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THE SOLUTION OF THE HEAT EQUATION WITHOUT BOUNDARY CONDITIONS

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ABSTRACT: We would like to propose the solution of the heat equation without boundary conditions. The methodology used is Laplace transform approach, and the transform can be changed another ones. This attempt is more advanced than the existing method and has a meaning in that it is approached in a general way without restricting the boundary conditions. The solution of heat equation is presented by using the property of integrability of transform.

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1. INTRODUCTION

The solution of the partial differential equations is a general method to obtain the solutions by using Fourier series which is the basis of Fourier transform, which is one of the integral transforms. The efforts have been pursued to solve the PDEs using other integral transform other than Fourier series. A representative example of these transforms is Laplace transform, and Laplace-typed transform can be used as an alternative to Laplace transform. This-typed integral transforms are mainly Sumudu[1-3], Elzaki[4-6] and G-transform[7-8], and these transforms are not yet well-resolved to solve the heat equation. For this reason, this research is deemed necessary.

To find the solution of the heat equation in which the boundary conditions are not given is not well treated in this way so far. In this research, the solution of the heat equation in which the boundary conditions are not given is obtained by using Laplace transform. The strong point of integral transform is well-represented in computed tomography (CT) or magnetic resonance imaging (MRI) which obtain the projection data by integral transform and produce the image with the inverse transform.

Ang [9] has dealt with the problem of solving the one-dimensional heat equation subject to given initial and nonlocal conditions. Using the method of the Laplace transform, Mainardi[10] showed that the fundamental solutions of the basic Cauchy and signaling problems can be expressed in terms of an auxiliary function. In the process of solving this diffusion problem numerically, the Laplace transform method is used to eliminate the dependence on time[11]. On the other hand, Begacem[12] has mentioned applications of integral transform in nonlinear fractional and stochastic differential equations and systems.

The main objective of this paper is to give a solution of the heat equation without boundary conditions by using the method of integral transforms. This study is meaningful in that it suggests points to be considered in existing studies.

Theorem 5 deals with finding the solution using Laplace-typed transforms when boundary conditions are not restricted. The obtained solution is as follows; We consider

$$\frac{\partial u}{\partial t} = c^2 \, \frac{\partial^2 w}{\partial x^2} \tag{1}$$

subject to $u(x,0) = f(x)$ where $u(x,y,z,t)$ is the temperature at a point (x,y,z) and time t, and $f(x)$ is the initial temperature. Of course, $c^2 = K/\rho\sigma$, where K is the thermal conductivity, ρ is the density, and σ is the specific heat. Then the solution $u(x,t)$ equals to

$$\frac{c}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(|x-\tau|^2/4tc^2)} f(\tau)\ d\tau\ .$$

If we changed the tool of transform to G-transform, a generalized Laplace-typed transform, then the representation of transform has to be changed to

$$\frac{c}{2} U^{\alpha+\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{c\sqrt{u}}|x-\tau|} f(\tau)\ d\tau$$

for an integer α and for $U = G(u)$.

2. THE SOLUTION OF THE HEAT EQUATION WITHOUT BOUNDARY CONDITIONS

Lemma 1. *(Boundedness theorem) Let $I = [a,b]$ be a closed bounded interval and let $f : I \to R$ be continuous on I. Then f is bounded on I [13].*

Lemma 2. *Let V_1 and V_2 be normed linear spaces, and let $T : V_1 \to V_2$ be a linear operator. Then T is continuous if and only if there is a non-negative number A such* *that*

$$||T(v)|| \leq A||v||$$

holds for each v in V_1[14].

Lemma 3. *(Lebesgue dominated convergence theorem(LDCT) [14]). Let (X, M, μ) be a measure space and suppose $\{f_n\}$ is a sequence of extended real-valued measurable functions defined on X such that*

a) $\lim_{n\to\infty} f_n(x) = f(x)$ *exists μ-a.e.*

b) There is an integrable function g so that for each n, $|f_n| \leq g$ μ-a.e.

Then f is integrable and

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

We note that the above lemma gives a validity to the following equality

$$\int \sum_{n=1}^{\infty} g_n \; d\mu = \sum_{n=1}^{\infty} \int g_n \; d\mu$$

for (g_n) is a nondecreasing sequence.

