

CONTINUITY OF EIGENVALUES IN WEAK TOPOLOGY FOR REGULAR STURM-LIOUVILLE PROBLEMS

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ABSTRACT: This paper is concerned with the eigenvalue problems of Sturm-Liouville differential expressions with general separated boundary conditions. With the aid of properties of analytic functions, only under the standard integrability conditions, we obtain the continuity of eigenvalues in the weak topology of $L^1[a, b]$ on all the coefficient functions of the differential expressions.

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1. INTRODUCTION

Consider the regular Sturm-Liouville eigenvalue problem

$$-(p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x), \quad x \in (a, b), \tag{1}$$

with the boundary conditions

$$y(a) \cos \theta_1 - py'(a) \sin \theta_1 = 0, \quad y(b) \cos \theta_2 - py'(b) \sin \theta_2 = 0, \tag{2}$$

where λ is the spectral parameter, and $\theta_1, \theta_2 \in [0, \pi)$. The coefficient functions of equation (1), p, q, w are real-valued functions satisfying the standard integrability

conditions

$$\frac{1}{p}, q, w \in L^1[a, b], p(x) > 0, w(x) > 0 \text{ a.e. on } [a, b], \quad (3)$$

where $L^1[a, b]$ denotes the integrable function space on $[a, b]$.

By the theory of boundary value problem (cf., [4, 8]), it is known that the eigenvalue problem (1) and (2) has only discrete and real (algebraic) simple eigenvalues. These eigenvalues satisfy (cf., [17, Theorem 13.2])

$$\begin{aligned} -\infty < \lambda_1\left(\frac{1}{p}, q, w\right) < \cdots < \lambda_n\left(\frac{1}{p}, q, w\right) < \cdots, \\ \text{and } \lambda_n\left(\frac{1}{p}, q, w\right) \rightarrow \infty, \text{ as } n \rightarrow \infty, \end{aligned} \quad (4)$$

where $\lambda_n(1/p, q, w)$ is the n -th eigenvalue of (1) and (2).

Many papers (cf., [1, 7, 9, 22]) have studied the continuity of the n -th eigenvalue of (1) and (2) with respect to the boundary condition and coefficient functions $1/p, q, w$, in the norm topology of $L^1[a, b]$ (cf., [6, (1.14)]).

Recently, some papers study the strong continuity of the n -th eigenvalues of (1) and (2) in the weak topology of $L^1[a, b]$ (cf., [10, 13, 14]) on coefficient functions of the differential expressions. In [20, 12], the authors study (1) and (2) in the case $p \equiv w \equiv 1$, i.e.,

$$-y'' + qy = \lambda y, \text{ on } [0, 1], q \in L^1[0, 1],$$

with the Dirichlet or the Neumann boundary condition. By considering eigenvalues $\lambda_n(q)$ as functionals of potentials $q \in L^1[0, 1]$, based on the dependence results of solutions and the generalized Prüfer transformation, it has been proved in [12, Theorem 1.3] that $\lambda_n(q)$ are continuous when the weak topology for potentials are considered. Furthermore, [12, Theorem 1.4] obtains $\lambda_n(q)$ has continuously Fréchet differentiable in weak topology of $L^1[a, b]$.

Furthermore, in [21], the authors study (1) and (2) in the case $p \equiv 1$ and $q \equiv 0$, i.e.,

$$-y'' = \lambda wy, \text{ on } [0, 1], w \in L^1[0, 1],$$

with the Dirichlet condition. And consider the n -th eigenvalue as a functional of weight functions $\lambda_n(w)$. Also using the argument approach, just the same as above case, the authors prove that $\lambda_n(w)$ are continuous in the weak topology for weight functions w in [21, Theorem 4.1]. In this paper, we will consider the general case (1) and (2), and study the continuity of the n -th eigenvalue $\lambda_n(1/p, q, w)$ in the weak topology of $L^1[a, b]$ on all the coefficient functions jointly of the differential expressions. However, when the Prüfer transformation is applied to the general case (1) and (2), we find the properties of argument are so complicated that we can't get the conclusions we need as in [12, Section 4.1, p.2211], [21, (2.9)-(2.12)]. Hence for

the general case, another method must be used to get the continuity of $\lambda_n(1/p, q, w)$ with respect to $1/p, q, w$ in the weak topology of $L^1[a, b]$.

