DYNAMICS OF GLOBAL ATTRACTOR FOR A SEMILINEAR DEGENERATE PARABOLIC EQUATION INVOLVING GRUSHIN OPERATORS

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ABSTRACT: In this paper we study the dynamics of global attractor of a semilinear parabolic equation involving Grushin operators. First we show that the global attractor is bounded in $L^\infty(\Omega)$ and $D(A)$. Then we investigate the existence of a Lyapunov function, the injectivity on the global attractor and the squeezing property. Finally, we obtain estimates on upper bound and lower bound of the fractal dimension of the global attractor.

1. INTRODUCTION

One of the most important problems in modern mathematical physics is to understand the asymptotic behavior of the trajectories of infinite-dimensional dynamical systems induced by PDEs. One way to investigate the problem for dissipative dynamical systems is to study their global attractors which contain much of the relevant information about the systems and often some finite-dimensional characters. The first step is to study the existence of a global attractor. After that, we study the dynamics of the global attractor such as the regularity, structure, continuous dependence on parameters, fractal dimension estimates, etc.

Recently, the class of equations involving an operator of *Grushin type*

$$G_s u = \Delta_x u + |x|^{2s} \Delta_y u, \quad s \geq 0$$

has been studied widely by many mathematicians (for examples, see [1, 2, 4, 5, 6, 10, 11, 14, 15, 16]). Here $(x, y) \in \Omega$ and Ω is a bounded domain in $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ ($N_1, N_2 \geq 1$) with smooth boundary $\partial\Omega$. The operator G_s was first introduced by Grushin in [7] when s is a positive integer, and later by Franchi and Lanconelli in [6] when s is not an integer. Note that G_0 is the *Laplace operator* denoted by Δ, and G_s, when $s > 0$, is not elliptic (but hypoelliptic) in domains of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ intersecting the surface $\{x = 0\}$.

In this paper, we study the dynamics of the global attractor for the following semilinear degenerate parabolic problem

$$\begin{cases} u_t - G_s u + f(u) = g(x), & x \in \Omega, \ t > 0, \\ u(x, t) = 0, & x \in \partial\Omega, \ t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega. \end{cases} \quad (1.1)$$

Note that if $s = 0$, the equation (1.1) is just the nonlinear reaction-diffusion equation. Here we assume that $u_0 \in L^2(\Omega)$, $g \in L^\infty(\Omega)$, and $f : \mathbb{R} \to \mathbb{R}$ is a function of *polynomial type*, that is, f is a C^1 function satisfying the following conditions

$$c_1 |s|^p - c_0 \leq f(s)s \leq c_2 |s|^p + c_0, \quad p \geq 2, \quad (1.2)$$

$$f'(s) \geq -\ell, \quad (1.3)$$

where c_0, c_1, c_2 and ℓ are positive constants. A typical example of such functions is a polynomial of odd order with positive leading coefficient

$$f(u) = \sum_{i=0}^{2p+1} a_i u^i, \quad a_i \in \mathbb{R}, \quad a_{2p+1} > 0.$$

The existence and uniqueness of weak solutions, and the existence of a global attractor $\mathcal{A}$ in $L^2(\Omega)$ of the continuous semigroup $S(t)$ generated by (1.1) were proved by Anh and Ke in [2] under the assumption $g \in L^2(\Omega)$.

To study problem (1.1), we recall some weighted function spaces which will be used in the paper. Let $S_0^1(\Omega)$ be the completion of $C_0^1(\bar{\Omega})$ with the norm

$$\|u\|_{S_0^1} = \left(\int_\Omega |\nabla_x u|^2 + |x|^{2s} |\nabla_y u|^2 \ dxdy \right)^{\frac{1}{2}}.$$

Then we know that $S_0^1(\Omega)$ is a Banach space and the embedding $S_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Denote $A = -G_s$ with the homogeneous Dirichlet boundary condition. Then its domain

$$D(A) = \left\{ u \in S_0^1(\Omega) \mid Au \in L^2(\Omega) \right\}$$

is a Banach space with the graph norm

$$\|u\|_{D(A)} = \left(\int_\Omega |G_s u|^2 dxdy \right)^{\frac{1}{2}}.$$

We also know that the embedding $D(A) \hookrightarrow S_0^1(\Omega)$ is compact.

The paper is organized as follows. In Section 2, we show that the global attractor of the semigroup $S(t)$ induced by (1.1) is bounded in $L^\infty(\Omega)$ and $D(A)$. In Section 3, we investigate the existence of a Lyapunov function, the injectivity on the global attractor, the squeezing property and its consequences. In Section 4, we obtain estimates on upper bound and lower bound of the fractal dimension of the global attractor. These are an extension of some previous results for the nonlinear reaction-diffusion equations in [9, 12, 13].