As a study of comprehensive forms, we have proposed the intrinsic structure and properties of Laplace-typed integral transforms in [7] as

$$F(u) = G(f) = u^{\alpha} \int_0^{\infty} e^{-\frac{t}{u}} f(t)dt \tag{1}$$

for α is an integer and for G is a generalized integral transform. In general, Laplace transform has a strong point in the transforms of derivatives, that is, the differentiation of a function $f(t)$ corresponds to multiplication of its transform $\mathcal{L}(f)$ by s. While, if we choose $G_{-2}(f)$ as

$$G_{-2}(f) = \frac{1}{u^2} \int_0^{\infty} e^{-\frac{t}{u}} f(t)dt, \tag{2}$$

then this transform is giving a simple tool for transforms of integrals. That is, the integration of a function $f(t)$ corresponds to multiplication of $G_{-2}(f)$ by u, whereas, the differentiation of $f(t)$ corresponds to division of $G_{-2}(f)$ by u. This means that the integer α is applicable to -2 in (1). This transform is meaningful in that it can select values of various α. In (2), Laplace transform has a value $\alpha = 0$, Sumudu one has $\alpha = -1$, and Elzaki one has $\alpha = 1$.

Lemma 4. *The following properties are valid in G-transform[8].*

(A) (u-shifting) If $f(t)$ has the transform $F(u)$, then $e^{at}f(t)$ has the transform

$$F(\frac{u}{1-au}).$$

That is,

$$G_{-2}[e^{at}f(t)] = F(\frac{u}{1-au}).$$

(B) (t-shifting) If $f(t)$ has the transform $F(u)$, then the shifted function $f(t-a)h(t-a)$ has the transform $e^{-a/u}F(u)$. In formulas,

$$G_{-2}[f(t-a)h(t-a)] = e^{-a/u}F(u)$$

for $h(t-a)$ is Heaviside function(We write h since we need u to denote u-space.)

(Ca)

$$G_{-2}(f') = \frac{1}{u}Y - \frac{1}{u^2}f(0)$$

(Cb)

$$G_{-2}(f'') = \frac{1}{u^2}Y - \frac{1}{u^3}f(0) - \frac{1}{u^2}f'(0)$$

for $Y = G_{-2}(f)$ and for f is n-th differentiable.

(Cc)

$$G_{-2}(f^{(n)}) = \frac{1}{u^n}Y - \frac{1}{u^{n+1}}f(0) - \frac{1}{u^n}f'(0) - \frac{1}{u^{n-1}}f''(0) - \cdots$$
$$-\frac{1}{u^2}f^{(n-1)}(0)$$

for n is an arbitrary natural number.

(D) Let $F(u)$ denote the transform of an integrable function $f(t)$ i.e., $F(u) = G_{-2}[f(t)]$. Then

$$G_{-2}[\int_0^t f(\tau)d\tau] = uF(u)$$

holds for $t > 0$.

(E)

$$G_{-2}(f * g) = u^2 G_{-2}(f)G_{-2}(g)$$

where $*$ is the convolution of f and g.

(Fa)

$$G_{-2}(f)'(u) = \frac{dG}{du} = -\frac{2}{u}Y + \frac{1}{u^2}G_{-2}(tf(t))$$

(Fb)

$$G_{-2}(f)''(u) = \frac{6}{u^2}Y - \frac{2}{u}G_{-2}(tf(t))(1 + \frac{1}{u^2}) + \frac{1}{u^2}G_{-2}(t^2 f(t))$$

(Fc)

$$\int_u^\infty G_{-2}(f)(s)ds = a - u^2 G_{-2}(\frac{f(t)}{t})$$

for a constant $a = \int_0^\infty f(t)/t\ dt$ *and for* $Y = G_{-2}(f)(u)$ *under the condition of the limit of* $f(t)/t$*, as* t *approaches* 0 *from the right, exists.*

(Fd) $G_{-2}(ty') = Y + u(dY/du)$ *and* $G_{-2}(ty'') = dY/du + (1/u^2)y(0)$ *for* $Y = G_{-2}(f)(u)$.

From now on, let us check the solution of heat equation without the boundary conditions.

Theorem 5. ***(heat equation without the boundary conditions)***

Let us consider the heat equation

$$\frac{\partial u}{\partial t} = c^2\ \frac{\partial^2 w}{\partial x^2} \tag{1}$$

subject to $u(x, 0) = f(x)$ *where* $u(x, y, z, t)$ *is the temperature at a point* (x, y, z) *and time* t*, and* $f(x)$ *is the initial temperature. Of course,* $c^2 = K/\rho\sigma$*, where* K *is the thermal conductivity,* ρ *is the density, and* σ *is the specific heat. Then the solution* $u(x, t)$ *equals to*

$$\frac{c}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(|x-\tau|^2/4tc^2)} f(\tau)\ d\tau\ .$$

In this equation, if we changed the tool of transform to G-transform, a generalized Laplace-typed transform, then the representation of transform has to be changed to

$$\frac{c}{2} U^{\alpha+\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{c\sqrt{u}}|x-\tau|} f(\tau)\ d\tau$$

for an integer α *and for* $U = G(u)$.

