_{x_0,y_0}(g)) \leq \frac{M}{\alpha}|f - g|_1,$$

hence $T_{x_0,y_0}(.)$ is a contraction on $L^1(I, X)$.

Consequently $T_{x_0,y_0}(.)$ admits a fixed point $f(.) \in L^1(I, X)$. We define $x(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)y_0 + \int_0^t \mathcal{U}(t,s)f(s)ds$.

We have that $\mathcal{S}(x_0, y_0) \subset C(I, X) \times L^1(I, X)$ is a closed subset. Let $(x_n, f_n) \in \mathcal{S}(x_0, y_0)$, $|(x_n, f_n) - (x, f)|_{C\times L} \to 0$. Thus, $f_n \in Fix(T_{x_0,y_0})$, which is a closed set, and thus $f(.) \in Fix(T_{x_0,y_0})$. Set $y(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)y_0 + \int_0^t \mathcal{U}(t,s)f(s)ds$ and we prove that $y(.) = x(.)$. One may write

$$|y - x_n|_C = \sup_{t\in I}|y(t) - x_n(t)| \leq \sup_{t\in I} M\int_0^t |f_n(u) - f(u)|du \leq Me^{\alpha m(T)}|f_n - f|_1$$

and finally we get that $y(.) = x(.)$.

We prove next the following inequality

$$d_H(T_{x_1,y_1}(f), T_{x_2,y_2}(f)) \leq \frac{1}{\alpha}(M_0|x_1 - x_2| + M|y_1 - y_2|) \tag{3.3}$$

$\forall f \in L^1(I,X), x_1, x_2, y_1, y_2 \in X$. Let us consider the set-valued map

$$G_1(t) = M_{x_1,x_2,f}(t) \cap \{z \in X;\ |\phi(t) - z| \le L(t)(|\frac{\partial}{\partial s}\mathcal{U}(t,0)||x_1 - x_2| + |\mathcal{U}(t,0)||y_1 - y_2| + \varepsilon\},$$

$t \in I$, where $\phi(.)$ is a measurable selection of $M_{x_1,y_1,f}(.)$ and $\varepsilon > 0$.

With the same arguments used for the set valued map $G(.)$, we deduce that $G_1(.)$ is measurable with nonempty closed values. Let $\psi(.)$ be a measurable selection of $G_1(.)$. It follows that $\psi(.) \in T_{x_2,y_2}(f)$ and

$$\begin{aligned} |\phi - \psi|_1 = &\int_0^T e^{-\alpha m(t)} |\phi(t) - \psi(t)| dt \le \int_0^T e^{-\alpha m(t)} L(t)(|\frac{\partial}{\partial s}\mathcal{U}(t,0)| \\ &\cdot |x_1 - x_2| + |\mathcal{U}(t,0)||y_1 - y_2|)dt + \varepsilon \int_0^T e^{-\alpha m(t)} dt \le \frac{M_0}{\alpha}|x_1 - x_2| \\ &+ \frac{M}{\alpha}|y_1 - y_2| + \varepsilon \int_0^T e^{-\alpha m(t)} dt. \end{aligned}$$

Since ε was arbitrary, we deduce that

$$d(\phi, T_{x_2,y_2}(f)) \le \frac{1}{\alpha}(M_0|x_1 - x_2| + M|y_1 - y_2|).$$

Replacing (x_1, y_1) by (x_2, y_2) we obtain (3.3).

