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POSITIVE SOLUTIONS OF A FOURTH-ORDER PERIODIC BOUNDARY VALUE PROBLEM WITH PARAMETER

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ABSTRACT: In this paper, we study the following periodic boundary value problem of fourth-order ordinary differential equation

$$\begin{cases} u^{(4)}(t) + \alpha u''(t) - \rho^4 u(t) + \lambda f(t, u(t)) = 0, \ t \in [0, 2\pi], \\ u^{(i)}(0) = u^{(i)}(2\pi), \ i = 0, 1, 2, 3, \end{cases}$$

where α and ρ are constants satisfying $\rho \neq 0$ and $4\alpha + 16\rho^4 < 1$, and $\lambda > 0$ is a parameter. By imposing some conditions on the nonlinear term f, we obtain the existence and multiplicity of positive solutions to the above problem for suitable λ. The main tool used is Guo-Krasnoselskii fixed point theorem.

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1. INTRODUCTION

Boundary value problems (BVPs for short) of fourth-order ordinary differential equations have received much attention due to their striking applications to engineering, physics, material mechanics and fluid mechanics (see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11] and the references therein). However, in the existing literature, most of the boundary conditions were separated or the second derivative term of unknown function (i.e., bending term) was not included in the discussed equation. For example, in 1995, by applying Guo-Krasnoselskii fixed point theorem, Ma and Wang (see [2]) established some existence results of positive solutions for the BVP consisting of the fourth-order equation without bending term

$$\frac{d^4y}{dx^4} - h(x)f(y(x)) = 0$$

and the separated boundary conditions

$$y(0) = y(1) = y''(0) = y''(1) = 0$$

or

$$y(0) = y'(1) = y''(0) = y'''(1) = 0.$$

In 2003, Li (see [3]) studied the existence of positive solutions for the BVP formed by the fourth-order equation with bending term

$$u^{(4)}(t) + \beta u''(t) - \alpha u(t) = f(t, u(t))$$

and the separated boundary conditions

$$u(0) = u(1) = u''(0) = u''(1) = 0.$$

The main tool used was the fixed point index theory.

It is worth mentioning that, in 2011, by using the theory of the fixed point index in cones, Li (see [8]) obtained the existence of at least one positive solution for the periodic BVP (PBVP for short) consisting of the fourth-order equation with bending term

$$u^{(4)}(t) - \beta u''(t) + \alpha u(t) = f(t, u(t), u''(t))$$

and the periodic boundary conditions

$$u^{(i)}(0) = u^{(i)}(1),\ i = 0, 1, 2, 3,$$

where $0 < \alpha < (\frac{\beta}{2} + 2\pi^2)^2$, $\beta > -2\pi^2$ and $\frac{\alpha}{\pi^4} + \frac{\beta}{\pi^2} + 1 > 0$.

In 2015, Pei and Chang (see [11]) considered the existence of positive solutions for the PBVP formed by the fourth-order equation without bending term

$$u^{(4)}(t) - \rho^4 u(t) + \lambda f(t, u(t)) = 0,\ t \in [0, 2\pi]$$

and the periodic boundary conditions

$$u^{(i)}(0) = u^{(i)}(2\pi),\ i = 0, 1, 2, 3,$$

where $\rho \in (0, \frac{1}{2})$ was a constant and $\lambda > 0$ was a parameter. Their main tool was the Guo-Krasnoselskii fixed point theorem.

Motivated greatly by the above-mentioned works, in this paper, we consider the existence and multiplicity of positive solutions for the following PBVP consisting of the fourth-order equation with bending term and periodic boundary conditions

$$\begin{cases} u^{(4)}(t) + \alpha u''(t) - \rho^4 u(t) + \lambda f(t, u(t)) = 0, \ t \in [0, 2\pi], \\ u^{(i)}(0) = u^{(i)}(2\pi), \ i = 0, 1, 2, 3. \end{cases} \tag{1}$$

Throughout this paper, we always assume that α and ρ are constants satisfying $\rho \neq 0$ and $4\alpha + 16\rho^4 < 1$, $f \in C([0, 2\pi] \times [0, +\infty), [0, +\infty))$ and $\lambda > 0$ is a parameter. Obviously, the problem in [11] is a special case of the PBVP (1). Moreover, if we let $\alpha = 0$, then $\rho \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$, which is different from the restriction in [11].

2. PRELIMINARIES

For convenience, we denote

$$A_1 = \frac{\sqrt{\alpha^2 + 4\rho^4} + \alpha}{2} \text{ and } A_2 = \frac{\sqrt{\alpha^2 + 4\rho^4} - \alpha}{2}.$$

Since $\rho \neq 0$ and $4\alpha + 16\rho^4 < 1$, it is easy to know that $0 < A_1 < \frac{1}{4}$ and $A_2 > 0$.