In the present paper, using the properties of analytic functions, we will prove that eigenvalues of (1) and (2) are continuous with respect to all the coefficient functions, $1/p, q, w$, in the weak topology of $L^1[a, b]$. The paper is organized as follows. In §2, we will first study the properties of the zero points of analytic functions when a family of analytic functions converges to an analytic function, see Lemma 2.2. Then in Lemma 2.5, under the condition that the eigenvalues are bounded below, we get the continuity of eigenvalues in the weak topology of $L^1[a, b]$ about all the coefficients functions of the differential equation (1). In §3, the main result will be given. Firstly, the lower bound of the first eigenvalue will be given in Lemma 3.1 and Lemma 3.3. Then, in Theorem 3.5, only under the standard integrability conditions (3), we will obtain the continuity of eigenvalues in the weak topology of $L^1[a, b]$ on all the coefficient functions of the differential expressions (1).

2. THE PROPERTIES OF ANALYTIC FUNCTIONS AND PRELIMINARIES

Some symbols and lemmas will be given in this section. The main work of this section is using the properties about the zero points of analytic functions to prove the eigenvalues' continuity with respect to the coefficient functions of problem (1) and (2), see Lemma 2.5.

In Lemma 2.2, we give the properties about the zero points of analytic functions. In the proof of this lemma, Montels Theorem will be used.

Proposition 2.1. (*Montels Theorem, see [15, p.225, Theorem 3.3]*) *Suppose \mathcal{F} is a family of holomorphic functions on Ω that is uniformly bounded on compact subsets of Ω , where Ω is an open subset of \mathbb{C} . Then,*

- (i) \mathcal{F} is equicontinuous on every compact subset of Ω .
- (ii) every sequence in \mathcal{F} has a subsequence that converges uniformly on every compact subset of Ω (the limit need not be in \mathcal{F}).

Here, see [15, p.225], the family \mathcal{F} is called to be uniformly bounded on compact subsets of Ω if for each compact set $K \subset \Omega$ there exists $B > 0$, such that $|f(z)| \leq B$ for all $z \in K$ and $f \in \mathcal{F}$. Also, the family \mathcal{F} is called to be equicontinuous on a compact set K if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $z_1, z_2 \in K$ and $|z_1 - z_2| < \delta$, then $|f(z_1) - f(z_2)| < \varepsilon$ for all $f \in \mathcal{F}$.

Lemma 2.2. *Let $F(\lambda) \not\equiv 0$, $F_n(\lambda)$, $n = 1, 2, \dots$, be analytic functions on \mathbb{C} . Suppose $\{|F_n(\lambda)|, n = 1, 2, \dots\}$ is uniformly bounded on compact subsets of \mathbb{C} and*

$F_n(\lambda) \rightarrow F(\lambda)$ on $\lambda \in \mathbb{C}$.

Denote Σ_n and Σ be the zero point sets of F_n and F , respectively. Set

$$\Sigma_\infty = \{\lambda : \exists \lambda_n \in \Sigma_n, \text{ s.t. } \lambda_n \rightarrow \lambda, n \rightarrow +\infty\}.$$

We have the next two conclusions.

(i) $\Sigma = \Sigma_\infty$.

(ii) If $\lambda_0 \in \Sigma$ and there exist $\lambda_{n,1}, \lambda_{n,2} \in \Sigma_n, n = 1, 2, \dots$, such that $\lambda_{n,j} \rightarrow \lambda_0$ as $n \rightarrow \infty$ for $j = 1, 2$, then $F'(\lambda_0) = 0$.

Proof. Using the classical analysis approach, Proposition 2.1 can lead to $F_n(\lambda) \rightarrow F(\lambda)$ uniformly on every compact subset of \mathbb{C} .

(i) By the definition, for $\lambda_0 \in \Sigma_\infty$, there exists $\{\lambda_n \in \Sigma_n, n = 1, 2, \dots\}$ such that $\lambda_n \rightarrow \lambda_0$ as $n \rightarrow \infty$. By Proposition 2.1 (i), we know $\{F_n(\lambda), n = 1, 2, \dots\}$ is equicontinuous on every compact subset of \mathbb{C} , hence

$$|F_n(\lambda_n) - F_n(\lambda_0)| \rightarrow 0, \text{ as } n \rightarrow \infty.$$

As a result, $F_n(\lambda_n) = 0$ yields that $F_n(\lambda_0) \rightarrow 0$, and hence $F(\lambda_0) = 0$, i.e., $\lambda_0 \in \Sigma$.