Throughout the paper, for brevity, we denote by $|\cdot|$, $\|\cdot\|_\infty$ and $\|\cdot\|_{S_0^1}$ the norms in the spaces $L^2(\Omega), L^\infty(\Omega)$ and $S_0^1(\Omega)$, respectively.

2. BOUNDEDNESS OF GLOBAL ATTRACTORS

In this section, we prove that the global attractor $\mathcal{A}$ of the semigroup $S(t)$ induced by (1.1) is bounded in both $L^\infty(\Omega)$ and $D(A)$, which will be used in the next sections.

Theorem 2.1. *The global attractor $\mathcal{A}$ of the equation* (1.1) *is bounded in $L^\infty(\Omega)$.*

Proof. For $u \in L^2(\Omega)$, we denote

$$u_+(x,y) = \begin{cases} u(x,y), & u(x,y) > 0, \\ 0, & \text{otherwise,} \end{cases}$$

and similarly

$$u_-(x,y) = \begin{cases} u(x,y), & u(x,y) < 0, \\ 0, & \text{otherwise.} \end{cases}$$

By (1.2), we have

$$f(s)s \geq c_1|s|^p - c_0$$

for all $s \in \mathbb{R}$. Hence we get

$$f(s) \geq \|g\|_\infty \text{ when } s \geq M \text{ for sufficiently large } M > 0. \tag{2.1}$$

Multiplying the first equation in (1.1) by $(u - M)_+$ and integrating over Ω yield

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u - M)_+^2 dxdy + \int_\Omega \left(|\nabla_x(u - M)_+|^2 + |x|^{2s}|\nabla_y(u - M)_+|^2 \right) dxdy$$

$$+ \int_\Omega f(u)(u - M)_+ dxdy = \int_\Omega g(u - M)_+ dxdy.$$

By (2.1), we see that

$$\frac{1}{2}\frac{d}{dt} \int_\Omega (u - M)_+^2 dxdy + \int_\Omega \left(|\nabla_x(u - M)_+|^2 + |x|^{2s}|\nabla_y(u - M)_+|^2\right) dxdy \le 0.$$

Since the space $S_0^1(\Omega)$ is embedded into $L^2(\Omega)$, we have

$$\int_\Omega \left(|\nabla_x(u - M)_+|^2 + |x|^{2s}|\nabla_y(u - M)_+|^2\right) dxdy \ge C \int_\Omega (u - M)_+^2 dxdy.$$

It follows that

$$\frac{d}{dt} \int_\Omega (u - M)_+^2 dxdy \le -C \int_\Omega (u - M)_+^2 dxdy,$$

and so

$$\int_\Omega (u - M)_+^2 dxdy \le e^{-Ct} \int_\Omega (u_0 - M)_+^2 dxdy.$$

Since $\mathcal{A}$ is bounded in $L^2(\Omega)$ (Theorem 2.6 in [2]) and invariant under $S(t)$, we obtain

$$\int_\Omega (u - M)_+^2 dxdy = 0$$

for all $u \in \mathcal{A}$. We apply a similar argument to $(u + M)_-$ to deduce that

$$\int_\Omega (u + M)_-^2 dxdy = 0$$

for all $u \in \mathcal{A}$. This shows that

$$\|u\|_\infty \le M \text{ for all } u \in \mathcal{A}.$$

$\square$

Using the boundedness of $\mathcal{A}$ in $L^\infty(\Omega)$, we now show that $\mathcal{A}$ is bounded in $D(A)$.

Theorem 2.2. *The global attractor $\mathcal{A}$ of the equation* (1.1) *is bounded in $D(A)$.*

Proof. We rewrite the equation (1.1) as

$$Au = -\frac{du}{dt} - f(u) + g, \quad u(0) = u_0. \tag{2.2}$$

To prove the theorem, it is enough to show that the right-hand side of the first equation of (2.2) is bounded in $L^2(\Omega)$ on $\mathcal{A}$. Since $u \in L^\infty(\Omega)$, we see that $f(u) \in L^2(\Omega)$, and we have had $g \in L^2(\Omega)$.