Let

$$u''(t) + A_1 u(t) = x(t), \ t \in [0, 2\pi].$$

Then it is not difficult to verify that the PBVP (1) is equivalent to the PBVP

$$\begin{cases} -x''(t) + A_2 x(t) = \lambda f\left(t, \int_0^{2\pi} H(t,s) x(s) ds\right), \ t \in [0, 2\pi], \\ x^{(i)}(0) = x^{(i)}(2\pi), \ i = 0, 1, \end{cases} \tag{2}$$

where

$$H(t,s) = \frac{1}{2\sqrt{A_1}\left(1 - cos\left(2\sqrt{A_1}\pi\right)\right)} \begin{cases} sin\left(\sqrt{A_1}\,(s-t)\right) + \\ sin\left(\sqrt{A_1}(2\pi + t - s)\right), \\ 0 \leq t \leq s \leq 2\pi, \\ \\ sin\left(\sqrt{A_1}(t-s)\right) + \\ sin\left(\sqrt{A_1}(2\pi + s - t)\right), \\ 0 \leq s \leq t \leq 2\pi. \end{cases} \tag{3}$$

Lemma 1. *$H(t,s)$ defined by (3) has the following properties:*

(1) $H(t,s) > 0, \ (t,s) \in [0, 2\pi] \times [0, 2\pi]$;

(2) $\int_0^{2\pi} H(t,s) ds = \frac{1}{A_1}, \ t \in [0, 2\pi]$.

Proof. (1) Since the proof of the case when $0 \le s \le t \le 2\pi$ is similar, we only prove the case when $0 \le t \le s \le 2\pi$. Let $0 \le t \le s \le 2\pi$. Then in view of $0 < \sqrt{A_1}\pi < \frac{\pi}{2}$ and $-\frac{\pi}{2} < \sqrt{A_1}(\pi + t - s) < \frac{\pi}{2}$, we have

$$\begin{aligned} H(t,s) &= \frac{sin\left(\sqrt{A_1}(s-t)\right) + sin\left(\sqrt{A_1}(2\pi + t - s)\right)}{2\sqrt{A_1}\left(1 - cos\left(2\sqrt{A_1}\pi\right)\right)} \\ &= \frac{sin\left(\sqrt{A_1}\pi\right) cos\left(\sqrt{A_1}(\pi + t - s)\right)}{\sqrt{A_1}\left(1 - cos\left(2\sqrt{A_1}\pi\right)\right)} \\ &= \frac{cos\left(\sqrt{A_1}(\pi + t - s)\right)}{2\sqrt{A_1} sin\left(\sqrt{A_1}\pi\right)} \\ &> 0. \end{aligned}$$

(2) For any $t \in [0, 2\pi]$,

$$\begin{aligned} &\int_0^{2\pi} H(t,s)ds \\ = &\frac{\int_0^t sin\left(\sqrt{A_1}(t-s)\right) + sin\left(\sqrt{A_1}(2\pi + s - t)\right) ds}{2\sqrt{A_1}\left(1 - cos\left(2\sqrt{A_1}\pi\right)\right)} + \\ &\frac{\int_t^{2\pi} sin\left(\sqrt{A_1}(s-t)\right) + sin\left(\sqrt{A_1}(2\pi + t - s)\right) ds}{2\sqrt{A_1}\left(1 - cos\left(2\sqrt{A_1}\pi\right)\right)} \\ = &\frac{1 - cos\left(2\sqrt{A_1}\pi\right)}{A_1\left(1 - cos\left(2\sqrt{A_1}\pi\right)\right)} \\ = &\frac{1}{A_1}. \end{aligned}$$

□

Lemma 2. *If the PBVP (2) has a positive solution, then the PBVP (1) has also a positive solution.*

Proof. Suppose that x is a positive solution of the PBVP (2). Let

$$u(t) = \int_0^{2\pi} H(t,s)x(s)ds, \; t \in [0, 2\pi].$$

Then it follows from (1) of Lemma 1 that u is a positive solution of the PBVP (1). □