From (3.3) and Theorem 3 we obtain

$$d_H(Fix(T_{x_1,y_1}), Fix(T_{x_2,y_2})) \le \frac{1}{\alpha - M}(M_0|x_1 - x_2| + M|y_1 - y_2|).$$

Let $x_1, x_2, y_1, y_2 \in X$ and $(x(.), f(.)) \in \mathcal{S}(x_1, y_1)$. In particular, $f(.) \in Fix(T_{x_1,y_1})$ and thus, for every $\varepsilon > 0$ there exists $g(.) \in Fix(T_{x_2,y_2})$ such that

$$|f - g|_1 \le \frac{1}{\alpha - M}(M_0|x_1 - x_2| + M|y_1 - y_2|) + \varepsilon. \tag{3.4}$$

Put $z(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)y_0 + \int_0^t \mathcal{U}(t,s)g(s)ds$. One has

$$\begin{aligned} |x - z|_C = &\sup_{t \in I} |x(t) - z(t)| \le M_0|x_1 - x_2| + M|y_1 - y_2| \\ &+ \sup_{t \in I} \int_0^t M|f(s) - g(s)|ds \le M_0|x_1 - x_2| + M|y_1 - y_2| \\ &+ MTe^{\alpha m(t)}|f - g|_1 \le (1 + \frac{Me^{\alpha m(t)}}{\alpha - M})(M_0|x_1 - x_2| + M|y_1 - y_2|) \\ &+ \frac{Me^{\alpha m(t)}}{\alpha - M}\varepsilon. \end{aligned}$$

If we denote $k = \max\{M_0 + \frac{MM_0e^{\alpha m(T)}}{\alpha - M}, M + \frac{M^2e^{\alpha m(T)}}{\alpha - M}\}$ we deduce first that

$$d((x,f), \mathcal{S}(x_2, y_2)) \le k[|x_1 - x_2| + |y_1 - y_2|]$$

and by interchanging (x_1,y_1) and (x_2,y_2) we obtain

$$d_H(\mathcal{S}(x_1,y_1),\mathcal{S}(x_2,y_2)) \le k[|x_1-x_2|+|y_1-y_2|]$$

and the proof is complete. □

Obviously, from Theorem 4 we also obtain

Corollary 5. *Let Hypothesis H1 be satisfied and let $\alpha > M$. Then the map $(x_0,y_0) \to \mathcal{S}_1(x_0,y_0)$ is Lipschitz continuous on $X \times X$ with nonempty values in $C(I,X)$.*

In general, under the hypothesis of Theorem 4 the set $\mathcal{S}_1(x_0,y_0)$ is not closed in $C(I,X)$. The next result shows that if X is reflexive and the multifunction $F(.,.)$ is convex valued and integrably bounded then $\mathcal{S}_1(x_0,y_0) \subset C(I,X)$ is closed.

Let B be the closed unit ball in X.

Theorem 6. *Assume that X is reflexive, $\alpha > M$ and let $F(.,.) : I \times X \to \mathcal{P}(X)$ be a convex valued set valued map that satisfies Hypothesis H1. Assume that there exists $k(.) \in L^1(I,X)$ such that for almost all $t \in I$ and for all $x \in X$, $F(t,x) \subset k(t)B$.*

Then for every $x_0, y_0 \in X$, the set $\mathcal{S}_1(x_0,y_0) \subset C(I,X)$ is closed.

Proof. Let $x_n(.) \in \mathcal{S}_1(x_0,y_0)$ such that $|x_n - x|_C \to 0$. There exists $h_n(.) \in L^1(I,X)$ such that $(x_n(.), h_n(.))$ is a trajectory-selection pair of (1.1) $\forall n \in N$. We define $f_n(t) = e^{-\alpha m(t)} h_n(t)$, $t \in I$.

The set valued map $F(.,.)$ being integrably bounded, we have that $f_n(.)$ is bounded in $L^1(I,X)$ and $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall E \subset I$, $\mu(E) < \delta$ $\;|\int_E f_n(s)ds| < \varepsilon$ uniformly with respect to n. Moreover, X is reflexive and so by the Dunford-Pettis criterion ([9]), taking a subsequence and keeping the same notations, we may assume that $f_n(.)$ converges weakly in $L^1(I,X)$ to some $f(.) \in L^1(I,X)$.