First let us estimate $\int_0^t |u_t|^2 ds$. Multiply $u_t := \frac{d}{dt}u$ to both sides of the following equation

$$u_t + Au + f(u) = g. \tag{2.3}$$

Then we get

$$|u_t|^2 + \frac{1}{2}\frac{d}{dt}\|u\|_{S_0^1}^2 + \int_\Omega f(u)u_t dxdy = \int_\Omega g(x,y)u_t dxdy,$$

and so

$$|u_t|^2 + \frac{1}{2}\frac{d}{dt}\|u\|_{S_0^1}^2 + \frac{d}{dt}\int_\Omega F(u)dxdy = \frac{d}{dt}\int_\Omega g(x,y)udxdy, \qquad (2.4)$$

where $F(s) = \int_0^s f(\sigma)d\sigma$ is a primitive of f. Integrating the equality (2.4) from 0 to t gives

$$\int_0^t |u_t|^2 dt + \frac{1}{2}\|u(t)\|_{S_0^1}^2 = \frac{1}{2}\|u_0\|_{S_0^1}^2 - \int_\Omega F(u(x,y,t))dxdy + \int_\Omega F(u(x,y,0))dxdy$$
$$+ \int_\Omega g(x,y)u(x,y,t)dxdy - \int_\Omega g(x,y)u(x,y,0)dxdy.$$

Since $\mathcal{A}$ is bounded in both $S_0^1(\Omega)$ and $L^\infty(\Omega)$, we see that

$$\int_0^t |u_t|^2 dt + \frac{1}{2}\|u(t)\|_{S_0^1}^2 \le K \qquad (2.5)$$

for some $K > 0$.

Next we obtain the bound on u_t in $L^2(\Omega)$. Differentiate (2.3) to obtain

$$\frac{d}{dt}u_t + Au_t + f'(u)u_t = 0.$$

If we take the inner product with $t^2 u_t$, then we get

$$(t^2 u_t, \partial_t u_t) - t^2(u_t, G_s u_t) = -t^2(f'(u)u_t, u_t).$$

Noting that $f'(s) \ge -\ell$ for all $s \in \mathbb{R}$, it follows that

$$\frac{1}{2}\frac{d}{dt}|tu_t|^2 - t|u_t|^2 + t^2\|u_t\|_{S_0^1}^2 \le t^2\ell|u_t|^2.$$

Integrating between 0 and t gives

$$|tu_t|^2 + \int_0^t t^2\|u_t\|_{S_0^1}^2 ds \le \int_0^t (s + \ell s^2)|u_t|^2 ds.$$

For $t = 1$, we have $\sup_{s\in[0,1]}(s + \ell s^2) = 1 + \ell$. Employing this and by (2.5), we have

$$|u_t(1)|^2 \le (1+\ell)\int_0^1 |u_t|^2 ds \le (1+\ell)K.$$

Hence we obtain the uniform $L^2(\Omega)$ bound on $u_t(1)$ for all $u \in \mathcal{A}$. Since $\mathcal{A}$ is invariant, we can deduce that $-du/dt - f(u) + g$ is uniformly bounded in $L^2(\Omega)$ over $\mathcal{A}$. Since Au is uniformly bounded in $L^2(\Omega)$, we conclude that u is uniformly bounded in $D(A)$ for all $u \in \mathcal{A}$. □

Remark 2.1. Using the asymptotic *a priori* estimate method as in [17], one can show the existence of global attractors for the semigroup $S(t)$ in the spaces $L^p(\Omega)$, $S_0^1(\Omega)$ and $D(A) \cap L^{2p-2}(\Omega)$. Of course, these global attractors coincide with the global attractor $\mathcal{A}$ in $L^2(\Omega)$ obtained in [2].

3. DYNAMICS OF GLOBAL ATTRACTORS

In this section, we prove the existence of a Lyapunov function on $\mathcal{A}$ and the injectivity of the semigroup $S(t)$ on $\mathcal{A}$ using the boundedness of $\mathcal{A}$. Moreover, we show that the squeezing property for (1.1) holds.

First, we recall the concept of Lyapunov functions.

Definition 3.1. *Let $S(t)$ be a continuous semigroup on a Banach space X. A Lyapunov function for a positively invariant set $\mathcal{M} \subset X$ is a functional $\Phi : \mathcal{M} \to \mathbb{R}$ such that*

(i) Φ is continuous on $\mathcal{M}$;

(ii) Φ is nonincreasing along trajectories;

(iii) if $\Phi(S(t)u_0) = \Phi(u_0)$ for some $t > 0$, then u_0 is a fixed point of $S(t)$.

We now prove the following result.

Theorem 3.1. *The functional Φ on the global attractor $\mathcal{A}$ of the equation (1.1) given by*

$$\Phi(u) = \int_\Omega \left(\frac{1}{2}|\nabla_x u|^2 + \frac{1}{2}|x|^{2s}|\nabla_y u|^2 + F(u) - gu \right) dxdy$$

is a Lyapunov function on $\mathcal{A}$, where $F(s) = \int_0^s f(\sigma)d\sigma$.