Lemma 3. *For any $y \in C[0, 2\pi]$, the PBVP*

$$\begin{cases} -x''(t) + A_2 x(t) = y(t), \; t \in [0, 2\pi], \\ x^{(i)}(0) = x^{(i)}(2\pi), \; i = 0, 1 \end{cases} \tag{4}$$

has a unique solution

$$x(t) = \int_0^{2\pi} G(t,s)y(s)ds, \; t \in [0, 2\pi],$$

where

$$G(t,s)=\frac{1}{2\sqrt{A_2}(e^{2\sqrt{A_2}\pi}-1)}\begin{cases} e^{\sqrt{A_2}(2\pi+t-s)}+e^{\sqrt{A_2}(s-t)}, \\ 0\le t\le s\le 2\pi, \\ \\ e^{\sqrt{A_2}(2\pi+s-t)}+e^{\sqrt{A_2}(t-s)}, \\ 0\le s\le t\le 2\pi. \end{cases} \tag{5}$$

Proof. In view of the equation in (4), we can suppose that

$$x(t)=C_1(t)e^{\sqrt{A_2}t}+C_2(t)e^{-\sqrt{A_2}t},\ t\in[0,2\pi]. \tag{6}$$

So,

$$x'(t)=C_1'(t)e^{\sqrt{A_2}t}+\sqrt{A_2}C_1(t)e^{\sqrt{A_2}t}+ \\ C_2'(t)e^{-\sqrt{A_2}t}-\sqrt{A_2}C_2(t)e^{-\sqrt{A_2}t},\ t\in[0,2\pi].$$

If we let

$$C_1'(t)e^{\sqrt{A_2}t}+C_2'(t)e^{-\sqrt{A_2}t}=0,\ t\in[0,2\pi], \tag{7}$$

then

$$x'(t)=\sqrt{A_2}C_1(t)e^{\sqrt{A_2}t}-\sqrt{A_2}C_2(t)e^{-\sqrt{A_2}t},\ t\in[0,2\pi]$$

and

$$x''(t)=\sqrt{A_2}C_1'(t)e^{\sqrt{A_2}t}+A_2C_1(t)e^{\sqrt{A_2}t}- \\ \sqrt{A_2}C_2'(t)e^{-\sqrt{A_2}t}+A_2C_2(t)e^{-\sqrt{A_2}t},\ t\in[0,2\pi]. \tag{8}$$

Therefore, it follows from the equation in (4), (6) and (8) that

$$-\sqrt{A_2}C_1'(t)e^{\sqrt{A_2}t}+\sqrt{A_2}C_2'(t)e^{-\sqrt{A_2}t}=y(t),\ t\in[0,2\pi]. \tag{9}$$

By (7) and (9), we get

$$C_1'(t)=-\frac{e^{-\sqrt{A_2}t}}{2\sqrt{A_2}}y(t),\ C_2'(t)=\frac{e^{\sqrt{A_2}t}}{2\sqrt{A_2}}y(t),\ t\in[0,2\pi].$$

So,

$$C_1(t)=C_1(0)-\frac{\int_0^t e^{-\sqrt{A_2}s}y(s)ds}{2\sqrt{A_2}},$$

$$C_2(t)=C_2(0)+\frac{\int_0^t e^{\sqrt{A_2}s}y(s)ds}{2\sqrt{A_2}},\ t\in[0,2\pi],$$

and so,

$$x(t)=C_1(0)e^{\sqrt{A_2}t}+C_2(0)e^{-\sqrt{A_2}t}-$$

$$\frac{\int_0^t [e^{\sqrt{A_2}(t-s)} - e^{\sqrt{A_2}(s-t)}]y(s)ds}{2\sqrt{A_2}},\ t \in [0, 2\pi],$$

which together with the boundary conditions in (4) implies that

$$C_1(0) = \frac{\int_0^{2\pi} e^{\sqrt{A_2}(2\pi-s)}y(s)ds}{2\sqrt{A_2}(e^{2\sqrt{A_2}\pi} - 1)},\ C_2(0) = -\frac{\int_0^{2\pi} e^{\sqrt{A_2}(s-2\pi)}y(s)ds}{2\sqrt{A_2}(e^{-2\sqrt{A_2}\pi} - 1)}.$$

Therefore,

$$\begin{aligned} x(t) &= \frac{\int_0^{2\pi} [e^{\sqrt{A_2}(2\pi+t-s)} + e^{\sqrt{A_2}(s-t)}]y(s)ds}{2\sqrt{A_2}(e^{2\sqrt{A_2}\pi} - 1)} - \\ &\quad \frac{\int_0^t [e^{\sqrt{A_2}(t-s)} - e^{\sqrt{A_2}(s-t)}]y(s)ds}{2\sqrt{A_2}} \\ &= \int_0^{2\pi} G(t, s)y(s)ds,\ t \in [0, 2\pi]. \end{aligned}$$