We recall that for convex subsets of a Banach space the strong closure coincides with the weak closure. We apply this result. Since $f_n(.)$ converges weakly in $L^1(I,X)$ to $f(.) \in L^1(I,X)$ for all $h \ge 0$, $f(.)$ belongs to the weak closure of the convex hull $co\{f_n(.)\}_{n\ge h}$ of the subset $\{f_n(.)\}_{n\ge h}$. It coincides with the strong closure of $co\{f_n(.)\}_{n\ge h}$. Hence there exist $\lambda_i^n > 0, i = n, \dots k(n)$ such that

$$\sum_{i=1}^{k(n)} \lambda_i^n = 1, \quad g_n(.) = \sum_{i=n}^{k(n)} \lambda_i^n f_i(.) \in co\{f_n(.)\}_{n\ge h}$$

and such that $g_n(.)$ converges strongly to $f(.)$ in $L^1(I,X)$. Let

$$l_n(.) = \sum_{i=n}^{k(n)} \lambda_i^n h_i(.)$$

Then there exists a subsequence $g_{n_j}(.)$ that converges to $f(.)$ almost everywhere. Hence, $l_{n_j}(.)$ converges almost everywhere to $l(.) = e^{\alpha m(.)}f(.) \in L^1(I, X)$. Hence using the Lebesque dominated convergence theorem, for every $t \in I$ we obtain

$$\lim_{j\to\infty} \int_0^t \mathcal{U}(t,s)l_{n_j}(s)ds = \int_0^t \mathcal{U}(t,s)l(s)ds$$

We define

$$y(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)y_0 + \int_0^t \mathcal{U}(t,s)l(s)ds, \quad t \in I$$

and observe that

$$|x(t) - y(t)| \le |x(.) - x_{n_j}(.)|_C + |\int_0^t \mathcal{U}(t,s)l_{n_j}(s)ds - \int_0^t \mathcal{U}(t,s)l(s)ds|$$

which yields $x(t) = y(t)\ \forall t \in I$.

Let us observe now that for almost every $t \in I$

$$l_{n_j}(t) \in \sum_{i=n_j}^{k(n_j)} \lambda_i^{n_j} F(t, x_i(t)) \subset F(t, x(t)) + L(t) \sum_{i=n_j}^{k(n_j)} \lambda_i^{n_j} |x(t) - x_i(t)|B.$$

Since $\lim_{i\to\infty} |x(t)-x_i(t)| = 0$, we deduce that $f(t) \in F(t, x(t))$ *a.e.*(I) and the proof is complete. □

Using the same idea as in the proof of Theorem 4 one may obtain a Filippov type existence result for problem (1.1).

Theorem 7. *Let Hypothesis H1 be satisfied and let $\alpha > M$ and let $y(.)$ be a mild solution of the problem*

$$x'' = A(t)x + g(t) \quad x(0) = x_1, \quad x'(0) = y_1,$$

where $g(.) \in L^1(I, X)$ and there exists $p(.) \in L^1(I, \mathbf{R})$ such that

$$d(g(t), F(t, y(t))) \le p(t), \quad a.e.\ (I).$$

Then for every $\varepsilon > 0$ there exists $x(.)$ a mild solution of (1.1) satisfying for all $t \in I$

$$|x(t) - y(t)| \le (M_0 + \frac{MM_0T}{\alpha - M}e^{\alpha m(t)})|x_0 - y_0| + (M+ \\ \frac{M^2}{\alpha - M}e^{\alpha m(t)})|x_1 - y_1| + \frac{\alpha M e^{\alpha m(t)}}{\alpha - M}\int_0^T e^{-\alpha m(s)}p(s)ds + \varepsilon. \quad (3.5)$$

Proof. We keep the same notations as in the proof of Theorem 4. Consider the following set-valued maps

$$\tilde{F}(t,x) = F(t,x) + p(t)B, \quad (t,x) \in I \times X,$$

$$\tilde{M}_{x_1,y_1,f}(t) = \tilde{F}(t, -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)y_0 + \int_0^t \mathcal{U}(t,s)f(s)ds), t \in I$$

$$\tilde{T}_{x_1,y_1}(f) = \{\phi(.) \in L^1(I,X); \quad \phi(t) \in \tilde{M}_{x_1,y_1,f}(t) \quad a.e.\ (I)\},$$

$f \in L^1(I,X)$, where B is the closed unit ball in X. Obviously, $\tilde{F}(.,.)$ satisfies Hypothesis H1.