Proof. By (2.4), we have

$$\frac{d}{dt}\Phi(u) = -|u_t|^2,$$

which implies that Φ is strictly decreasing along the trajectories. Hence if $\Phi(u(T)) = \Phi(u_0)$ for some $T > 0$, then we see that $u(t) \equiv u_0$ for all $0 \leq t \leq T$, and so u_0 is a fixed point. We now show the continuity of Φ on $\mathcal{A}$. Since

$$\int_\Omega \left(|\nabla_x u|^2 + |x|^{2s}|\nabla_y u|^2 \right) dxdy = \|u\|_{S_0^1}^2 = (Au, u) \leq |Au||u|,$$

we see that $\|u\|_{S_0^1}^2$ is Lipschitz continuous on $\mathcal{A}$. Furthermore, we have

$$\left| \int_\Omega (F(u) - F(v)) \, dxdy \right| = \left| \int_\Omega \int_{v(x,y)}^{u(x,y)} f(s) \, dsdxdy \right|$$

$$\leq \int_{\Omega} \|f\|_{\infty}|u(x) - v(x)| \ dxdy$$

$$\leq \|f\|_{\infty}|u - v|$$

where $\|f\|_{\infty} := \sup_{u \in \mathcal{A}} \|f(u)\|_{\infty} < \infty$ since $\mathcal{A}$ is bounded in $L^{\infty}(\Omega)$, and

$$\left| \int_{\Omega} (gu - gv) \ dxdy \right| \leq \|g\|_{\infty}|u - v| \text{ for } u, v \in \mathcal{A}.$$

This implies that Φ is a Lyapunov function on $\mathcal{A}$. □

Because the existence of a Lyapunov function, we can say more about the structure of the global attractor $\mathcal{A}$. It is known that a global attractor is the union of all bounded complete trajactories. Moreover, in our case due to the existence of a Lyapunov function, the global attractor $\mathcal{A}$ is the unstable manifold of the set of equilibria

$$\mathcal{A} = W^u(E),$$

where E is the set of all fixed points of the semigroup $S(t)$.

To show the injectivity of the semigroup $S(t)$ on $\mathcal{A}$, we need the following lemma whose proof can be found in [12, Chapter 11].

Lemma 3.1. *Let H and V be Hilbert spaces, and H^* and V^* be their duals, respectively, with $V \subset\subset H \cong H^* \subset V^*$. Suppose that $w \in L^{\infty}(0, T; V) \cap L^2(0, T; D(A))$ satisfies*

$$\frac{dw}{dt} + Aw = h(t, w(t)) \text{ in } L^2(0, T; H), \text{ and}$$

$$|h(t, w(t))| \leq k(t)\|w(t)\|, \ k(t) \in L^2(0, T),$$

where A is a bounded linear operator from V into V^*. If $w(T) = 0$, then $w(t) = 0$ for all $0 \leq t \leq T$.

Theorem 3.2. *The semigroup $S(t)$ has the injectivity property on $\mathcal{A}$. More precisely, if $u(t)$ and $v(t)$ are two trajectories on $\mathcal{A}$ with $u(T) = v(T)$ for some $T > 0$, then $u(t) = v(t)$ for all $0 \leq t \leq T$.*

Proof. Let $w = u - v$. Then we have

$$\frac{dw}{dt} + Aw + f(u) - f(v) = 0. \tag{3.1}$$

Since the global attractor $\mathcal{A}$ is bounded in both $S_0^1(\Omega)$ and $D(A)$, we obtain

$$w \in L^{\infty}(0, T; S_0^1(\Omega)) \cap L^2(0, T; D(A)).$$

Let

$$h(t, w(t)) = f(v(t)) - f(u(t)).$$

Then we see that

$$
\begin{aligned}
|f(u) - f(v)|^2 &= \int_\Omega |f(u) - f(v)|^2 dx dy \\
&= \int_\Omega \Big| \int_{v(x,y)}^{u(x,y)} f'(s)\ ds \Big|^2 dx dy \\
&\leq \int_\Omega \|f'\|_\infty^2 |u(x,y) - v(x,y)|^2 dx dy \\
&\leq \|f'\|_\infty^2 |u - v|_{L^2}^2,
\end{aligned}
\tag{3.2}
$$

where $\|f'\|_\infty := \sup_{u \in \mathcal{A}} \|f'(u)\|_\infty < \infty$ since $\mathcal{A}$ is bounded in $L^\infty(\Omega)$. Consequently, we get

$$
|h(t, w(t))| \leq \|f'\|_\infty |w(t)| \leq C\|w(t)\|_{S_0^1}.
$$

Hence we complete the proof by applying Lemma 3.1. $\square$

By Theorem 3.2, we can see that every trajectory on $\mathcal{A}$ is defined for all $t \in \mathbb{R}$, and $S(t)\mathcal{A} = \mathcal{A}$ for all $t \in \mathbb{R}$. Hence $\{S(t)\}_{t \in \mathbb{R}}$ is a dynamical system on $\mathcal{A}$.