□

Lemma 4. *$G(t, s)$ defined by (5) has the following property:*

$$0 < m := \frac{e^{\sqrt{A_2}\pi}}{\sqrt{A_2}(e^{2\sqrt{A_2}\pi} - 1)} \le G(t, s) \le \frac{e^{2\sqrt{A_2}\pi} + 1}{2\sqrt{A_2}(e^{2\sqrt{A_2}\pi} - 1)} =: M,$$
$$(t, s) \in [0, 2\pi] \times [0, 2\pi].$$

Proof. Since the proof of the case when $0 \le s \le t \le 2\pi$ is similar, we only prove the case when $0 \le t \le s \le 2\pi$. Let $0 \le t \le s \le 2\pi$. Then

$$G(t, s) = \frac{e^{\sqrt{A_2}(2\pi+t-s)} + e^{\sqrt{A_2}(s-t)}}{2\sqrt{A_2}(e^{2\sqrt{A_2}\pi} - 1)}.$$

Define

$$g(x) = \frac{e^{\sqrt{A_2}(2\pi-x)} + e^{\sqrt{A_2}x}}{2\sqrt{A_2}(e^{2\sqrt{A_2}\pi} - 1)},\ x \in [0, 2\pi].$$

Then it is easy to verify that $g(x)$ is decreasing on $[0, \pi]$ and monotonically increasing on $[\pi, 2\pi]$, which together with $g(\pi) = m$ and $g(0) = g(2\pi) = M$ shows that

$$m \le g(x) \le M,\ x \in [0, 2\pi].$$

In view of $0 \le t \le s \le 2\pi$, we have $0 \le s - t \le 2\pi$. So,

$$m \le G(t, s) = g(s - t) \le M.$$

□

Let $C[0, 2\pi]$ be equipped with the maximum norm and

$$K = \{x \in C[0, 2\pi] : x(t) \ge 0,\ t \in [0, 2\pi], \min_{t\in[0,2\pi]} x(t) \ge \sigma\|x\|\},$$

where $0 < \sigma = \frac{2e^{\sqrt{A_2}\pi}}{e^{2\sqrt{A_2}\pi}+1} < 1$. Then K is a cone in Banach space $C[0, 2\pi]$.

Now, we define an operator T_λ as follow:

$$(T_\lambda x)(t) = \lambda \int_0^{2\pi} G(t,s) f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds, \ x \in K, \ t \in [0, 2\pi].$$

Obviously, if x is a nontrivial fixed point of T_λ, then x is a positive solution of the PBVP (2).

Lemma 5. *$T_\lambda : K \to K$ is completely continuous.*

Proof. For any $x \in K$, by Lemma 4, we know that

$$\begin{aligned} 0 \le (T_\lambda x)(t) &= \lambda \int_0^{2\pi} G(t,s) f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds \\ &\le M\lambda \int_0^{2\pi} f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds, \ t \in [0, 2\pi], \end{aligned}$$

so,

$$\|T_\lambda x\| \le M\lambda \int_0^{2\pi} f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds,$$

which together with Lemma 4 shows that

$$\begin{aligned} (T_\lambda x)(t) &= \lambda \int_0^{2\pi} G(t,s) f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds \\ &\ge m\lambda \int_0^{2\pi} f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds \\ &\ge \frac{m}{M}\|T_\lambda x\| \\ &= \sigma\|T_\lambda x\|, \ t \in [0, 2\pi], \end{aligned}$$

and so,

$$\min_{t\in[0,2\pi]} (T_\lambda x)(t) \ge \sigma\|T_\lambda x\|.$$

This indicates that $T_\lambda(K) \subset K$. Moreover, it follows from Arzela-Ascoli theorem that $T_\lambda : K \to K$ is completely continuous. □

Lemma 6. *(Guo-Krasnoselskii fixed point theorem (see [12], [13])) Let X be a Banach space and K be a cone in X. Assume that Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(1) $\|Tx\| \le \|x\|$, $x \in K \cap \partial\Omega_1$, and $\|Tx\| \ge \|x\|$, $x \in K \cap \partial\Omega_2$; or

(2) $\|Tx\| \ge \|x\|$, $x \in K \cap \partial\Omega_1$, and $\|Tx\| \le \|x\|$, $x \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

In the remainder of this paper, for any constant $c > 0$, we define

$$\Omega_c = \{x \in C[0, 2\pi] : \|x\| < c\}.$$

Lemma 7. *Let $c > 0$ be a constant. Then for any $x \in K \cap \partial\Omega_c$,*

$$\frac{\sigma c}{A_1} \leq \int_0^{2\pi} H(s, \tau)x(\tau)d\tau \leq \frac{c}{A_1}, \; s \in [0, 2\pi].$$

