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SOLUTION PROFILES BEYOND QUENCHING FOR SINGULAR SEMILINEAR PARABOLIC PROBLEMS

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ABSTRACT: Let $T \leq \infty$, $a > 0, 0 < r < 1$, $D = (0, a)$, $\Omega = D \times (0, T]$ and $\chi(S)$ be the characteristic function of the set S. This article studies the steady-state solution after quenching has occured of the semilinear parabolic equation with singularity

$$\begin{aligned}
&u_t - u_{xx} - \frac{r}{x} u_x = f(u)\chi(\{u < c\}) \text{ in } \Omega,\\
&u(x, 0) = 0 \text{ on } \bar{D},\\
&u(0, t) = 0 = u(a, t) \text{ for } 0 < t < T.
\end{aligned}$$

It is shown that as t tends to ∞, all weak solutions $u\,(x, t)$ tend to a unique steady-state solution $U(x)$ for $0 < b_s^* \leq x \leq B_s^* < a$. The numerical methods are developed for computing b_s^* and B_s^*.

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1. INTRODUCTION

Quenching concept was first introduced by Kawarada [12] in 1975. After that, quenching phenomena, beyond quenching and quenching profiles have been studied by many researchers (e.g. Acker and Walter [1], 1976; Chan and Kaper [8], 1989; Chan and Ke [9], 1994; Escobar [4], 2007, and references cited there).

Let $T \leq \infty,\ a > 0,\ 0 < r < 1,\ D = (0, a),\ \Omega = D \times (0, T]$ and

$$\chi(S) = \begin{cases} 1 & ; \text{if } u \in S, \\ 0 & ; \text{if } u \notin S, \end{cases}$$

be the characteristic function of the set S. This article studies the steady-state solution after quenching has occurred of the semilinear parabolic equation with singularity

$$Mu = u_t - u_{xx} - \frac{r}{x}u_x = f(u)\chi(\{u < c\}) \text{ in } \Omega, \tag{1}$$

$$u(x, 0) = 0 \text{ on } \bar{D}, \tag{2}$$

$$u(0, t) = 0 = u(a, t) \text{ for } 0 < t < T, \tag{3}$$

where f is a twice continuously differentiable function on $[0, c)$ for some constant c with $f(0) > 0,\ f' > 0,\ f'' \geq 0$ and $\lim_{u \to c^-} f(u) = \infty$.

The linear operator $M = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} - \frac{r}{x}\frac{\partial}{\partial x}$ represents several physical phenomena. For example, if r is a noninteger, it can be regarded as a first step in the approach to a theory of generalized axially symmetric heat potentials (cf. Alexiades [2] and Arena [3]). For $0 \leq r \leq 2$, it can describe the conduction of heat in a body with r being a geometric parameter concerning to the shape of the body; for instance, $r = 0.5$ refers to heat transfer into one face of a flat cylinder with a small ratio of depth to diameter (cf. Solomon [13]). Under some appropriate transformations, the operator M can be transformed to operators which correspond to a stochastic process and problems in the theory of probability (cf. Chan and Chen [7]).

A solution u is said to *quench* if there exists an extended real number $t_q \in (0, \infty]$ such that

$$\max\{u(x, t) : 0 \leq x \leq a\} \to c^- \text{ as } t \to t_q$$

(cf. Chan and Liu [11]). If t_q is finite, then u is said to *quench in a finite time*. On the other hand, if $t_q = \infty$, then u is said to *quench in infinite time*.

Let $Hu = x^r u_t - (x^r u_x)_x$. Then, the problem (1) with the conditions (2) and (3) can be rewritten as the following problem:

$$\left.\begin{aligned} &Hu = x^r u_t - (x^r u_x)_x = x^r f(u)\chi(\{u < c\}) \text{ in } \Omega, \\ &u(x, 0) = 0 \text{ on } \bar{D}, \\ &u(0, t) = 0 = u(a, t) \text{ for } 0 < t \leq T. \end{aligned}\right\} \tag{4}$$

Definition 1. A function u is a *weak solution* of (4) if and only if

(i) $u \in C([0,t_0]; L^1((0,a))) \cap L^\infty((0,a)\times(0,t_0))$ for each $t_0 > 0$;

(ii) for any $g \in C^{2,1}(\bar{\Omega})$ such that g has a compact support with respect to t and $g(0,t) = 0 = g(a,t)$,

$$\int_0^\infty \int_0^a uH^*g\; dxdt + \int_0^\infty \int_0^a x^r f(u)\chi(\{u<c\})g\; dxdt = 0, \tag{5}$$

where $H^* = x^r \frac{\partial}{\partial t} + \frac{\partial}{\partial x}\left(x^r \frac{\partial}{\partial x}\right)$ is the adjoint operator of H.

By hiring the idea of [8] for a problem with one insulated boundary condition to the problem (4), where $f(u)\,\chi(\{u<c\})$ in (4) is being replaced with $f(u)$, we can deduce that there exists a critical length a^* such that the solution may exist for all $t > 0$ if $a < a^*$, but the solution approaches c in a finite time if $a > a^*$. Let τ be the first finite quenching time of such problem. By modifying the techniques presented in [10] with the assumptions

$$f'(\mu)\left(\frac{c-\mu}{f(\mu)}\right)^2 \le K_1 \text{ and } \int_\mu^c f(s)ds \le \min\{K_2(c-\mu)f(\mu), K_3(c-\mu)^\gamma\},$$

for $0 \le \mu < c$ and some positive constants K_1, K_2, K_3 and γ is such that $0 < \gamma < 2$, we have the following theorem. As the proof to this theorem is analogous to the proof presented in [10], we omit the details.