Proof. Taking Laplace transform on both sides, we have

$$sU(x, s) - u(x, 0) = c^2 \frac{\partial^2 U}{\partial x^2}$$

for $U(x, s) = \pounds[u(x, t)]$. Organizing the equality, we have

$$c^2 \frac{\partial^2 U}{\partial x^2} - sU(x, s) = -f(x). \tag{4}$$

At first, let us find the solution U_h of

$$c^2 \frac{\partial^2 U}{\partial x^2} - sU(x, s) = 0.$$

Since the trial solution is $u = e^{\lambda x}$, we can easily find the solution U_h of the form of

$$U_h(x, s) = c_1 e^{\sqrt{s}x/c} + c_2 e^{-\sqrt{s}x/c}. \tag{5}$$

Next, let us find the particular solution U_p of (5). Clearly, the Wronskian of bases is $-2\sqrt{s}/c$, and so,

$$\begin{aligned} U_p(x,s) &= -e^{\frac{\sqrt{s}x}{c}} \int_0^x \frac{e^{-\frac{\sqrt{s}\tau}{c}}(-f(\tau))}{\frac{-2\sqrt{s}}{c}}\, d\tau + e^{-\frac{\sqrt{s}x}{c}} \int_0^x \frac{e^{\frac{\sqrt{s}\tau}{c}}(-f(\tau))}{\frac{-2\sqrt{s}}{c}}\, d\tau \\ &= -\frac{c}{2\sqrt{s}} \cdot e^{\frac{\sqrt{s}x}{c}} \int_0^x e^{-\frac{\sqrt{s}\tau}{c}} f(\tau)\, d\tau + \frac{c}{2\sqrt{s}} e^{-\frac{\sqrt{s}x}{c}} \int_0^x e^{\frac{\sqrt{s}\tau}{c}} f(\tau)\, d\tau \qquad (6) \end{aligned}$$

From (4) and (5), we have

$$\begin{aligned} U(x,s) &= c_1 e^{\frac{\sqrt{s}x}{c}} + c_2 e^{-\frac{\sqrt{s}x}{c}} - \frac{c}{2\sqrt{s}} \cdot e^{\frac{\sqrt{s}x}{c}} \int_0^x e^{-\frac{\sqrt{s}\tau}{c}} f(\tau)\, d\tau \\ &\quad + \frac{c}{2\sqrt{s}} e^{-\frac{\sqrt{s}x}{c}} \int_0^x e^{\frac{\sqrt{s}\tau}{c}} f(\tau)\, d\tau \\ &= (c_1 - \frac{c}{2\sqrt{s}} \int_0^x e^{-\frac{\sqrt{s}\tau}{c}} f(\tau)\, d\tau) \cdot e^{\frac{\sqrt{s}x}{c}} + (c_2 + \frac{c}{2\sqrt{s}} \int_0^x e^{\frac{\sqrt{s}\tau}{c}} f(\tau)\, d\tau) \cdot e^{-\frac{\sqrt{s}x}{c}}. \end{aligned}$$

Note that the integrability of U ensures the boundedness of it. Since U is bounded as $x \to \infty$,

$$c_1 = \frac{c}{2\sqrt{s}} \int_0^\infty e^{\frac{\sqrt{s}\tau}{c}} f(\tau)\, d\tau.$$

Similarly, since U is bounded as $x \to -\infty$,

$$c_2 = -\frac{c}{2\sqrt{s}} \int_0^{-\infty} e^{\frac{\sqrt{s}\tau}{c}} f(\tau)\, d\tau = \frac{c}{2\sqrt{s}} \int_{-\infty}^0 e^{\frac{\sqrt{s}\tau}{c}} f(\tau)\, d\tau.$$

Thus

$$\begin{aligned} U(x,s) &= \frac{c}{2\sqrt{s}} e^{\frac{\sqrt{s}x}{c}} \int_x^\infty e^{-\frac{\sqrt{s}x}{c}} f(\tau)\, d\tau + \frac{c}{2\sqrt{s}} e^{-\frac{\sqrt{s}x}{c}} \int_{-\infty}^x e^{\frac{\sqrt{s}\tau}{c}} f(\tau)\, d\tau \\ &= \frac{c}{2\sqrt{s}} \int_x^\infty e^{\frac{\sqrt{s}}{c}(x-\tau)} f(\tau)\, d\tau + \frac{c}{2\sqrt{s}} \int_{-\infty}^x e^{-\frac{\sqrt{s}}{c}(x-\tau)} f(\tau)\, d\tau \\ &= \frac{c}{2\sqrt{s}} \int_{-\infty}^\infty e^{-\frac{\sqrt{s}}{c}|x-\tau|} f(\tau)\, d\tau \qquad (7) \end{aligned}$$