Proof. It is easy to prove from the definition of K and Lemma 1. □

3. MAIN RESULTS

For convenience, we denote

$$f^0 = \limsup_{u\to 0^+} \max_{t\in[0,2\pi]} \frac{f(t,u)}{u}, \; f_0 = \liminf_{u\to 0^+} \min_{t\in[0,2\pi]} \frac{f(t,u)}{u},$$

$$f^\infty = \limsup_{u\to +\infty} \max_{t\in[0,2\pi]} \frac{f(t,u)}{u}, \; f_\infty = \liminf_{u\to +\infty} \min_{t\in[0,2\pi]} \frac{f(t,u)}{u},$$

$$i_0 = \begin{cases} 0, \; f^0 \neq 0 \text{ and } f^\infty \neq 0, \\ 1, \; f^0 = 0 \text{ and } f^\infty \neq 0, \text{ or } f^0 \neq 0 \text{ and } f^\infty = 0, \\ 2, \; f^0 = f^\infty = 0, \end{cases}$$

$$i_\infty = \begin{cases} 0, \; f_0 \neq +\infty \text{ and } f_\infty \neq +\infty, \\ 1, \; f_0 = +\infty \text{ and } f_\infty \neq +\infty, \text{ or } f_0 \neq +\infty \text{ and } f_\infty = +\infty, \\ 2, \; f_0 = f_\infty = +\infty. \end{cases}$$

Theorem 8. *Assume that $i_0 \neq 0$ and there exist $[a, b] \subset [0, 2\pi]$ and $r > 0$ such that $f(t, u) > 0$ for $(t, u) \in [a, b] \times [\frac{\sigma^2 r}{A_1}, \frac{r}{A_1}]$. Then there exists $\lambda^* > 0$ such that the PBVP (1) has at least i_0 positive solution(s) for any $\lambda > \lambda^*$.*

Proof. Let

$$m_0 = min\{f(t, u) : (t, u) \in [a, b] \times [\frac{\sigma^2 r}{A_1}, \frac{r}{A_1}]\}. \tag{10}$$

Obviously, $m_0 > 0$.

Choose $\lambda^* = \frac{r}{mm_0(b-a)}$. In what follows, for any $\lambda > \lambda^*$, we prove that the PBVP (1) has at least i_0 positive solution(s). Since $i_0 \neq 0$, we divide the proof into two cases:

Case 1: $i_0 = 1$. At this time, $f^0 = 0$ and $f^\infty \neq 0$, or $f^0 \neq 0$ and $f^\infty = 0$.

(i) $f^0 = 0$ and $f^\infty \neq 0$:

Since $f^0 = 0$, there exists $r_1 \in (0, \sigma r)$ such that

$$f(t,u) < \frac{A_1}{2\pi M\lambda} u,\ (t,u) \in [0,2\pi] \times (0, \frac{r_1}{A_1}]. \tag{11}$$

For any $x \in K \cap \partial\Omega_{r_1}$, in view of Lemma 7, we have

$$0 < \frac{\sigma r_1}{A_1} \le \int_0^{2\pi} H(s,\tau)x(\tau)d\tau \le \frac{r_1}{A_1},\ s \in [0,2\pi], \tag{12}$$

so, by (11) and (12), we get

$$\begin{aligned} f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) &< \frac{A_1}{2\pi M\lambda} \int_0^{2\pi} H(s,\tau)x(\tau)d\tau \\ &\le \frac{r_1}{2\pi M\lambda},\ s \in [0,2\pi], \end{aligned} \tag{13}$$

and so, it follows from Lemma 4 and (13) that

$$\begin{aligned} (T_\lambda x)(t) &= \lambda \int_0^{2\pi} G(t,s) f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds \\ &\le M\lambda \int_0^{2\pi} f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds \\ &< r_1 = \|x\|,\ t \in [0,2\pi], \end{aligned}$$

which indicates that

$$\|T_\lambda x\| < \|x\|,\ x \in K \cap \partial\Omega_{r_1}. \tag{14}$$

On the other hand, for any $x \in K \cap \partial\Omega_{\sigma r}$, in view of Lemma 7, we have

$$\frac{\sigma^2 r}{A_1} \le \int_0^{2\pi} H(s,\tau)x(\tau)d\tau \le \frac{\sigma r}{A_1} < \frac{r}{A_1},\ s \in [0,2\pi], \tag{15}$$

so, by (10) and (15), we get

$$f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) \ge m_0,\ s \in [a,b], \tag{16}$$