As in the proof of Theorem 4 we obtain that $\tilde{T}_{x_1,y_1}(.)$ is also a $\frac{M}{\alpha}$-contraction on $L^1(I,X)$ with closed nonempty values.

We prove next the following estimate

$$d_H(T_{x_0,y_0}(f), \tilde{T}_{x_1,y_1}(f)) \leq \frac{M_0}{\alpha}|x_0 - x_1| + \frac{M}{\alpha}|y_0 - y_1| + \int_0^T e^{-\alpha m(t)}p(t)dt \tag{3.6}$$

$\forall\ f(.) \in L^1(I,X)$.

Let $\phi \in T_{x_0,y_0}(f)$, $\delta > 0$ and, for $t \in I$, define

$$G_1(t) = \tilde{M}_{x_1,y_1,f}(t) \ \cap\ \{z \in X;\ |\phi(t) - z| \leq L(t)(|\frac{\partial}{\partial s}\mathcal{U}(t,0)||x_1 - x_0| + |\mathcal{U}(t,0)||y_1 - y_0|) + p(t) + \delta\}.$$

With the same arguments used for the set-valued map $G(.)$ in the proof of Theorem 4, we deduce that $G_1(.)$ is measurable with nonempty closed values. Let $\psi(.)$ be a measurable selection of $G_1(.)$. It follows that $\psi(.) \in \tilde{T}_{y_0,y_1}(f)$ and one has

$$|\phi - \psi|_1 = \int_0^T e^{-\alpha m(t)}|\phi(t) - \psi(t)|dt \leq \int_0^T e^{-\alpha m(t)}[L(t)(|\frac{\partial}{\partial s}\mathcal{U}(t,0)||x_1 - x_2| + |\mathcal{U}(t,0)||y_1 - y_2|) + p(t) + \delta]dt \leq \frac{M_0}{\alpha}|x_0 - x_1| + \frac{M}{\alpha}|y_0 - y_1| + \int_0^T e^{-\alpha m(t)}p(t)dt + \delta \int_0^T e^{-\alpha m(t)}p(t)dt.$$

Since $\delta > 0$ was arbitrary, as above, we obtain (3.6). Applying Theorem 3 we get

$$d_H(Fix(T_{x_0,y_0}), Fix(\tilde{T}_{x_1,y_1})) \leq \frac{M_0}{\alpha - M}|x_0 - y_0| + \frac{M}{\alpha - M}|x_1 - y_1| + \frac{\alpha}{\alpha - M}\int_0^T e^{-\alpha m(t)}p(t)dt.$$

Since $g(.) \in Fix(\tilde{T}_{x_1,y_1})$ it follows that there exists $f(.) \in Fix(T_{x_0,y_0})$ such that for any $\varepsilon > 0$

$$|g-f|_1 \leq \frac{M_0}{\alpha - M}|x_0 - x_1| + \frac{M}{\alpha - M}|y_0 - y_1| + \frac{\alpha}{\alpha - M}\int_0^T e^{-\alpha m(t)}p(t)dt + \frac{\varepsilon}{Me^{\alpha m(T)}}. \quad (3.7)$$

We define $x(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)y_0 + \int_0^t \mathcal{U}(t,s)f(s)ds$, $t \in I$ and we have

$$|x(t) - y(t)| \leq M_0|x_0 - x_1| + M|y_0 - y_1| + Me^{\alpha m(t)}|f-g|_1.$$

Combining the last inequality with (3.7) we obtain (3.5). □

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