Conversely, for $\lambda_0 \in \Sigma$, if $\lambda_0 \notin \Sigma_\infty$, then by the Zero Isolation Theorem of analytic functions, there exists $\varepsilon_0 > 0$ such that

$$B(\lambda_0, \varepsilon_0) \cap \Sigma = \{\lambda_0\} \text{ and } d(\lambda_0, \Sigma_n) \geq 2\varepsilon_0 \tag{5}$$

as $n \geq N$ for some $N > 0$, where $B(\lambda_0, \varepsilon_0) := \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| \leq \varepsilon_0\}$ and $d(\lambda_0, \Sigma_n) := \inf_{\lambda_n \in \Sigma_n} |\lambda_0 - \lambda_n|$. From the inequality in (5), we have

$$F_n(\lambda) \neq 0, \text{ for any } \lambda \in B(\lambda_0, \varepsilon_0), n \geq N.$$

Therefore, $F_n^{-1}(\lambda) := 1/F_n(\lambda)$ is analytic on $B(\lambda_0, \varepsilon_0)$ for any $n > N$. By the Cauchy integral formula,

$$F_n^{-1}(\lambda_0) = \frac{1}{2\pi i} \int_{\partial B(\lambda_0, \varepsilon_0)} \frac{F_n^{-1}(z)}{z - \lambda_0} dz. \tag{6}$$

Moreover, $F(z) \neq 0$ on $\partial B(\lambda_0, \varepsilon_0)$ implies for $z \in \partial B(\lambda_0, \varepsilon_0)$,

$$|F^{-1}(z)| \leq \frac{1}{A}, \text{ } A := \min_{\partial B(\lambda_0, \varepsilon_0)} |F(z)| > 0.$$

Since $F_n \rightarrow F$ as $n \rightarrow \infty$ uniformly on $\partial B(\lambda_0, \varepsilon_0)$, we know there exists a sufficiently large number $N_1 (> N)$ such that

$$|F_n^{-1}(z)| \leq \frac{2}{A}, \text{ } z \in \partial B(\lambda_0, \varepsilon_0),$$

for $n \geq N_1$. This together with (6) gives

$$|F_n^{-1}(\lambda_0)| \leq \frac{1}{2\pi} 2\pi\varepsilon_0 \frac{1}{\varepsilon_0} \frac{2}{A} = \frac{2}{A}, \text{ for } n \geq N_1.$$

This clearly contradicts $F_n(\lambda_0) \rightarrow F(\lambda_0) = 0$ as $n \rightarrow \infty$. Hence $\lambda_0 \in \Sigma_\infty$ and $\Sigma_\infty = \Sigma$ has been proven.

(ii) For the proof of the second part we note that for every $\lambda_0 \in \Sigma$, there exists $\varepsilon > 0$ such that $B(\lambda_0, \varepsilon) \cap \Sigma = \{\lambda_0\}$, just as (5). Since every set Σ_n is countable, $\bigcup_{n=1}^\infty \Sigma_n$ is countable. Therefore, there exists $\varepsilon_0 \in (0, \varepsilon]$ such that $\partial B(\lambda_0, \varepsilon_0) \cap \Sigma_n = \emptyset$, $n \geq 1$. That is for $n \geq 1$,

$$B(\lambda_0, \varepsilon_0) \cap \Sigma = \{\lambda_0\} \text{ and } \partial B(\lambda_0, \varepsilon_0) \cap \Sigma_n = \emptyset. \tag{7}$$

Hence $\inf\{|F(\lambda)| : \lambda \in \partial B(\lambda_0, \varepsilon_0)\} > 0$. This fact and $F_n(\lambda) \rightarrow F(\lambda)$ uniformly on every compact subset of \mathbb{C} lead that for enough large number n , we have

$$|F - F_n| \Big|_{\partial B(\lambda_0, \varepsilon_0)} < |F| \Big|_{\partial B(\lambda_0, \varepsilon_0)}.$$

From Rouché Theorem, we know F and F_n have the same number of zeros in $B(\lambda_0, \varepsilon_0)$. By the condition of Lemma 2.2 (ii), for enough large number n , we have $\{\lambda_{1n}, \lambda_{2n}\} \subset B(\lambda_0, \varepsilon_0)$. Hence F has two zeros in $B(\lambda_0, \varepsilon_0)$ with respect to multiplicity. This and $B(\lambda_0, \varepsilon_0) \cap \Sigma = \{\lambda_0\}$ imply that $F'(\lambda_0) = 0$. The proof is completed. \square

Let us give some symbols of the eigenvalues and analytic functions about problem (1) and (2). Set the k -th eigenvalue of (1) and (2) as

$$\lambda_k\left(\frac{1}{p}, q, w\right) := \lambda_k(\mathcal{A}), \mathcal{A} := \left(\frac{1}{p}, q, w\right), k = 1, 2, \dots \tag{8}$$

Moreover, let $\phi(\cdot; \lambda, \mathcal{A}) \in \mathcal{D}$ be the solution of the Cauchy problem

$$\tau\phi := -(p\phi')' + q\phi = \lambda w\phi, \phi(a) = \sin\theta_1, p\phi'(a) = \cos\theta_1. \tag{9}$$