and so, it follows from Lemma 4 and (16) that

$$\begin{aligned} (T_\lambda x)(t) &= \lambda \int_0^{2\pi} G(t,s) f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds \\ &\ge m\lambda \int_0^{2\pi} f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds \\ &\ge m\lambda \int_a^b f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds \\ &\ge m\lambda m_0(b-a) \\ &> m\lambda^* m_0(b-a) \\ &= r > \sigma r = \|x\|,\ t \in [0,2\pi], \end{aligned}$$

which indicates that

$$\|T_\lambda x\| > \|x\|,\ x \in K \cap \partial\Omega_{\sigma r}. \tag{17}$$

Therefore, it follows from Lemma 6, (14) and (17) that T_λ has a fixed point $x_1 \in K$ satisfying

$$r_1 < \|x_1\| < \sigma r.$$

This shows that x_1 is a positive solution of the PBVP (2). By Lemma 2, we know that $u_1(t) = \int_0^{2\pi} H(t,s)x_1(s)ds$, $t \in [0, 2\pi]$ is a positive solution of the PBVP (1).

(ii) $f^0 \neq 0$ and $f^\infty = 0$:

Since $f^\infty = 0$, there exists $r_2 \in (r, +\infty)$ such that

$$f(t,u) < \frac{A_1}{2\pi M\lambda}u,\ (t,u) \in [0,2\pi] \times [\frac{\sigma r_2}{A_1}, +\infty). \tag{18}$$

For any $x \in K \cap \partial\Omega_{r_2}$, in view of Lemma 7, we have

$$\frac{\sigma r_2}{A_1} \leq \int_0^{2\pi} H(s,\tau)x(\tau)d\tau \leq \frac{r_2}{A_1},\ s \in [0,2\pi], \tag{19}$$

so, by (18) and (19), we get

$$\begin{aligned} f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) &< \frac{A_1}{2\pi M\lambda} \int_0^{2\pi} H(s,\tau)x(\tau)d\tau \\ &\leq \frac{r_2}{2\pi M\lambda},\ s \in [0,2\pi], \end{aligned} \tag{20}$$

and so, it follows from Lemma 4 and (20) that

$$\begin{aligned} (T_\lambda x)(t) &= \lambda \int_0^{2\pi} G(t,s) f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds \\ &\leq M\lambda \int_0^{2\pi} f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds \\ &< r_2 = \|x\|,\ t \in [0,2\pi], \end{aligned}$$

which indicates that

$$\|T_\lambda x\| < \|x\|,\ x \in K \cap \partial\Omega_{r_2}. \tag{21}$$

On the other hand, for any $x \in K \cap \partial\Omega_r$, in view of Lemma 7, we have

$$\frac{\sigma^2 r}{A_1} < \frac{\sigma r}{A_1} \leq \int_0^{2\pi} H(s,\tau)x(\tau)d\tau \leq \frac{r}{A_1},\ s \in [0,2\pi], \tag{22}$$

so, by (10) and (22), we get

$$f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) \geq m_0,\ s \in [a,b], \tag{23}$$

and so, it follows from Lemma 4 and (23) that

$$\begin{aligned}(T_\lambda x)(t) &= \lambda \int_0^{2\pi} G(t,s) f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds \\ &\geq m\lambda \int_0^{2\pi} f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds \\ &\geq m\lambda \int_a^{b} f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds \\ &\geq m\lambda m_0(b-a) \\ &> m\lambda^* m_0(b-a) \\ &= r = \|x\|, \ t \in [0,2\pi],\end{aligned}$$

which indicates that

$$\|T_\lambda x\| > \|x\|, \ x \in K \cap \partial\Omega_r. \tag{24}$$

Therefore, it follows from Lemma 6, (21) and (24) that T_λ has a fixed point $x_2 \in K$ satisfying

$$r < \|x_2\| < r_2.$$

This shows that x_2 is a positive solution of the PBVP (2). By Lemma 2, we know that $u_2(t) = \int_0^{2\pi} H(t,s)x_2(s)ds$, $t \in [0,2\pi]$ is a positive solution of the PBVP (1).

Case 2: $i_0 = 2$. At this time, $f^0 = f^\infty = 0$.

First, it follows from the proof of Case 1 that there exist $x_i \in K$ $(i = 1,2)$ such that

$$r_1 < \|x_1\| < \sigma r < r < \|x_2\| < r_2 \tag{25}$$

and $u_i(t) = \int_0^{2\pi} H(t,s)x_i(s)ds$, $t \in [0,2\pi]$ $(i = 1,2)$ are positive solutions of the PBVP (1).