Theorem 2. *A weak solution u of (4) exists and it has the following properties*

(i) $u(x,t) \in C^{2,1}(\{u<c\} \cap \Omega) \cap C^{1,0}(\Omega) \cap C(\bar{\Omega})$*;*

(ii) $u \le c$ *in* Ω*;*

(iii) If $u(x,t_0) = c$ *for some* $x \in (0,a)$ *and* $t_0 \in [\tau,\infty)$*, then* $u(x,t) = c$ *for* $t \in [t_0,\infty)$*;*

(iv) u *is nondecreasing with respect to* t *in* $\{u<c\} \cap \Omega$*;*

(v) $u_x = 0$ *at the point* (x,t) *where* $u(x,t) = c$*.*

In Section 2, we prove that as t tends to infinity, all weak solutions of (4) tend to the unique solution of the steady-state problem

$$W(x) = c \text{ for } b_s^* \le x \le B_s^*, \tag{6}$$

$$-(x^r W'(x))' = x^r f(W(x)) \text{ for } 0 < x < b_s^*,\ W(0) = 0,\ W(b_s^*) = c, \tag{7}$$

$$-(x^r W'(x))' = x^r f(W(x)) \text{ for } B_s^* < x < a,\ W(B_s^*) = c,\ W(a) = 0, \tag{8}$$

where b_s^* and B_s^* are positive constants to be calculated. These constants determine the solution profile beyond quenching to the problem (4). We disscuss the bounds for b_s^* and B_s^*. Due to the singularity term, $(x^r u_x)_x$, the exact values for b_s^* and B_s^* cannot be obtained. Therefore, the numerical methods for computing them and some numerical examples are given in Section 3. Finally, conclusion and discussion are made in Section 4.

2. BEYOND QUENCHING

The main purpose of this section is to study the behavior of all solutions of the problem (4) beyond the finite quenching time τ. Let u denote any weak solution of (4). For $t \geq \tau$, let $b(t) = \inf\{x : u(x,t) = c\}$, $B(t) = \sup\{x : u(x,t) = c\}$, $b^* = \lim_{t\to\infty} b(t)$ and $B^* = \lim_{t\to\infty} B(t)$. A proof similar to those of Lemmas 2 and 6 of Chan and Ke [9] gives the following result.

Lemma 1. *The function $b(t)$ is nonincreasing while the function $B(t)$ is nondecreasing; furthermore, $b^* \geq b_s^* > 0$ and $B^* \leq B_s^* < a$.*

Since $u(\leq c)$ is continuous and nondecreasing with respect to t, it follows from the Dini Theorem ([14], p.143) that $u(x,t)$ converges uniformly to the limit, $\lim_{t\to\ \infty} u(x,t)$, which is continuous on $[0,a]$. Let $U(x) = \lim_{t\to\infty} u(x,t)$.

Intuitively, Lemma 2.1 implies that, as $t \to \infty$, the portion in the domain $(0,a)$ where u reaches c is broadened. However, since our problem concerns the Dirichlet boundary conditions at both ends, u cannot reach c on the entire domain. Thus, the solution profile separates into three segments determined by b_s^* and B_s^* as shown in the following lemma.

Lemma 2. *(i) For $x \in (0,b^*)$, $u(x,t)$ converges uniformly to a solution of (7) as $t \to \infty$ with $b^* = b_s^*$.*

(ii) For $x \in (B^*,a)$, $u(x,t)$ converges uniformly to a solution of (8) as $t \to \infty$ with $B^* = B_s^*$.

(iii) For $x \in [b_s^*, B_s^*]$, $U(x) \equiv c$.

(iv) $U_s'(b_s^*) = 0 = U'(B_s^*)$.

Proof. (i) Let us consider u in the region $[0,\tilde{b}] \times (0,\infty)$ where $\tilde{b} \in [0,b^*]$. Let

$$F(x,t) = \int_0^{\tilde{b}} \rho^r \tilde{G}(x;\rho)\, u(\rho,t)\, d\rho,$$

where $\tilde{G}(x;\rho)$ is Green's function corresponding to (7) with b_s^* being replaced by $\tilde{b}$. Note that

$$\tilde{G}(x;\rho)=\begin{cases}\dfrac{x^{1-r}\left(\tilde{b}^{1-r}-\rho^{1-r}\right)}{\tilde{b}^{1-r}(1-r)} & \text{for } 0\le x\le\rho,\\[2ex] \dfrac{\rho^{1-r}\left(\tilde{b}^{1-r}-x^{1-r}\right)}{\tilde{b}^{1-r}(1-r)} & \text{for } \rho<x\le\tilde{b}.\end{cases}$$

Since the operator is self-adjoint, $\tilde{G}(x;\rho)=\tilde{G}(\rho;x)$. We have

$$\begin{aligned}F_t(x,t)&=\int_0^{\tilde{b}}\rho^r\tilde{G}(x;\rho)\,u_t(\rho,t)\,d\rho\\ &=\int_0^{\tilde{b}}\tilde{G}(x;\rho)\left(\rho^r u_\rho(\rho,t)\right)_\rho d\rho+\int_0^{\tilde{b}}\tilde{G}(x;\rho)\,\rho^r f(u(\rho,t))\,d\rho.\end{aligned}$$

Using integration by parts to the first integral on the right hand side of the above equation, we obtain

$$F_t(x,t)=-u(\tilde{b},t)\rho^r\tilde{G}_\rho(\tilde{b};x)-u(x,t)+\int_0^{\tilde{b}}\tilde{G}(x;\rho)\,\rho^r f(u(\rho,t))\,d\rho.$$