Here $\mathcal{D} = \left\{ \int_a^b w|y|^2 < \infty : y, py' \in AC[a, b], \int_a^b |\tau y|^2/|w| < \infty \right\}$ and $AC[a, b]$ is the set of all absolutely continuous functions on $[a, b]$. And set

$$F(\lambda, \mathcal{A}) := \cos(\theta_2)\phi(b; \lambda, \mathcal{A}) - \sin(\theta_2)p\frac{\partial\phi}{\partial x}(b; \lambda, \mathcal{A}), \tag{10}$$

on $\lambda \in \mathbb{C}$. It is well-known that $F(\lambda, \mathcal{A})$ is analytic on $\lambda \in \mathbb{C}$ and λ is an eigenvalue of (1) and (2) if and only if $F(\lambda, \mathcal{A}) = 0$ (cf., [2, p.33] or [18, (3.8)]).

Moreover, we will prove that for any fixed $\lambda \in \mathbb{C}$, $F(\lambda, \mathcal{A})$ is continuous about the coefficients $\mathcal{A} = (1/p, q, w)$ in the meaning of weak topology, see Lemma 2.4. Here we call that $f_n \in L^1[a, b]$ is weakly convergent to $f \in L^1[a, b]$ if for each $g \in L^\infty[a, b]$ one has

$$\lim_{n \rightarrow \infty} \int_a^b g f_n = \int_a^b g f, \text{ and take this as } f_n \xrightarrow{w} f,$$

where

$$L^\infty[a, b] := \{f \text{ is a measurable function on } [a, b] : \text{esssup}|f| < \infty\}.$$

This topology is just called the weak topology of $L^1[a, b]$. For more details about weak topology, one can refer to [12, p.2202], [19].

Now we will show the following continuity and continuous differentiability results, which is related to the solutions of Cauchy problem (9) in the weak topology of $L^1[a, b]$.

Proposition 2.3. *(cf., [11, Theorem 6, p.1292], [12, Theorem 1.1]) For any fixed $\lambda \in \mathbb{C}$, $\phi(x; \lambda, \mathcal{A})$ and $p \frac{\partial \phi}{\partial x}(x; \lambda, \mathcal{A})$ are uniformly continuous on $x \in [a, b]$, with respect to the coefficient functions \mathcal{A} in the weak topology of $L^1[a, b]$, where the definition of $\phi(x; \lambda, \mathcal{A})$ is in (9).*

In this paper, a weaker case is enough, see Lemma 2.4(i). Furthermore, under the condition of Lemma 2.2, we will prove Lemma 2.4(ii).

Lemma 2.4. *Suppose $\mathcal{A}_n = (1/p_n, q_n, w_n) \xrightarrow{w} \mathcal{A} = (1/p, q, w)$, as $n \rightarrow \infty$, where $\mathcal{A}_n \xrightarrow{w} \mathcal{A}$ means $1/p_n \xrightarrow{w} 1/p$, $q_n \xrightarrow{w} q$ and $w_n \xrightarrow{w} w$ as $n \rightarrow \infty$. Then,*

- (i) $\phi(b; \lambda, \mathcal{A}_n) \rightarrow \phi(b; \lambda, \mathcal{A})$ and $p_n \phi'(b; \lambda, \mathcal{A}_n) \rightarrow p \phi'(b; \lambda, \mathcal{A})$.
- (ii) $\{|\phi(b; \lambda, \mathcal{A}_n)|, |p_n \phi'(b; \lambda, \mathcal{A}_n)|, n \geq 1\}$ are uniformly bounded about λ on any compact subset of \mathbb{C} .

Proof. We only need to prove (ii). For $\mathcal{A}_n, n \geq 1$, the Cauchy problem (9) can be rewritten as a system,

$$\begin{aligned} \frac{\partial}{\partial x} \begin{pmatrix} \phi_n \\ p_n \phi'_n \end{pmatrix} &= \begin{pmatrix} 0 & 1/p_n \\ q_n - \lambda w_n & 0 \end{pmatrix} \begin{pmatrix} \phi_n \\ p_n \phi'_n \end{pmatrix}, \\ \begin{pmatrix} \phi_n \\ p_n \phi'_n \end{pmatrix} (a) &= \begin{pmatrix} \sin \theta_1 \\ \cos \theta_1 \end{pmatrix}. \end{aligned}$$