Since

$$\pounds\left(\frac{e^{-\frac{a^2}{4t}}}{\sqrt{\pi t}}\right) = \frac{e^{-a\sqrt{s}}}{\sqrt{s}},$$

setting $a = |x-\tau|/c$, we have

$$\pounds\left(\frac{e^{-\frac{|x-\tau|^2}{4tc^2}}}{\sqrt{4\pi t}}\right) = \frac{e^{\frac{-\sqrt{s}}{c}}|x-\tau|}{2\sqrt{s}},$$

and hence,

$$u(x,t) = \pounds^{-1}(U(x,s)) = \pounds^{-1}\left(\frac{c}{2\sqrt{s}} \int_{-\infty}^\infty e^{-\frac{\sqrt{s}}{c}(x-\tau)} f(\tau)\, d\tau\right)$$

$$= c \int_{-\infty}^{\infty} \pounds^{-1}\left(\frac{e^{-\frac{\sqrt{s}}{c}(x-\tau)}}{2\sqrt{s}}\right) f(\tau)\ d\tau$$

$$= \frac{c}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(|x-\tau|^2/4tc^2)} f(\tau)\ d\tau\ .$$

In this theorem, the tool of transform can be changed to another transform such as a generalized transform, G-transform. Then the above (7) has to be changed to

$$\frac{c}{2} U^{\alpha+\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{c\sqrt{u}}|x-\tau|} f(\tau)\ d\tau$$

for an suitable integer α and for $U = G(u)$. Note that $F(u)$ is defined by

$$F(u) = G(f) = u^{\alpha} \int_0^{\infty} e^{-\frac{t}{u}} f(t) dt.$$

□

In theorem 5, if the temperature $u(x, y, z, t)$ is a continuous on $0 \le u \le L$, by lemma 1, u is bounded on the interval. The boundedness of the transform U follows from the integrability of it. Let us check some related things.

Corollary 6. *In theorem 5, if the temperature*

$$u(x,t) = \frac{c}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(|x-\tau|^2/4tc^2)} f(\tau)\ d\tau$$

is a continuous on $0 \le u \le L$*, then* $||u||_1$ *is a norm and* u *is bounded.*

Proof. Since a continuous function on $[0, L]$ that vanishes almost everywhere must vanish everywhere, we have $f = 0$, and so $u = 0$. The remaining conditions of normed space are clearly established. On the other hand, the linearity of u is established from the linearity of f. Hence, by lemma 2, u is bounded. □

Example 1. Let us consider a semi-infinite insulated bar

$$\frac{\partial u}{\partial t} = c^2\ \frac{\partial^2 w}{\partial x^2}$$

subject to $w(x, 0) = T_0$, $w(0, t) = 0$ and $w(x, t) = 0$ as x approaches 0.

Solution. In a similar way to theorem 5, we obtain the solution

$$w(x,t) = T_0\ erf(x/2c\sqrt{t})$$

where

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-t^2}\ dt.$$

In the above example, $erf(x)$ is called the error function.

Of course, Laplace transform can be changed to another transform, and it is well adapted to wave equation. Hence, now that let us change to integral transform to a generalized transform, G-transform[7]. Let us check the solution of semi-infinite string by G-transform in terms of a typical example as appearing in [15].

We consider the semi-infinite string subject to the following conditions.

a) The string is initially at rest on the x-axis from $x = 0$ to ∞.

b) For $t > 0$ the left end of the string is moved in a given fashion, namely, according to a single sine wave $w(0,t) = f(t) = \sin t$ if $0 \leq t \leq 2\pi$, and zero otherwise.

c) Furthermore, $\lim w(x,t) = 0$ as $x \to \infty$ for $t \geq 0$.

Of course there is no infinite string, but our model describes a long string or rope(of negligible weight) with its right end fixed far out on the x-axis[15]. If so, let us find the displacement $w(x,t)$ of the above elastic string. It is well-known fact that the equation of semi-infinite string can be expressed by

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \tag{8}$$

subject to $w(0,t) = f(t)$, $\lim\ w(x,t) = 0$ as $x \to \infty$, $w(x,0) = 0$ and $w_t(x,0) = 0$.