By using the fact that f is increasing, $u(\rho,t)$ is increasing with respect to t, and $\lim_{t\to\infty}u(\rho,t)=U(\rho)$, we have from the Monotone Convergence Theorem and the continuity of f that

$$\lim_{t\to\infty}F_t(x,t)=-U(\tilde{b})\rho^r\tilde{G}_\rho(\tilde{b};x)-U(x)+\int_0^{\tilde{b}}\tilde{G}(x;\rho)\,\rho^r f(U(\rho))\,d\rho.$$

Because u is nondecreasing with respect to t, we have from the definition of $F(x,t)$ that $\lim_{t\to\infty}F_t(x,t)\ge 0$. If this limit value were positive at some point x, then $\lim_{t\to\infty}F_t(x,t)=\infty$, which contradicts to $\lim_{t\to\infty}u(x,t)\le c$. Therefore,

$$\lim_{t\to\infty}F_t(x,t)=0.$$

Thus,

$$U(x)=-U(\tilde{b})\rho^r\tilde{G}_\rho(\tilde{b};x)+\int_0^{\tilde{b}}\tilde{G}(x;\rho)\,\rho^r f(U(\rho))\,d\rho.$$

Since $\tilde{G}(x;\rho)=\tilde{G}(\rho;x)$, we have

$$\rho^r\tilde{G}_\rho(\tilde{b};x)=\begin{cases}\dfrac{\tilde{b}^{1-r}-x^{1-r}}{\tilde{b}^{1-r}} & \text{for } 0\le\rho\le x,\\[2ex] -\dfrac{x^{1-r}}{\tilde{b}^{1-r}} & \text{for } x<\rho\le\tilde{b}.\end{cases}$$

Therefore,

$$U(x) = \frac{x^{1-r}}{\tilde{b}^{1-r}} U(\tilde{b}) + \int_0^{\tilde{b}} \tilde{G}(x;\rho)\,\rho^r f(U(\rho))\,d\rho. \tag{9}$$

Then, $U(0) = 0$. For $0 < x < \tilde{b}$,

$$U'(x) = \frac{(1-r)\,x^{-r}}{\tilde{b}^{1-r}} U(\tilde{b}) + \int_0^{\tilde{b}} \tilde{G}_x(x;\rho)\,\rho^r f(U(\rho))\,d\rho.$$

Multiplying with x^r and differentiate with respect to x on both sides yields

$$\begin{aligned} -\left(x^r U'(x)\right)' &= -\int_0^{\tilde{b}} \left(x^r \tilde{G}_x(x;\rho)\right)_x \rho^r f(U(\rho))\,d\rho \\ &= \int_0^{\tilde{b}} \delta(\rho - x)\,\rho^r f(U(\rho))\,d\rho = x^r f(U(x)). \end{aligned}$$

Since $\tilde{b}$ was arbitrary and U is continuous on $[0,a]$, the proof is complete.
(ii) Let us consider u in the region $[\tilde{B}, a] \times (0,\infty)$ where $\tilde{B} \in [B^*, a]$. Let

$$F(x,t) = \int_{\tilde{B}}^a \rho^r \tilde{G}(x;\rho)\,u(\rho,t)\,d\rho,$$

where $\tilde{G}(x;\rho)$ is Green's function corresponding to (8) with B_s^* being replaced by $\tilde{B}$. Note that

$$\tilde{G}(x;\rho) = \begin{cases} \left(\dfrac{x^{1-r} - \tilde{B}^{1-r}}{1-r}\right)\left(\dfrac{a^{1-r} - \rho^{1-r}}{a^{1-r} - \tilde{B}^{1-r}}\right) & \text{for } \tilde{B} \le x \le \rho, \\[2ex] \left(\dfrac{x^{1-r} - a^{1-r}}{1-r}\right)\left(\dfrac{\tilde{B}^{1-r} - \rho^{1-r}}{a^{1-r} - \tilde{B}^{1-r}}\right) & \text{for } \rho < x \le a. \end{cases}$$

Since the operator is self-adjoint, $\tilde{G}(x;\rho) = \tilde{G}(\rho;x)$. We have

$$\begin{aligned} F_t(x,t) &= \int_{\tilde{B}}^a \rho^r \tilde{G}(x;\rho)\,u_t(\rho,t)\,d\rho \\ &= \int_{\tilde{B}}^a \tilde{G}(x;\rho)\left(\rho^r u_\rho(\rho,t)\right)_\rho d\rho + \int_{\tilde{B}}^a \tilde{G}(x;\rho)\,\rho^r f(u(\rho,t))\,d\rho. \end{aligned}$$

Using integration by parts to the first integral on the right hand side of the above equation, we obtain

$$F_t(x,t) = u(\tilde{B},t)\rho^r \tilde{G}_\rho(\tilde{B};x) - u(x,t) + \int_{\tilde{B}}^a \tilde{G}(x;\rho)\,\rho^r f(u(\rho,t))\,d\rho.$$

By using the fact that f is increasing, $u(\rho,t)$ is increasing with respect to t, and $\lim_{t\to\infty} u(\rho,t) = U(\rho)$, we have from the Monotone Convergence Theorem and the continuity of f that

$$\lim_{t\to\infty} F_t(x,t) = U(\tilde{B})\rho^r \tilde{G}_\rho(\tilde{B};x) - U(x) + \int_{\tilde{B}}^a \tilde{G}(x;\rho)\,\rho^r f(U(\rho))\,d\rho.$$

Because u is nondecreasing with respect to t, we have from the definition of $F(x,t)$ that $\lim_{t\to\infty} F_t(x,t) \geq 0$. If this limit value were positive at some point x, then $\lim_{t\to\infty} F_t(x,t) = \infty$, which contradicts to $\lim_{t\to\infty} u(x,t) \leq c$. Therefore,

$$\lim_{t\to\infty} F_t(x,t) = 0.$$

Thus,

$$U(x) = U(\tilde{B})\rho^r \tilde{G}_\rho(\tilde{B}; x) + \int_{\tilde{B}}^a \tilde{G}(x;\rho)\,\rho^r f(U(\rho))\,d\rho.$$