Then, we have the estimate

$$\begin{aligned} (|\phi_n| + |p_n \phi'_n|)(x) &\leq 2 + \int_a^x \left(\frac{1}{p_n} + |q_n| + |\lambda| w_n \right) (|\phi_n| + |p_n \phi'_n|) \\ &\leq 2 + (1 + |\lambda|) \int_a^x |\mathcal{A}_n| (|\phi_n| + |p_n \phi'_n|), \end{aligned}$$

where $|\mathcal{A}_n| := |1/p_n| + |q_n| + |w_n|$. Since $\mathcal{A}_n \xrightarrow{w} \mathcal{A}$, there exists a constant $M > 0$, such that for any $n \geq 1, |\mathcal{A}_n| < M$ by [5, Theorem 1.27] or Proposition 3.2 (i). Hence the Gronwall inequality can lead that for any $x \in [a, b]$,

$$(|\phi_n| + |p_n \phi'_n|)(x) \leq 2e^{(1+|\lambda|)M(x-a)}. \tag{11}$$

Especially, for any $n \geq 1$, we get

$$|\phi(b; \lambda, \mathcal{A}_n)| + |p_n \phi'(b; \lambda, \mathcal{A}_n)| \leq 2e^{(1+|\lambda|)M(b-a)}.$$

This inequality can lead to (ii) and the proof is finished. □

From Lemma 2.4 (ii) we know $\{F(\lambda, \mathcal{A}_n), n \geq 1\}$ is uniformly bounded on compact subsets of \mathbb{C} . Moreover, according to Lemma 2.4 (i), if $\mathcal{A}_n \xrightarrow{w} \mathcal{A}$, then $F(\lambda, \mathcal{A}_n) \rightarrow F(\lambda, \mathcal{A})$, as $n \rightarrow \infty$. Hence the conditions of Lemma 2.2 are satisfied. Using Lemma 2.2, we can prove the next lemma.

Lemma 2.5. *If $\mathcal{A}_n = (1/p_n, q_n, w_n) \xrightarrow{w} \mathcal{A}$ and $\{\lambda_1(\mathcal{A}_n) : n \geq 1\}$ are bounded below, then for any $k \geq 1$, $\lambda_k(\mathcal{A}_n) \rightarrow \lambda_k(\mathcal{A})$, $n \rightarrow \infty$.*

Proof. Set $F_n(\lambda) := F(\lambda, \mathcal{A}_n)$ and $F(\lambda) := F(\lambda, \mathcal{A})$, where the definition of $F(\lambda, \mathcal{A})$ is in (10). Then by Lemma 2.4 we know

$$F \not\equiv 0 \text{ and } F_n(\lambda) \rightarrow F(\lambda), \text{ as } n \rightarrow \infty.$$

In the proof, the definitions of Σ_n, Σ_∞ and Σ are the same as in Lemma 2.2. Therefore, $\Sigma_n = \{\lambda_k(\mathcal{A}_n), k \geq 1\}$ and $\Sigma = \{\lambda_k(\mathcal{A}), k \geq 1\}$.

To begin with, we prove that $\{\lambda_1(\mathcal{A}_n) : n \geq 1\}$ is bounded above. Suppose on the contrary, then without losing generality we can assume that $\lambda_1(\mathcal{A}_n) \rightarrow \infty$ as $n \rightarrow \infty$. Since $\lambda_k(\mathcal{A}_n) \geq \lambda_1(\mathcal{A}_n)$ for any $k \geq 1$, one sees that $\Sigma_\infty = \emptyset \neq \Sigma$, which contradicts Lemma 2.2 (i). With the same method, it can be deduced that for any $k \geq 1$, $\{\lambda_k(\mathcal{A}_n) : n \geq 1\}$ is bounded above.

For the case $k = 1$, from the condition of Lemma 2.5 we know that $\{\lambda_1(\mathcal{A}_n), n \geq 1\}$ are bounded below. Then $\{\lambda_1(\mathcal{A}_n) : n \geq 1\}$ is bounded. Hence there exists $\lambda_1 \in \mathbb{R}$ such that $\lambda_1(\mathcal{A}_n) \rightarrow \lambda_1 \in \Sigma_\infty = \Sigma$ as $n \rightarrow \infty$. This fact and $\Sigma = \Sigma_\infty$ in Lemma 2.2 (i) can lead to

$$\lim_{n \rightarrow \infty} \lambda_1(\mathcal{A}_n) = \lambda_1 = \inf \Sigma_\infty = \inf \Sigma = \lambda_1(\mathcal{A}).$$