Definition 3.2. *We say that the squeezing property holds for a semigroup $\{S(t)\}_{t \geq 0}$ if, for each $0 < \delta < 1$, there is a finite-rank orthogonal projection $P = P(\delta)$, with orthogonal complement $Q = Q(\delta)$, such that for all $u, v \in \mathcal{A}$, either*

$$
|Q(Su - Sv)| \leq |P(Su - Sv)|
\tag{3.3}
$$

or if not, then

$$
|Su - Sv| \leq \delta|u - v|,
\tag{3.4}
$$

where $S = S(1)$.

We will verify that the squeezing property indeed holds for the equation (1.1). The main idea is to consider $w(t) = u(t) - v(t)$ and to estimate the equation that bound $d(Pw)/dt$ below and $d(Qw)/dt$ above. The following lemma which will be used in our proof is clear.

Lemma 3.2. *For $t \in [0, T]$, let $x(t) \in \mathbb{R}$ satisfy the following inequality*

$$
\frac{1}{2}\frac{d}{dt}|x|^2 \leq C(t)|x|,
$$

where $C(t)$ is continuous. Then we have

$$
\frac{d}{dt_+}|x| \leq C(t).
$$

Theorem 3.3. *The squeezing property holds for the semigroup $S(t)$ induced by the equation (1.1).*

Proof. Let ω_j be eigenfunctions of A which are orthonormal in $L^2(\Omega)$ with corresponding eigenvalues λ_j, increasing in j with

$$A\omega_j = \lambda_j\omega_j, \ \lambda_{j+1} \geq \lambda_j, \ \text{for } j = 1, 2, \ldots$$

(for more details, see [3]). Let $u, v \in \mathcal{A}$, and $w = u - v$. Then the equation (3.1) is satisfied. For each $n \in \mathbb{N}$, define

$$P_n w = \sum_{j=1}^{n}(w, \omega_j)\omega_j \text{ and } Q_n w = \sum_{j=n+1}^{\infty}(w, \omega_j)\omega_j.$$

Put $p = P_n w$ and $q = Q_n w$. Then we have $w = p + q$. Take the inner product of (3.1) with p to get

$$\frac{1}{2}\frac{d}{dt}|p|^2 + \|p\|^2_{S_0^1} + (f(u) - f(v), p) = 0.$$

By (3.2), we have

$$\frac{1}{2}\frac{d}{dt}|p|^2 + \|p\|^2_{S_0^1} \geq -C|w||p|,$$

where $C = \|f'\|_{\infty}$. Due to the eigenvalue expansion of u and the fact that $\|u\|^2_{S_0^1} = (Au, u)$, we have

$$\|p\|_{S_0^1} \leq \lambda_n^{\frac{1}{2}}|p| \tag{3.5}$$

and

$$\|q\|_{S_0^1} \geq \lambda_{n+1}^{\frac{1}{2}}|q|. \tag{3.6}$$

By (3.5), we obtain

$$\frac{1}{2}\frac{d}{dt}|p|^2 \geq -\lambda_n|p|^2 - C|w||p|.$$

By Lemma 3.2, this becomes

$$\frac{d}{dt_+}|p| \geq -\lambda_n|p| - C(|p| + |q|).$$

Similarly, we take the inner product of (3.1) with q to obtain, by applying (3.6),

$$\frac{d}{dt_+}|q| \leq -\lambda_n|q| + C(|p| + |q|). \tag{3.7}$$

Choose n sufficiently large so that

$$\lambda_n - C > 2C. \tag{3.8}$$

If (3.3) holds, then we are done. If not, we have

$$|Q_n w(1)| > |P_n w(1)|. \tag{3.9}$$

Using (3.8), we see that

$$(\lambda_n - C)|Q_n w(t)| > 2C|P_n w(t)| \tag{3.10}$$

for $t = 1$. Since $w(t)$ is continuous into $L^2(\Omega)$, (3.10) is satisfied in a neighborhood of $t = 1$. Consider the following two cases.

Case 1. Suppose (3.10) holds for all $t \in [\frac{1}{2}, 1]$. By (3.8), we have

$$(\lambda_n - C)|q| - C|p| > \frac{1}{2}(\lambda_n - C)|q| > C|q|$$

for all $t \in [\frac{1}{2}, 1]$. Then (3.7) becomes

$$\frac{d}{dt_+}|q| \leq -\lambda_n C|q|, \text{ and so } |q(1)| \leq e^{-\frac{1}{2}\lambda_n C}\left|q\left(\frac{1}{2}\right)\right|.$$

By (3.9), we get

$$|w(1)| \leq \sqrt{2}e^{-\frac{1}{2}\lambda_n C}\left|q\left(\frac{1}{2}\right)\right| \leq \sqrt{2}e^{-\frac{1}{2}\lambda_n C}\left|w\left(\frac{1}{2}\right)\right|.$$

Since the solutions u and v satisfy the Lipschitz property (see Theorem 2.4 in [2]), we obtain that

$$\left|w\left(\frac{1}{2}\right)\right| \leq L\left(\frac{1}{2}\right)|w(0)|.$$

Hence

$$|w(1)| \leq \sqrt{2}L\left(\frac{1}{2}\right)e^{-\frac{1}{2}\lambda_n C}|w(0)|.$$

This induces (3.4) if λ_n is chosen to be sufficiently large enough.