Since $\tilde{G}(x;\rho) = \tilde{G}(\rho;x)$, we have

$$\rho^r \tilde{G}_\rho(\rho;x) = \begin{cases} \dfrac{a^{1-r} - x^{1-r}}{a^{1-r} - \tilde{B}^{1-r}} & \text{for } \tilde{B} \leq \rho \leq x, \\[2ex] \dfrac{\tilde{B}^{1-r} - x^{1-r}}{a^{1-r} - \tilde{B}^{1-r}} & \text{for } x < \rho \leq a. \end{cases}$$

Therefore,

$$U(x) = \frac{a^{1-r} - x^{1-r}}{a^{1-r} - \tilde{B}^{1-r}} U(\tilde{B}) + \int_{\tilde{B}}^a \tilde{G}(x;\rho)\,\rho^r f(U(\rho))\,d\rho. \tag{10}$$

Then, $U(a) = 0$. For $\tilde{B} < x < a$,

$$U'(x) = \frac{(r-1)\,x^{-r}}{a^{1-r} - \tilde{B}^{1-r}} U(\tilde{B}) + \int_{\tilde{B}}^a \tilde{G}_x(x;\rho)\,\rho^r f(U(\rho))\,d\rho.$$

Multiplying with x^r and differentiate with respect to x on both sides yields

$$\begin{aligned} -\left(x^r U'(x)\right)' &= -\int_{\tilde{B}}^a \left(x^r \tilde{G}_x(x;\rho)\right)_x \rho^r f(U(\rho))\,d\rho \\ &= \int_{\tilde{B}}^a \delta(\rho - x)\,\rho^r f(U(\rho))\,d\rho = x^r f(U(x)). \end{aligned}$$

Since $\tilde{B}$ was arbitrary chosen and U is continuous on $[0,a]$, the proof is complete.

(iii) Let us suppose that there exists some $x_0 \in [b_s^*, B_s^*]$ such that $U(x_0) < c$. By the continuity of U, there exists an interval (x_1, x_2) with $b_s^* < x_1 < x_0 < x_2 < B_s^*$ such that $U(x_1) = c = U(x_2)$ and $U(x) < c$ for $x \in (x_1, x_2)$. Since $u_t \geq 0$ for $u < c$, we have $u(x,t) < c$ in $\{(x,t) : x \in (x_1,x_2) \text{ and } t > 0\}$. This implies $Hu = x^r f(u)$ for $x_1 < x < x_2$ and $t > 0$. Let

$$F(x,t) = \int_{x_1}^{x_2} \rho^r \tilde{G}(x;\rho)\,u(\rho,t)\,d\rho, \tag{11}$$

where

$$\tilde{G}(x;\rho) = \begin{cases} \left(\dfrac{x_2^{1-r} - x^{1-r}}{1-r}\right)\left(\dfrac{\rho^{1-r} - x_1^{1-r}}{x_2^{1-r} - x_1^{1-r}}\right) & \text{for } x_1 \leq \rho \leq x, \\[2ex] \left(\dfrac{x_1^{1-r} - x^{1-r}}{1-r}\right)\left(\dfrac{\rho^{1-r} - x_2^{1-r}}{x_2^{1-r} - x_1^{1-r}}\right) & \text{for } x < \rho \leq x_2. \end{cases}$$

Since $u(x,t)$ is nondecreasing with respect to t, it follows from (11) that $\lim_{t\to\infty} F_t(x,t) \geq 0$. By direct calculation, we obtain

$$F_t(x,t) = \left(\frac{x_2^{1-r} - x^{1-r}}{x_2^{1-r} - x_1^{1-r}}\right) u(x_1,t) - u(x,t) - \left(\frac{x_1^{1-r} - x^{1-r}}{x_2^{1-r} - x_1^{1-r}}\right) u(x_2,t) + \int_{x_1}^{x_2} \tilde{G}(x;\rho)\,\rho^r f(u(\rho,t))\,d\rho.$$

Since f is nondecreasing, it follows from the Monotone Convergence Theorem, the continuity of f and $U(x_1) = c = U(x_2)$ that

$$\lim_{t\to\infty} F_t(x,t) = -U(x) + c + \int_{x_1}^{x_2} \tilde{G}(x;\rho)\,\rho^r f(U(\rho))\,d\rho. \tag{12}$$

To show that $\lim_{t\to\infty} F_t(x,t) = 0$, let us suppose that $\lim_{t\to\infty} F_t(x,t) > 0$ at some point $x \in (x_1, x_2)$. Then, as t tends to infinity, $F(x,t)$ increases without bound there. This implies that u reaches c at some finite time. This contradicts to $U(x) < c$ for $x \in (x_1, x_2)$. Thus, $\lim_{t\to\infty} F_t(x,t) = 0$. From (12),

$$U(x) = c + \int_{x_1}^{x_2} \tilde{G}(x;\rho)\,\rho^r f(U(\rho))\,d\rho > c$$

for $x \in (x_1, x_2)$. This contradiction leads us to the conclusion $U(x) \equiv c$ for $x \in (b_s^*, B_s^*)$.
(iv) By applying Theorem 1.2 (i) and (v), we obtain

$$\lim_{(x,t)\to(b_s^*,\infty)} u_x(x,t) = 0 = \lim_{(x,t)\to(B_s^*,\infty)} u_x(x,t).$$

Thus, $U'(b^*) = 0 = U'(B^*)$. □

By modifying the proof of Lemma 3.4 of Chan and Boonklurb [6], we have the uniqueness of the solutions of (7) and (8).