For the case $k = 2$, the same method can be used to obtain $\lambda_2(\mathcal{A}_n) \rightarrow \lambda_2 \geq \lambda_1(\mathcal{A})$, $n \rightarrow \infty$. If $\lambda_2 = \lambda_1(\mathcal{A})$, by Lemma 2.2 (ii) we know $F'(\lambda_1(\mathcal{A})) = 0$. It is a contradiction with the fact that the eigenvalue $\lambda_1(\mathcal{A})$ is algebraic simple. Hence $\lambda_2 > \lambda_1(\mathcal{A}) = \lambda_1$ and

$$\lim_{n \rightarrow \infty} \lambda_2(\mathcal{A}_n) = \lambda_2 = \inf \Sigma_\infty \setminus \{\lambda_1\} = \inf \Sigma \setminus \{\lambda_1(\mathcal{A})\} = \lambda_2(\mathcal{A}).$$

By mathematical deduction, we get for the general case $\lambda_k(\mathcal{A}_n) \rightarrow \lambda_k(\mathcal{A})$ as $n \rightarrow \infty$, for any $k \geq 1$. The proof is finished. □

3. THE CONTINUITY OF EIGENVALUES IN WEAK TOPOLOGY

The main conclusion of this paper will be given in this section. In Theorem 3.5, we will prove the continuities of eigenvalues in the weak topology of $L^1[a, b]$ on all the

coefficient functions of problem (1) and (2). First, in Lemma 3.3, we will prove the first eigenvalue of (1) and (2) is bounded below about the the coefficient functions.

Firstly, δ and ε will be defined. By $1/p \in L^1[a, b]$, we know there exists a number larger than zero, which is defined as $\delta := \delta(1/p, q) > 0$ such that

$$\sup \left\{ \int_t^{t+\delta} \frac{1}{p} : t \in [a, b - \delta] \right\} \leq \frac{1}{8 + 4\|q\|_1}, \tag{12}$$

where $\|q\|_1 := \int_a^b |q|$. Then for any $t \in [a, b - \delta]$, $\int_t^{t+\delta} 1/p \leq \frac{1}{8+4\|q\|_1}$.

Furthermore, set

$$\varepsilon := \varepsilon(\delta, w) := \varepsilon(\mathcal{A}) = \frac{1}{4} \inf \left\{ \int_t^{t+\delta} w : t \in [a, b - \delta] \right\}, \tag{13}$$

where $\mathcal{A} = (1/p, q, w)$ and the definition of δ is in (12). Clearly, $\varepsilon > 0$ from $w > 0$ a.e. on $[a, b]$.

Lemma 3.1. *Let $\lambda_1 := \lambda_1(\mathcal{A})$ be the first eigenvalue of (1) and (2). Then*

$$\lambda_1 \geq -\frac{4 + 2\|q\|_1}{\varepsilon}, \tag{14}$$

where the definition of $\varepsilon = \varepsilon(\mathcal{A})$ is in (13).

Proof. Let ϕ with

$$\max\{|\phi|\} := \max\{|\phi(x)| : x \in [a, b]\} = 1, \tag{15}$$

be the corresponding eigenfunction about the first eigenvalue λ_1 of the problem (1) and (2). Without losing generality, we can assume $\lambda_1 < 0$.

Integrating (1) from a to b and using the boundary condition (2), we can get

$$\begin{aligned} & \cot(\theta_1)\phi^2(a) - \cot(\theta_2)\phi^2(b) + \int_a^b p|\phi'|^2 + \int_a^b q|\phi|^2 \\ &= \lambda_1 \int_a^b w|\phi|^2. \end{aligned} \tag{16}$$

Plug (15) into (16) and notice $\lambda_1 < 0$, we have

$$\int_a^b p|\phi'|^2 \leq |\cot(\theta_1)| + |\cot(\theta_2)| + \|q\|_1 = 2 + \|q\|_1. \tag{17}$$

Then (15) and (17) can lead to

$$\begin{aligned} & \left| \cot(\theta_1)\phi^2(a) - \cot(\theta_2)\phi^2(b) + \int_a^b p|\phi'|^2 + \int_a^b q|\phi|^2 \right| \\ & \leq 4 + 2\|q\|_1. \end{aligned} \tag{18}$$

Since eigenfunction ϕ is continuous, there exists $x_0 \in [a, b]$ such that $\max\{|\phi|\} = |\phi(x_0)| = 1$. From the definition of δ in (12), we know it holds for at least one of $[x_0, x_0 + \delta] \subset [a, b]$ and $[x_0 - \delta, x_0] \subset [a, b]$. Then we have

$$\int_{x_0}^{x_0+\delta} \frac{1}{p} \leq \frac{1}{8 + 4\|q\|_1} \quad \text{or} \quad \int_{x_0-\delta}^{x_0} \frac{1}{p} \leq \frac{1}{8 + 4\|q\|_1}. \tag{19}$$