Next, we prove that u_1 and u_2 are two different positive solutions of the PBVP (1).

In fact, by Lemma 1 and (25), we get

$$\begin{aligned}u_1(t) &= \int_0^{2\pi} H(t,s)x_1(s)ds \leq \frac{\|x_1\|}{A_1} < \frac{\sigma r}{A_1} < \frac{\sigma\|x_2\|}{A_1} \\ &\leq \int_0^{2\pi} H(t,s)x_2(s)ds = u_2(t), \ t \in [0,2\pi].\end{aligned}$$

This shows that u_1 and u_2 are two different positive solutions of the PBVP (1). □

Theorem 9. *If $i_\infty \neq 0$, then there exists $\lambda^{**} > 0$ such that the PBVP (1) has at least i_∞ positive solution(s) for any $0 < \lambda \leq \lambda^{**}$.*

Proof. Let $\gamma > 0$ be given and

$$M_0 = 1 + \max\{f(t,u) : (t,u) \in [0,2\pi] \times [\frac{\sigma^2\gamma}{A_1}, \frac{\gamma}{A_1}]\}. \tag{26}$$

Choose $\lambda^{**} = \frac{\sigma\gamma}{2\pi M M_0}$. In what follows, for any $0 < \lambda \le \lambda^{**}$, we prove that the PBVP (1) has at least i_∞ positive solution(s). Since $i_\infty \ne 0$, we divide the proof into two cases:

Case 1: $i_\infty = 1$. At this time, $f_0 = +\infty$ and $f_\infty \ne +\infty$, or $f_0 \ne +\infty$ and $f_\infty = +\infty$.

(i) $f_0 = +\infty$ and $f_\infty \ne +\infty$:

Since $f_0 = +\infty$, there exists $\gamma_1 \in (0, \sigma\gamma)$ such that

$$f(t,u) > \frac{A_1}{2\pi m\sigma\lambda}u,\ (t,u) \in [0,2\pi] \times (0, \frac{\gamma_1}{A_1}]. \tag{27}$$

For any $x \in K \cap \partial\Omega_{\gamma_1}$, in view of Lemma 7, we have

$$0 < \frac{\sigma\gamma_1}{A_1} \le \int_0^{2\pi} H(s,\tau)x(\tau)d\tau \le \frac{\gamma_1}{A_1},\ s \in [0,2\pi], \tag{28}$$

so, by (27) and (28), we get

$$\begin{aligned} f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) &> \frac{A_1}{2\pi m\sigma\lambda}\int_0^{2\pi} H(s,\tau)x(\tau)d\tau \\ &\ge \frac{\gamma_1}{2\pi m\lambda},\ s \in [0,2\pi], \end{aligned} \tag{29}$$

and so, it follows from Lemma 4 and (29) that

$$\begin{aligned} (T_\lambda x)(t) &= \lambda \int_0^{2\pi} G(t,s) f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds \\ &\ge m\lambda \int_0^{2\pi} f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds \\ &> \gamma_1 = \|x\|,\ t \in [0,2\pi], \end{aligned}$$

which indicates that

$$\|T_\lambda x\| > \|x\|,\ x \in K \cap \partial\Omega_{\gamma_1}. \tag{30}$$

On the other hand, for any $x \in K \cap \partial\Omega_{\sigma\gamma}$, in view of Lemma 7, we have

$$\frac{\sigma^2\gamma}{A_1} \le \int_0^{2\pi} H(s,\tau)x(\tau)d\tau \le \frac{\sigma\gamma}{A_1} < \frac{\gamma}{A_1},\ s \in [0,2\pi], \tag{31}$$

so, by (26) and (31), we get

$$f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) < M_0,\ s \in [0,2\pi], \tag{32}$$

and so, it follows from Lemma 4 and (32) that

$$\begin{aligned}(T_\lambda x)(t) &= \lambda \int_0^{2\pi} G(t,s) f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds \\ &\le M\lambda \int_0^{2\pi} f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds \\ &< 2\pi M\lambda M_0 \\ &\le 2\pi M\lambda^{**} M_0 \\ &= \sigma\gamma = \|x\|, \ t \in [0, 2\pi],\end{aligned}$$

which indicates that

$$\|T_\lambda x\| < \|x\|, \ x \in K \cap \partial\Omega_{\sigma\gamma}. \tag{33}$$

Therefore, it follows from Lemma 6, (30) and (33) that T_λ has a fixed point $x_1 \in K$ satisfying

$$\gamma_1 < \|x_1\| < \sigma\gamma.$$

This shows that x_1 is a positive solution of the PBVP (2). By Lemma 2, we know that $u_1(t) = \int_0^{2\pi} H(t,s)x_1(s)ds, \ t \in [0, 2\pi]$ is a positive solution of the PBVP (1).