Lemma 3. *Each of (7) and (8) has a unique solution.*

From Lemmas 2.2 and 2.3, we obtain the following result.

Theorem 3. *As t tends to infinity, all weak solutions of (4) tend to the unique steady-state solution given by (6) - (8).*

Now, we can find the fixed point representations for b_s^* and B_s^*. These representations enable us to find the bounds for b_s^* and B_s^*.

Theorem 4. *(i)* $b_s^* \geq \left[\frac{1-r}{\sqrt{2}(b_s^*)^r} \int_0^c \left(\int_\varsigma^c f(\eta)d\eta\right)^{-\frac{1}{2}} d\varsigma\right]^{\frac{1}{1-r}}$.

(ii) $B_s^* \leq a\left[1 + \frac{r-1}{\sqrt{2}a}\int_0^c \left(\int_\varsigma^c f(\eta)d\eta\right)^{-\frac{1}{2}} d\varsigma\right]^{\frac{1}{1-r}}$.

Proof. (i) Multiplying (7) by $x^r W'(x)$, integrating from x to b_s^*, and using $W'(b_s^*) = 0$, we have

$$\frac{1}{2}(x^r W'(x))^2 = \int_x^{b_s^*} \rho^{2r} f(W(\rho))W'(\rho)d\rho.$$

Since $W'(x) \geq 0$ for $x \in [0, b_s^*]$, we have

$$\frac{1}{x^r} = \frac{1}{\sqrt{2}}\left(\int_x^{b_s^*} \rho^{2r} f(W(\rho))W'(\rho)d\rho\right)^{-\frac{1}{2}} W'(x).$$

Therefore,

$$\int_0^x \frac{1}{\xi^r}d\xi = \frac{1}{\sqrt{2}}\int_0^x \left(\int_\xi^{b_s^*} \rho^{2r} f(W(\rho))W'(\rho)d\rho\right)^{-\frac{1}{2}} W'(\xi)d\xi, \tag{13}$$

which gives

$$b_s^* = \left[\frac{1-r}{\sqrt{2}}\int_0^{b_s^*} \left(\int_\xi^{b_s^*} \rho^{2r} f(W(\rho))W'(\rho)d\rho\right)^{-\frac{1}{2}} W'(\xi)d\xi\right]^{\frac{1}{1-r}}. \tag{14}$$

We would like to find a lower bound for b_s^*. Since $\rho^{2r} \leq (b_s^*)^{2r}$, where $\rho \in [\xi, b_s^*]$ and $f(W(\rho))W'(\rho) > 0$, we have

$$\int_\xi^{b_s^*} \rho^{2r} f(W(\rho))W'(\rho)d\rho \leq (b_s^*)^{2r}\int_\xi^{b_s^*} f(W(\rho))W'(\rho)d\rho = (b_s^*)^{2r}\int_{W(\xi)}^c f(\eta)d\eta,$$

which gives

$$\left(\int_\xi^{b_s^*} \rho^{2r} f(W(\rho))W'(\rho)d\rho\right)^{-\frac{1}{2}} \geq (b_s^*)^{-r}\left(\int_{W(\xi)}^c f(\eta)d\eta\right)^{-\frac{1}{2}}.$$

Since $W'(\xi) > 0$, we obtain

$$\begin{aligned}&\frac{1-r}{\sqrt{2}}\int_0^{b_s^*} \left(\int_\xi^{b_s^*} \rho^{2r} f(W(\rho))W'(\rho)d\rho\right)^{-\frac{1}{2}} W'(\xi)d\xi \\ &\geq \frac{1-r}{\sqrt{2}(b_s^*)^r}\int_0^{b_s^*} \left(\int_{W(\xi)}^c f(\eta)d\eta\right)^{-\frac{1}{2}} W'(\xi)d\xi.\end{aligned}$$

From (14), we have

$$b_s^* \geq \left[\frac{1-r}{\sqrt{2}(b_s^*)^r} \int_0^{b_s^*} \left(\int_{W(\xi)}^c f(\eta) d\eta \right)^{-\frac{1}{2}} W'(\xi) d\xi \right]^{\frac{1}{1-r}}$$

$$= \left[\frac{1-r}{\sqrt{2}(b_s^*)^r} \int_0^c \left(\int_\varsigma^c f(\eta) d\eta \right)^{-\frac{1}{2}} d\varsigma \right]^{\frac{1}{1-r}}. \tag{15}$$

(ii) Multiplying (8) by $x^r W'(x)$, integrating from B_s^* to x, and using $W'(B_s^*) = 0$, we have

$$-\int_{B_s^*}^x (\rho^r W'(\rho))(\rho^r W'(\rho))' d\rho = \int_{B_s^*}^x (\rho^r W'(\rho))(\rho^r f(W(\rho))) d\rho.$$

That is,

$$\frac{1}{2}(x^r W'(x))^2 = \int_x^{B_s^*} \rho^{2r} f(W(\rho)) W'(\rho) d\rho.$$

From $W'(x) \leq 0$ for $x \in [B_s^*, a]$, we have

$$\frac{1}{x^r} = -\frac{1}{\sqrt{2}} \left(\int_x^{B_s^*} \rho^{2r} f(W(\rho)) W'(\rho) d\rho \right)^{-\frac{1}{2}} W'(x).$$