In [16, 17], we have dealt with the validity on exchangeability of integral and limit in the solving process of PDEs by using dominated convergence theorem of lemma 3.

Example 2. We consider the above equation (8). Taking G-transform, we have the solution

$$w(x,t) = f(t - \frac{x}{c})h((t - \frac{x}{c}) = \sin(t - \frac{x}{c})$$

for $\frac{x}{c} < t < \frac{x}{c} + 2\pi$ and zero otherwise, where h is Heaviside function[7].

Let us check another heat equation without boundary conditions.

Example 3. Let us consider

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 w}{\partial x^2}$$

subject to $u(x,0) = 3sin(2\pi x)$.

Solution. Taking Laplace transform on both sides, we have

$$U(x,s) = c_1 e^{\frac{\sqrt{s}x}{c}} + c_2 e^{-\frac{\sqrt{s}x}{c}} + \frac{3}{s + 4\pi^2}\ sin(2\pi x).$$

The boundedness of U follows from the integrability of U. Hence, as $x \to -\infty$, we have $c_2 = 0$ and as $x \to \infty$, $c_1 = 0$. Thus

$$U(x,s) = \frac{3}{s + 4\pi^2}\ sin(2\pi x),$$

and so

$$u(x,t) = 3e^{-4\pi^2 t} sin(2\pi x).$$

REFERENCES

[1] G. K. Watugula, Sumudu Transform: a new integral transform to solve differential equations and control engineering problems, *Int. J. of Math. Edu. in Sci. & Tech.*, **24**, (1993), 409-421.

[2] H. Eltayeb, A. Kilicman, and M. B. Jleli, Fractional Integral Transform and Application, *Abst. & Appl. Anal.*, **2015**. (2015), 1-2.

[3] F. B. M. Belgacem and A. A. Karaballi, Sumudu transform fundamental properties investigations and applications, *J. Appl. Math. & Stocha. Anal.*, **2006**, (2006), 1-23.

[4] T. M. Elzaki and J. Biazar, Homotopy perturbation method and Elzaki transform for solving system of nonlinear partial differential equations, *Wor. Appl. Sci. J.*, **7**, (2013), 944-948.

[5] Tarig M. Elzaki and Hj. Kim, The solution of radial diffusivity and shock wave equations by Elzaki variational iteration method, *Int. J. of Math. Anal.*, **9**, (2015), 1065-1071.

[6] Hj. Kim, The time shifting theorem and the convolution for Elzaki transform, *Int. J. of Pure & Appl. Math.*, **87**, (2013), 261-271.

[7] Hj. Kim, The Intrinsic Structure and Properties of Laplace-Typed Integral Transforms, *Mathematical Problem in Engineering* **2017**, (2017),1–8.

[8] Hj. Kim, On the form and properties of an integral transform with strength in integral transforms. *Far East J. Math. Sci.* **102**, (2017), 2831–2844.

[9] W. T. Ang, A Method of Solution for the One-Dimensional Heat Equation Subject to Nonlocal Conditions, *Southeast Asian Bulletin of Mathematics* **26**, (2003),185–191.

[10] Mainardi, F. "The fundamental solutions for the fractional diffusion-wave equation, *Applied Mathematics Letters* **9**, (1996), 23-28.

[11] Sutradhar, A., Paulino, G. H., & Gray, L. J. (2002). Transient heat conduction in homogeneous and non-homogeneous materials by the Laplace transform Galerkin

boundary element method, *Engineering Analysis with Boundary Elements*, **26**, 119-132.

[12] F. B. M. Belgacem and S. Sivasundaram, New developments in computational techniques and transform theory applications to nonlinear fractional and stochastic differential equations and systems, *Nonlinear Studies*, **22**, (2015), 561-563.

[13] R. G. Bartle and D. R. Sherbert, *Real Analysis*, John Wiley & Sons, Inc., New York (1982).

[14] D. L. Cohn, *Measure theory*, Birkhäuser, Boston (1980).

[15] E. Kreyszig, *Advanced Engineering Mathematics*, Singapore, Wiley (2013).

[16] Hc. Chae and Hj. Kim, The Validity Checking on the Exchange of Integral and Limit in the Solving Process of PDEs, *Int. J. of Math. Anal.*, **8** (2014), 1089-1092.

[17] Jy. Jang and Hj. Kim, An application of monotone convergence theorem in PDEs and Fourier analysis, *Far East J. Math. Sci.*, **98**, (2015), 665-669.