Cauchy inequality and (17), (19) can lead that for any $x \in (a, b)$ and $|x - x_0| \leq \delta$, we have

$$|\phi(x)| \geq |\phi(x_0)| - \left| \int_{x_0}^x \phi' \right| \geq 1 - \left| \int_{x_0}^x p|\phi'|^2 \int_{x_0}^x \frac{1}{p} \right|^{1/2} \geq \frac{1}{2}.$$

This fact and the definition of ε can lead to the next estimate,

$$\int_a^b w|\phi|^2 \geq \int_{x_0}^{x_0+\delta} w|\phi|^2 \quad \text{or} \quad \int_{x_0-\delta}^{x_0} w|\phi|^2 \geq \frac{1}{4} \int_{x_0}^{x_0+\delta} w \quad \text{or} \quad \int_{x_0-\delta}^{x_0} w \geq \varepsilon. \tag{20}$$

Hence it follows from (16), (18) and (20) can lead to

$$\begin{aligned} |\lambda_1| &\leq \frac{1}{\int_a^b w|\phi|^2} \left| \cot(\theta_1)\phi^2(a) - \cot(\theta_2)\phi^2(b) + \int_a^b (p|\phi'|^2 + q|\phi|^2) \right| \\ &\leq \frac{4 + 2\|q\|_1}{\varepsilon} \end{aligned}$$

and the proof is finished. □

Fixing the weight function w and applying Lemma 3.1, we can give a lower bound of all the eigenvalues of problem (1) and (2), see Lemma 3.3. First, we need the next conclusion which is a special case of [3, p.294] or [11, Lemma 1, p.1289].

Proposition 3.2. *(cf., [3, p.294] or [11, Lemma 1, p.1289]) Suppose $1/p_n \xrightarrow{w} 1/p$ in $L^1[a, b]$. Then,*

- (i) *there exists $r > 0$ such that $\|q_n\|_1 < r$, for any $n \geq 1$.*
- (ii) *for every $\hat{\varepsilon} > 0$ there exists $\hat{\delta} > 0$ such that $\int_c^d 1/p_n < \hat{\varepsilon}$ for any $n \geq 1$ and any $[c, d] \subset [a, b]$ with $d - c < \hat{\delta}$.*

Lemma 3.3. *If $1/p_n \xrightarrow{w} 1/p$ and $q_n \xrightarrow{w} q$, then for any fixed weight function w , we have*

$$\lambda_1 \left(\frac{1}{p_n}, q_n, w \right) \geq -\frac{4 + 2r}{\varepsilon(1/p, q, w)}, \quad n = 1, 2, \dots,$$

where the definition of $\varepsilon(1/p, q, w) > 0$ is in (13) and independent of $1/p_n$ and q_n .

Proof. Note, by [5, Theorem 1.27] or Proposition 3.2 (i), $q_n \xrightarrow{w} q$ can deduce that there exists $r > 0$ such that

$$\|q_n\|_1 < r, \quad \text{for any } n \geq 1. \tag{21}$$

Since $1/p_n \xrightarrow{w} 1/p$, by Proposition 3.2, we can get that for the r in (21), there exists $\hat{\delta} := \hat{\delta}(1/p, r) > 0$ such that (cf., the definition of δ in (12))

$$\sup \left\{ \int_t^{t+\hat{\delta}} \frac{1}{p_n} : t \in [a, b - \hat{\delta}] \right\} \leq \frac{1}{8 + 4r}, \text{ for any } n \geq 1. \tag{22}$$

By $\|q_n\|_1 < r$, contrasting $\hat{\delta}(1/p, r)$ in (22) and the definition of $\delta(1/p_n, q_n)$ in (12), we can select $\delta(1/p_n, q_n)$ such that

$$\hat{\delta}(1/p, r) \leq \delta(1/p_n, q_n). \tag{23}$$

Then (23) and the definition of $\varepsilon := \varepsilon(\delta, w)$ in (13) lead that

$$\varepsilon \left(\hat{\delta}(1/p, r), w \right) \leq \varepsilon(\delta(1/p_n, q_n), w) := \varepsilon(1/p_n, q_n, w). \tag{24}$$

Recall (14) in Lemma 3.1, using $\|q_n\|_1 < r$ and (24), we get a lower bound of the first eigenvalues. For any $n = 1, 2, \dots$,

$$\lambda_1 \left(\frac{1}{p_n}, q_n, w \right) \geq -\frac{4 + 2\|q_n\|_1}{\varepsilon(1/p_n, q_n, w)} \geq -\frac{4 + 2r}{\varepsilon(1/p, r, w)}.$$

The proof is finished. □

According to Lemma 3.3, for any fixed w , the eigenvalues of problem (1) and (2) with coefficients $\{(1/p_n, q_n, w), n \geq 1\}$, are bounded below. Hence by Lemma 2.5 we obtain the next corollary.