(ii) $f_0 \ne +\infty$ and $f_\infty = +\infty$:

Since $f_\infty = +\infty$, there exists $\gamma_2 \in (\gamma, +\infty)$ such that

$$f(t,u) > \frac{A_1}{2\pi m\sigma\lambda} u, \ (t,u) \in [0, 2\pi] \times [\frac{\sigma\gamma_2}{A_1}, +\infty). \tag{34}$$

For any $x \in K \cap \partial\Omega_{\gamma_2}$, in view of Lemma 7, we have

$$\frac{\sigma\gamma_2}{A_1} \le \int_0^{2\pi} H(s,\tau)x(\tau)d\tau \le \frac{\gamma_2}{A_1}, \ s \in [0, 2\pi], \tag{35}$$

so, by (34) and (35), we get

$$\begin{aligned}f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) &> \frac{A_1}{2\pi m\sigma\lambda} \int_0^{2\pi} H(s,\tau)x(\tau)d\tau \\ &\ge \frac{\gamma_2}{2\pi m\lambda}, \ s \in [0, 2\pi],\end{aligned} \tag{36}$$

and so, it follows from Lemma 4 and (36) that

$$\begin{aligned}(T_\lambda x)(t) &= \lambda \int_0^{2\pi} G(t,s) f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds \\ &\ge m\lambda \int_0^{2\pi} f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds \\ &> \gamma_2 = \|x\|, \ t \in [0, 2\pi],\end{aligned}$$

which indicates that

$$\|T_\lambda x\| > \|x\|, \ x \in K \cap \partial\Omega_{\gamma_2}. \tag{37}$$

On the other hand, for any $x \in K \cap \partial\Omega_\gamma$, in view of Lemma 7, we have

$$\frac{\sigma^2\gamma}{A_1} < \frac{\sigma\gamma}{A_1} \le \int_0^{2\pi} H(s,\tau)x(\tau)d\tau \le \frac{\gamma}{A_1},\ s \in [0,2\pi], \tag{38}$$

so, by (26) and (38), we get

$$f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) < M_0,\ s \in [0,2\pi], \tag{39}$$

and so, it follows from Lemma 4 and (39) that

$$\begin{aligned}
(T_\lambda x)(t) &= \lambda \int_0^{2\pi} G(t,s) f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds \\
&\le M\lambda \int_0^{2\pi} f\left(s, \int_0^{2\pi} H(s,\tau)x(\tau)d\tau\right) ds \\
&< 2\pi M\lambda M_0 \\
&\le 2\pi M\lambda^{**} M_0 \\
&= \sigma\gamma < \gamma = \|x\|,\ t \in [0,2\pi],
\end{aligned}$$

which indicates that

$$\|T_\lambda x\| < \|x\|,\ x \in K \cap \partial\Omega_\gamma. \tag{40}$$

Therefore, it follows from Lemma 6, (37) and (40) that T_λ has a fixed point $x_2 \in K$ satisfying

$$\gamma < \|x_2\| < \gamma_2.$$

This shows that x_2 is a positive solution of the PBVP (2). By Lemma 2, we know that $u_2(t) = \int_0^{2\pi} H(t,s)x_2(s)ds,\ t \in [0,2\pi]$ is a positive solution of the PBVP (1).

Case 2: $i_\infty = 2$. At this time, $f_0 = f_\infty = +\infty$.

First, it follows from the proof of Case 1 that there exist $x_i \in K$ $(i = 1,2)$ such that

$$\gamma_1 < \|x_1\| < \sigma\gamma < \gamma < \|x_2\| < \gamma_2 \tag{41}$$

and $u_i(t) = \int_0^{2\pi} H(t,s)x_i(s)ds,\ t \in [0,2\pi]$ $(i = 1,2)$ are positive solutions of the PBVP (1).

Next, we prove that u_1 and u_2 are two different positive solutions of the PBVP (1).

In fact, by Lemma 1 and (41), we get

$$\begin{aligned}
u_1(t) &= \int_0^{2\pi} H(t,s)x_1(s)ds \le \frac{\|x_1\|}{A_1} < \frac{\sigma\gamma}{A_1} < \frac{\sigma\|x_2\|}{A_1} \\
&\le \int_0^{2\pi} H(t,s)x_2(s)ds = u_2(t),\ t \in [0,2\pi].
\end{aligned}$$

This shows that u_1 and u_2 are two different positive solutions of the PBVP (1). □

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