Case 2. Suppose there exists $t_0 \geq 1/2$ such that (3.10) is valid on $(t_0, 1]$ and

$$(\lambda_n - C)|Q_n w(t_0)| = 2C|P_n w(t_0)| \tag{3.11}$$

Define a map $\Phi : [t_0, 1] \to \mathbb{R}$ by

$$\Phi(t) = \Phi(p(t), q(t)) = (|p| + |q|)\exp\left(\frac{\lambda_n|q|}{C(|p| + |q|)}\right).$$

Then we have

$$\frac{d\Phi}{dt} \leq 0 \text{ for all } t \in [t_0, 1].$$

Hence

$$\Phi(1) \leq \Phi(t_0).$$

Since (3.10) holds at $t = 1$, we obtain

$$\Phi(1) \geq |q(1)|e^{\lambda_n/C}.$$

On the other hand, since (3.11) holds at $t = t_0$, it follows that

$$2C(|p(t_0)| + |q(t_0)|) = (\lambda_n + C)|q(t_0)|,$$

and so

$$\Phi(t_0) = \frac{\lambda_n + C}{2C}|q(t_0)|e^{2\lambda_n/(\lambda_n + C)}.$$

Consequently

$$|q(1)| \leq e^{-\lambda_n/C}\frac{\lambda + C}{2C}e^2|q(t_0)|.$$

Using the Lipschitz property of the solutions again, we see that

$$|q(1)| \leq e^{-\lambda_n/C}\frac{\lambda + C}{2C}e^2 L(1)|w(0)|.$$

Since $|p(1)| < |q(1)|$, we must have

$$|w(1)| \leq 2e^{-\lambda_n/C}\frac{\lambda_n + C}{2C}e^2 L(1)|w(0)|.$$

Hence if we choose n sufficiently large enough, we obtain (3.4), and so it completes the proof. $\square$

Remark 3.1. As pointed out in [12, Chapter 14], the squeezing property has many important consequences. First, using the squeezing property one can show that the global attractor $\mathcal{A}$ lies close to a finite-dimensional Lipschitz manifold as a graph over $PL^2(\Omega)$ and that the dynamics "projected" onto $PL^2(\Omega)$ (here $P = P(\delta)$ is the finite-rank orthogonal projection in Definition 3.2) give a good indication on the asymptotic behavior of solutions. Furthermore, one can prove that the flow on the global attractor $\mathcal{A}$ has a finite number of determining modes, i.e., for two solutions $u(t)$ and $v(t)$ on the the attractor, if

$$|P(u(t) - v(t))| \to 0 \quad \text{as} \quad t \to \infty$$

then

$$|u(t) - v(t)| \to 0 \quad \text{as} \quad t \to \infty.$$

4. FRACTAL DIMENSION ESTIMATES

In this section, we obtain an upper bound and a lower bound of the fractal dimension of the global attractor $\mathcal{A}$. For brevity, we denote $V = S_0^1(\Omega)$ and $H = L^2(\Omega)$. The scalar product on V and H will be denoted by $((\cdot, \cdot))$ and $(\cdot, \cdot)$, respectively. Then we have

$$V \subset H \cong H^* \subset V^*.$$

Let $F(u) = -G_s + f(u) - g$, where u is the solution of (1.1), $g \in L^\infty(\Omega)$, and $f : \mathbb{R} \to \mathbb{R}$ is a function of polynomial type satisfying (1.2)-(1.3) and an additional assumption:

$$f \in C^2(\mathbb{R}) \text{ and } f'' \text{ is uniformly bounded on the bounded sets of } \mathbb{R}. \qquad (4.1)$$