Corollary 3.4. *If $1/p_n \xrightarrow{w} 1/p$ and $q_n \xrightarrow{w} q$, then for any fixed weight function w and $k \geq 1$, we have*

$$\lambda_k \left(\frac{1}{p_n}, q_n, w \right) \rightarrow \lambda_k \left(\frac{1}{p}, q, w \right), \text{ as } n \rightarrow \infty.$$

Using this conclusion, we can prove the main result of this paper. Recall the symbols $\mathcal{A}_n = (1/p_n, q_n, w_n)$ and $\mathcal{A} = (1/p, q, w)$.

Theorem 3.5. *Suppose \mathcal{A}_n and \mathcal{A} satisfy the standard condition (3). If $\mathcal{A}_n \xrightarrow{w} \mathcal{A}$, then for any $k \geq 1$, $\lambda_k(\mathcal{A}_n) \rightarrow \lambda_k(\mathcal{A})$, as $n \rightarrow \infty$.*

Proof. We only need to prove that $\{\lambda_1(\mathcal{A}_n) : n = 1, 2, \dots\}$ is bounded below, by Lemma 2.5. Consider the next three problems

$$-(py')' + (q + (1 - \lambda_1(\mathcal{A}))w)y = \mu wy, \text{ on } (a, b), \tag{25}$$

$$-(p_n y')' + (q_n + (1 - \lambda_1(\mathcal{A}))w_n)y = \hat{\mu}(n)wy, \text{ on } (a, b), \tag{26}$$

$$-(p_n y')' + (q_n + (1 - \lambda_1(\mathcal{A}))w_n)y = \tilde{\mu}(n)w_n y, \text{ on } (a, b), \tag{27}$$

all with the boundary condition (2).

Clearly, the first eigenvalue of (25) satisfies $\mu_1 = 1$. Note

$$\frac{1}{p_n} \xrightarrow{w} \frac{1}{p}, \text{ and } q_n + (1 - \lambda_1(\mathcal{A}))w_n \xrightarrow{w} q + (1 - \lambda_1(\mathcal{A}))w,$$

as $n \rightarrow \infty$, hence we have $\hat{\mu}_1(n) \rightarrow \mu_1$, $n \rightarrow \infty$, by Corollary 3.4. This fact and $\mu_1 = 1$ lead that there exists $N > 0$, for any $n \geq N$, $\hat{\mu}_1(n) > 0$. Since the left differential expression of (26) is the same as (27), we obtain that

$$\text{for any } n \geq N, \tilde{\mu}_1(n) > 0. \tag{28}$$

Moreover, from (27) and the definition of $\lambda_1(\mathcal{A}_n)$, we get

$$-(p_n y')' + q_n y = \{\tilde{\mu}_1(n) + \lambda_1(\mathcal{A}) - 1\}w_n y = \lambda_1(\mathcal{A}_n)w_n y,$$

$y = y(x)$, $x \in (a, b)$. Hence, for any $n \geq N$, $\lambda_1(\mathcal{A}_n) = \tilde{\mu}_1(n) + \lambda_1(\mathcal{A}) - 1 > \lambda_1(\mathcal{A}) - 1$, by (28). Then $\lambda_1(\mathcal{A}_n)$ is bounded below and the proof is complete. \square

In this section, for obtaining the main result, we find the most important thing is that all the eigenvalues are bounded below. By Lemmas 3.1 and 3.3, we know the estimation of the lower bound of eigenvalues contains δ and ε . However the definitions of δ and ε in (12) and (13) are complicated, hence two examples, which can be easily calculated, will be given in the next example.

Example 1. (i) Consider the case $p \equiv w \equiv 1$, i.e.,

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in (a, b), \tag{29}$$

with the boundary condition (2).

In this case, we can select $\delta = \frac{1}{8+4\|q\|_1}$, and $\varepsilon = \delta/4$. Then Lemma 3.1 gives

$$\lambda_1 \geq -\frac{4 + 2\|q\|_1}{\varepsilon} = -32(2 + \|q\|_1)^2.$$

(ii) Consider the case $1/p(x) \leq 1/M < +\infty$, $w(x) \geq w_0 > 0$ a.e. on $x \in [a, b]$, and $\|q\|_1 \leq r < +\infty$.

Then we can select $\delta = \frac{M}{8+4r}$, and $\varepsilon = \delta w_0/4$. Then Lemma 3.3 gives

$$\lambda_1 \geq -\frac{4 + 2r}{\varepsilon} = -\frac{32(2 + r)^2}{Mw_0}.$$

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