Hence,

$$\int_a^x \frac{1}{\xi^r} d\xi = -\frac{1}{\sqrt{2}} \int_a^x \left(\int_\xi^{B_s^*} \rho^{2r} f(W(\rho)) W'(\rho) d\rho \right)^{-\frac{1}{2}} W'(\xi) d\xi, \tag{16}$$

which gives

$$(B_s^*)^{1-r} = a^{1-r} + \frac{r-1}{\sqrt{2}} \int_a^{B_s^*} \left(\int_\xi^{B_s^*} \rho^{2r} f(W(\rho)) W'(\rho) d\rho \right)^{-\frac{1}{2}} W'(\xi) d\xi.$$

Thus,

$$B_s^* = \left[a^{1-r} + \frac{r-1}{\sqrt{2}} \int_a^{B_s^*} \left(\int_\xi^{B_s^*} \rho^{2r} f(W(\rho)) W'(\rho) d\rho \right)^{-\frac{1}{2}} W'(\xi) d\xi \right]^{\frac{1}{1-r}}. \tag{17}$$

We would like to find an upper bound for B_s^*. Since $\rho^{2r} \leq \xi^{2r}$, where $\rho \in [B_s^*, \xi]$ and $-f(W(\rho))W'(\rho) > 0$, we have

$$\int_\xi^{B_s^*} \rho^{2r} f(W(\rho)) W'(\rho) d\rho \leq \xi^{2r} \int_\xi^{B_s^*} f(W(\rho)) W'(\rho) d\rho = \xi^{2r} \int_{W(\xi)}^c f(\eta) d\eta,$$

which gives

$$\left(\int_{\xi}^{B_s^*} \rho^{2r} f(W(\rho))W'(\rho)d\rho\right)^{-\frac{1}{2}} \geq \xi^{-r}\left(\int_{W(\xi)}^{c} f(\eta)d\eta\right)^{-\frac{1}{2}}.$$

For $\xi \in [B_s^*, a]$, we have $\xi^{-r} \geq a^{-r}$ and hence,

$$\left(\int_{\xi}^{B_s^*} \rho^{2r} f(W(\rho))W'(\rho)d\rho\right)^{-\frac{1}{2}} \geq a^{-r}\left(\int_{W(\xi)}^{c} f(\eta)d\eta\right)^{-\frac{1}{2}}.$$

Since $W'(\xi) < 0$, we obtain

$$a^{1-r} + \frac{r-1}{\sqrt{2}}\int_a^{B_s^*}\left(\int_{\xi}^{B_s^*} \rho^{2r} f(W(\rho))W'(\rho)d\rho\right)^{-\frac{1}{2}} W'(\xi)d\xi$$

$$\leq a^{1-r} + \frac{r-1}{\sqrt{2}a^r}\int_a^{B_s^*}\left(\int_{W(\xi)}^{c} f(\eta)d\eta\right)^{-\frac{1}{2}} W'(\xi)d\xi.$$

From (17), we have

$$B_s^* \leq \left[a^{1-r} + \frac{r-1}{\sqrt{2}a^r}\int_a^{B_s^*}\left(\int_{W(\xi)}^{c} f(\eta)d\eta\right)^{-\frac{1}{2}} W'(\xi)d\xi\right]^{\frac{1}{1-r}}$$

$$= a\left[1 + \frac{r-1}{\sqrt{2}a}\int_0^{c}\left(\int_{\varsigma}^{c} f(\eta)d\eta\right)^{-\frac{1}{2}} d\varsigma\right]^{\frac{1}{1-r}}. \tag{18}$$

□

3. NUMERICAL COMPUTATIONS FOR B_S^* AND B_S^*

By replacing $\tilde{b}$ in (9) with b_s^*, the integral equation corresponding to (7) becomes

$$W(x) = \left(\frac{x^{1-r}}{(b_s^*)^{1-r}}\right)c + \int_0^{b_s^*} \tilde{G}(x;\rho)\rho^r f(W(\rho))d\rho \quad \text{for } x \in (0, b_s^*), \tag{19}$$

where $\tilde{G}(x;\rho)$ is Green's function for $-(x^r W'(x))'$ in $(0, b_s^*)$ subject to $\tilde{G}(0;\rho) = 0 = \tilde{G}(b_s^*;\rho)$. Similarly, by replacing $\tilde{B}$ in (10) with B_s^*, the integral equation corresponding to (8) becomes

$$W(x) = \left(\frac{a^{1-r} - x^{1-r}}{a^{1-r} - (B_s^*)^{1-r}} \right) c \ + \int_{B_s^*}^{a} \tilde{G}(x;\rho)\rho^r f(W(\rho))d\rho \ \text{ for } x \in (B_s^*, a), \qquad (20)$$

where $\tilde{G}(x;\rho)$ is Green's function for $-(x^r W'(x))'$ in (B_s^*, a) subject to $\tilde{G}(B_s^*;\rho) = 0 = \tilde{G}(a;\rho)$.

Let us construct a sequence $\{W_i(x)\}$ with $W_0(x) = 0$ and for $i \in \{0, 1, 2, ...\}$,

$$W_{i+1}(x) = \left(\frac{x^{1-r}}{(b_s^*)^{1-r}} \right) c \ + \int_0^{b_s^*} \tilde{G}(x;\rho)\rho^r f(W_i(\rho))d\rho \ \text{ for } x \in (0, b_s^*), \qquad (21)$$

and

$$W_{i+1}(x) = \left(\frac{a^{1-r} - x^{1-r}}{a^{1-r} - (B_s^*)^{1-r}} \right) c \ + \int_{B_s^*}^{a} \tilde{G}(x;\rho)\rho^r f(W_i(\rho))d\rho \ \text{ for } x \in (B_s^*, a). \quad (22)$$

By using MatLab R2017b and modifying the computational method of Chan and Boonklurb [5], we can use (22) to compute B_s^* as follows:

Step 1: Let $B_l = \frac{a}{2}$, $B_u = a\left[1 + \frac{r-1}{\sqrt{2}a} \int_0^c \left(\int_\varsigma^c f(\eta)d\eta\right)^{-\frac{1}{2}} d\varsigma\right]^{\frac{1}{1-r}}$, $B = \frac{B_l + B_u}{2}$ and $h = \frac{a-B}{m}$ for some positive integer m. For $k \in \{0, 1, 2, ..., m\}$, we use the subroutine trapz to perform the numerical integration:

$$\begin{aligned} W_1(B+kh) = &\left(\frac{a^{1-r} - B + kh^{1-r}}{a^{1-r} - B^{1-r}} \right) c \ + f(0) \int_B^{B+kh} \tilde{G}(B+kh;\rho)\rho^r d\rho \\ &+ f(0) \int_{B+kh}^{a} \tilde{G}(B+kh;\rho)\rho^r d\rho. \end{aligned}$$

Step 2: Using subroutine polyfit, we construct an interpolation function $W_1(x)$ from $W_1(B + kh)$, where $k \in \{0, 1, 2, ..., m\}$. Then, for each $i \in \{0, 1, 2, ...\}$, we use the subroutine trapz to integrate

$$\begin{aligned} W_{i+1}(B+kh) = &\left(\frac{a^{1-r} - B + kh^{1-r}}{a^{1-r} - B^{1-r}} \right) c \ + \int_B^{B+kh} \tilde{G}(B+kh;\rho)\rho^r f(W_i(\rho))d\rho \\ &+ \int_{B+kh}^{a} \tilde{G}(B+kh;\rho)\rho^r f(W_i(\rho))d\rho, \end{aligned}$$

with $k \in \{0, 1, 2, ..., m\}$. After that, use subroutine polyfit to construct an interpolation function $W_{i+1}(x)$ from $W_{i+1}(B + kh)$, where $k \in \{0, 1, 2, ..., m\}$.

Step 3: Let $err = \max_{0 \le k < m} |W_{i+1}(B + kh) - W_i(B + kh)|$. If $err < tol$ for some given tolerance tol and $W_{i+1}(B + kh) \le c$ for $k \in \{0, 1, 2, ..., m\}$, then let $B = B_u$ and we repeat the process by going to Steps 1, 2 and 3. If $W_{i+1}(B + kh) > c$ for some k, then let $B = B_l$ and we repeat the process by going to Steps 1, 2 and 3.

Step 4: The process is stopped when $B_u - B_l \leq 10^{-5}$. Then, B_u is taken to be B_s^*.

To compute b_s^*, the same procedure as above can be used by replacing (21) with (22).

As an illustration, let $f(u) = (1-u)^{-\beta}$, where $\beta \in (0,1)$. We have $f(0) = 1$, $f'(u) = \beta(1-u)^{-\beta-1} > 0$, $f''(u) = \beta(\beta+1)(1-u)^{-\beta-2} > 0$, $\lim_{u\to 1^-} f(u) = \infty$ and $\int_0^1 f(u)du = (1-\beta)^{-1}$. We note that

$$\int_0^1 \left(\int_\varsigma^1 f(\eta)d\eta\right)^{-\frac{1}{2}} d\varsigma = \sqrt{1-\beta}\int_0^1 (1-\varsigma)^{\frac{\beta-1}{2}} d\varsigma = \frac{2\sqrt{1-\beta}}{1+\beta}.$$

From (15) and (18), we have

$$\left[\frac{(1-r)\sqrt{2(1-\beta)}}{(b_s^*)^r(1+\beta)}\right]^{\frac{1}{1-r}} \leq b_s^* \text{ and } B_s^* \leq a\left[1 + \frac{(r-1)\sqrt{2(1-\beta)}}{a(1+\beta)}\right]^{\frac{1}{1-r}}.$$

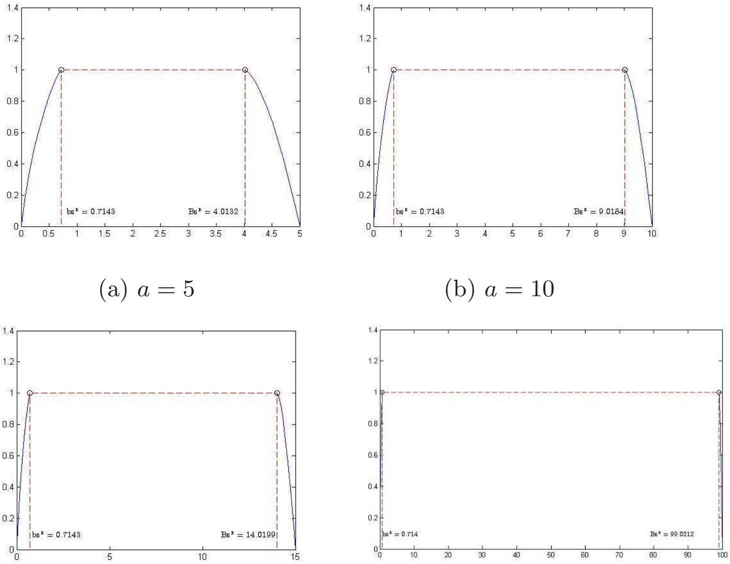
Tables 3.1 and 3.2 show the values of b_s^* and B_s^* for various values of a with $m = 10$ and $tol = 10^{-6}$

r	β	a	Lower bound	Upper bound	b_s^*
$\frac{1}{5}$	$\frac{1}{4}$	5	0.6515	$\frac{5}{2}$	0.7143
		10	0.5656	5	0.7143
		15	0.5170	$\frac{15}{2}$	0.7143
		100	0.3286	50	0.7140
$\frac{1}{5}$	$\frac{1}{3}$	4	0.5868	2	0.6395
		5	0.5623	$\frac{5}{2}$	0.6395
		9	0.4979	$\frac{9}{2}$	0.6395
		40	0.3524	20	0.6395