For any $t \in [0, T]$, define

$$A(t) : V \to V^* \text{ by } A(t) = F'(u(t)),$$

where F' is the Fréchet derivative of F with $F'(u) = -G_s + f'(u)$. Then $A(t)$ is an isomorphism sending $D(A(t))$ onto H. For each $t \in [0, T]$, define $a(t; \cdot, \cdot) : V \times V \to \mathbb{R}$ by

$$a(t; u, v) = \langle A(t)u, v \rangle,$$

where $\langle \cdot, \cdot \rangle$ is a dual pairing of V and V^*. Then $\{a(t; u, v) : t \in [0, T]\}$ is a family of bilinear continuous forms on V such that

$$\forall u, v \in V, \ t \mapsto a(t; u, v) \text{ is a measurable function,} \qquad (4.2)$$

$$\exists M = M_T < \infty \text{ s.t. } |a(t; u, v)| \leq M_T \|u\| \|v\|, \ \forall u, v \in V, \text{ a.e. } t \in [0, T], \qquad (4.3)$$

$$\exists \alpha > 0 \text{ s.t. } a(t; u, u) \geq \alpha \|u\|^2, \ u \in V, \text{ a.e. } t \in [0, T]. \qquad (4.4)$$

For any $\xi \in H$, consider the following initial value problem

$$\begin{cases} \dfrac{dU(t)}{dt} + A(t)U(t) = 0 \text{ on } (0, T), \\ U(0) = \xi. \end{cases} \qquad (4.5)$$

Then from (4.2)-(4.4) we can show that there exists a unique solution U of (4.5) such that

$$U \in L^2(0, T; V) \cap C([0, T]; H), \qquad (4.6)$$

$$U' \in L^2(0, T; V'). \qquad (4.7)$$

From the above setting, we have the following result on the uniform differentiability of $S(t)$ on $\mathcal{A}$.

Lemma 4.1. *For every $t > 0$, $S(t)$ is uniformly differentiable on $\mathcal{A}$. Its differential at $u_0 \in \mathcal{A}$ is the linear operator $L(t, u_0) : \xi \in H \mapsto U(t) \in H$, where $U(\cdot)$ is the solution of (4.5). Moreover, we have $\sup_{u_0 \in \mathcal{A}} |L(t, u_0)|_{\mathscr{L}(H)} < \infty$.*

Proof. Let $t_0 > 0$ and $u_0, v_0 \in \mathcal{A}$. Let u, v be the solutions of (1.1) with $u(0) = u_0$ and $v(0) = v_0$, and U be the solution of (4.5) with $\xi = u_0 - v_0$. For each $t \geq 0$, define a linear operator

$$L(t, u_0) : \xi \in H \mapsto U(t) \in H.$$

Let $\eta(t) = v(t) - u(t) - U(t)$ for $t \geq 0$. Then we see that $\eta(t)$ satisfies $\eta(0) = 0$ and

$$\frac{d\eta}{dt} - G_s\eta + f'(u)\eta = h, \tag{4.8}$$

where

$$h = f(u) - f(v) + f'(u)(v - u) = [f'(u) - f'(u + \theta(v - u))](u - v), \ \ 0 < \theta < 1.$$

By Theorem 2.1 and (4.1), we see that

$$|h(t)| \leq c|u - v|^2,$$

for some $c > 0$ and all $t \geq 0$. Hence, by Theorem 2.4 in [2], we get

$$|h(t)| \leq ce^{4\ell t}|u_0 - v_0|^2, \ \text{for all } t \geq 0. \tag{4.9}$$

We multiply (4.8) by η and integrate over Ω. Then (1.3) and Green's formula yield

$$\frac{1}{2}\frac{d}{dt}|\eta|^2 + \|\eta\|_{S_0^1}^2 \leq \ell|\eta|^2 + \int_\Omega h\eta \, dxdy \leq (\ell + \frac{1}{2})|\eta|^2 + \frac{1}{2}|h|^2.$$

Hence by (4.9),

$$\frac{d}{dt}|\eta|^2 \leq (2\ell + 1)|\eta|^2 + c^2 e^{8\ell t_0}|u_0 - v_0|^4, \ \text{for all } 0 \leq t \leq t_0.$$

By Gronwall's inequality, we obtain

$$|\eta(t_0)|^2 \leq \frac{e^{(2\ell+1)t_0} - 1}{2\ell + 1}c^2 e^{8\ell t_0}|u_0 - v_0|^4.$$

This implies that $S(t_0)$ is uniformly differentiable on $\mathcal{A}$, and its differential at $u_0 \in \mathcal{A}$ is $L(t_0, u_0)$. Moreover, by the continuous dependence of solution of (4.5) on the initial data, we can see that $\sup_{u_0 \in \mathcal{A}} |L(t, u_0)|_{\mathcal{L}(H)} < \infty$. $\qquad \square$

We now recall the definition of fractal dimension. The *fractal dimension* of $\mathcal{A}$ is defined by

$$d_F\mathcal{A} = \limsup_{\varepsilon \to 0} \frac{\log N(\mathcal{A}, \varepsilon)}{\log(1/\varepsilon)},$$

where $N(\mathcal{A}, \varepsilon)$ is the minimum number of the balls of radius $\varepsilon > 0$ to cover $\mathcal{A}$.