Table 1: Numerical examples of b_s^*.

r	β	a	Lower bound	Upper bound	B_s^*
$\frac{1}{5}$	$\frac{1}{4}$	5	$\frac{5}{2}$	4.0402	4.0132
		10	5	9.0300	9.0184
		15	$\frac{15}{2}$	14.0267	14.0199
		100	50	99.0212	99.0212
$\frac{1}{5}$	$\frac{1}{3}$	4	2	3.1536	3.1194
		5	$\frac{5}{2}$	4.1495	4.1218
		9	$\frac{9}{2}$	8.1425	8.1258
		40	20	39.1359	39.1292

Table 2: Numerical examples of B_s^*.



(a) $a = 5$ (b) $a = 10$

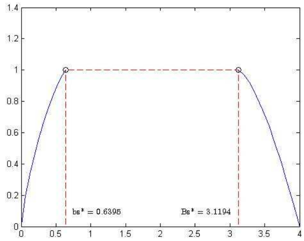
(c) $a = 15$ (d) $a = 100$

Figure 1: Solution profiles for $\beta = 1/4$ and $r = 1/5$.

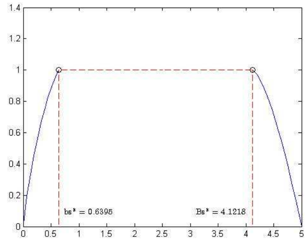
4. CONCLUSION AND DISCUSSION

The semilinear parabolic equation with singularity is studied. It was shown that under some conditions on the forcing term f, the problem has a weak solution $u(x,t)$. In the second section of this work, the beyond quenching profile of the solution was studied. We proved that as the time t approaches infinity, all weak solutions $u\,(x,t)$ on the space domain $[0,a]$ approach a unique steady-state solution $U\,(x)$. The steady-state solution profile $U(x)$ on $[0,a]$ consists of three segments determined by the values of b_s^* and B_s^* as $[0,a] = [0,b_s^*] \cup [b_s^*, B_s^*] \cup [B_s^*, a]$. The bounds for b_s^* and B_s^* were obtained and the numerical methods were established for finding the approximations of b_s^* and B_s^*. The numerical results show the length of the space domain plays and important role. The longer the space domain, the longer the segment $[b_s^*, B_s^*]$, where the steady-state solution $U(x)$ reaches the value c.

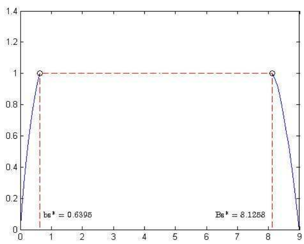
If we consider various values of r, where $r \in [0.05, 0.5]$, we obtain that the value of $B_s^* - b_s^*$ increases with repect to r as shown in Figure 4.1. Furthermore, the beyond quenching profile is asymmetric suggested by the fact that both B_s^* and b_s^* decrease with the different rates as r increases from 0.05 to 0.5 (Figure 4.2).



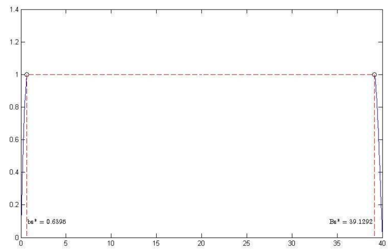
(a) $a = 4$



(b) $a = 5$



(c) $a = 9$



(d) $a = 40$

Figure 2: Solution profiles for $\beta = 1/3$ and $r = 1/5$.

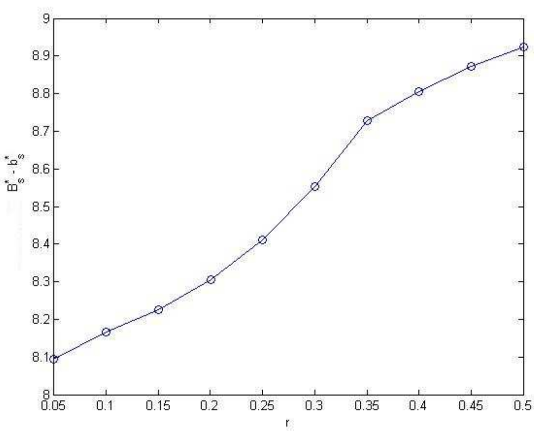
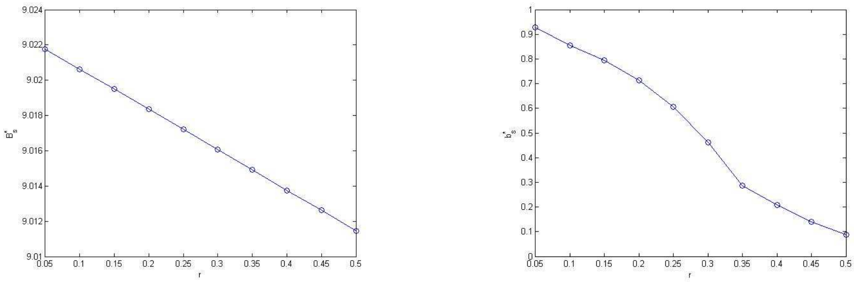


Figure 3: Plot of $B_s^* - b_s^*$ against r, where $r \in [0.05, 0.5]$, for $a = 10$ and $\beta = 1/4$.



(a) B_s^* (b) b_s^*

Figure 4: Plots of B_s^* and b_s^* against r, where $r \in [0.05, 0.5]$, for $a = 10$ and $\beta = 1/4$.

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