For $L \in \mathcal{L}(H)$ and $m \in \mathbb{N}$, let $\omega_m(L)$ be the norm of the exterior product $\wedge^m L$ in $\wedge^m H$, that is,

$$\omega_m(L) = \sup_{\substack{\xi_1, \cdots, \xi_m \in H \\ |\xi_i| \leq 1}} |L\xi_1 \wedge \cdots \wedge L\xi_m|. \tag{4.10}$$

For each $m \in \mathbb{N}$, let

$$\bar{\omega}_m(t) = \sup_{u_0 \in \mathcal{A}} \omega_m(L(t, u_0)), \ t \geq 0, \ \text{and} \ \Pi_m = \lim_{t \to \infty} \bar{\omega}_m(t)^{1/t}.$$

The *uniform Lyapunov numbers*, μ_m are given by

$$\mu_1 = \log \Pi_1, \ \mu_m = \log \Pi_m - \log \Pi_{m-1}, \ m \geq 2.$$

To estimate the fractal dimension of the global attractor, we use the following abstract result.

Lemma 4.2. *[9, Theorem 3.1] If $\mu_1 + \cdots + \mu_m < 0$ for some $m \in \mathbb{N}$, then*

$$d_F \mathcal{A} \leq m \max_{1 \leq j \leq m-1} \left(1 + \frac{(\mu_1 + \cdots + \mu_j)_+}{|\mu_1 + \cdots + \mu_m|} \right).$$

The crucial step to obtain an upper bound of the fractal dimension of $\mathcal{A}$ is to estimate the Lyapunov numbers of $\mathcal{A}$.

Theorem 4.1. *The fractal dimension of $\mathcal{A}$ is less than or equal to $2m$, where $m \in \mathbb{N}$ is such that*

$$m - 1 < \left(\frac{2\kappa_2}{\kappa_1} \right)^{N(s)/(2+N(s))} \leq m.$$

Here $\kappa_1 = \frac{1}{2}c_1, \kappa_2 = c_3(c_2 + \ell)^{1+(N(s)/2)}$ for some constants c_1, c_2 and c_3. More precisely, c_1, c_2 and c_3 are obtained in the proof below.

Proof. Note that

$$\mu_1 + \cdots + \mu_m = \log \Pi_m, \ \text{for } m \in \mathbb{N}.$$

According to (4.10), we obtain

$$\omega_m(L(t, u_0)) = \sup_{\substack{\xi_i \in H \\ |\xi_i| \leq 1}} |U_1 \wedge \cdots \wedge U_m|,$$

where U_i is the solution of (4.5) with $U_i(0) = \xi_i$ for $1 \leq i \leq m$. By Lemma 1.2 in [13, Chapter V], we see that

$$\frac{1}{2}\frac{d}{dt}|U_1 \wedge \cdots \wedge U_m|^2 = -\mathrm{Tr}(F'(u) \circ Q_m)|U_1 \wedge \cdots \wedge U_m|^2,$$

where Q_m is the orthogonal projection in H onto the space spanned by $U_1, \cdots, U_m$. So we get

$$\omega_m L(t, u_0) \leq \sup_{\substack{\xi_i \in H \\ |\xi_i| \leq 1}} \exp \left(- \int_0^t \mathrm{Tr} \ F'(S(\tau)u_0) \circ Q_m(\tau) \ d\tau \right). \qquad (4.11)$$

For each $m \in \mathbb{N}$, we let

$$q_m = \limsup_{t \to \infty} \left\{ \inf_{u_0 \in \mathcal{A}} \frac{1}{t} \int_0^t \inf_{\mathrm{rank}Q=m} \mathrm{Tr} \ F'(S(\tau)u_0) \circ Q(\tau) \ d\tau \right\}, \qquad (4.12)$$

where Q is any orthogonal projection in H with $QH \subset V$. From (4.11), we get

$$\bar{\omega}_m(t) = \sup_{u_0 \in \mathcal{A}} \omega_m(L(t, u_0)) \leq \sup_{u_0 \in \mathcal{A}} \exp \left\{ - \int_0^t \inf_{\text{rank}Q=m} \text{Tr } F'(S(\tau)u_0) \circ Q(\tau) \, d\tau \right\}.$$

Hence we obtain

$$\Pi_m = \lim_{t \to \infty} \left\{ \bar{\omega}_m(t) \right\}^{1/t} \leq \exp(-q_m).$$

Consequently, if $q_m > 0$ for some m, then by Lemma 4.2, we get

$$d_F \mathcal{A} \leq m \max_{1 \leq j \leq m-1} \left\{ 1 + \frac{(-q_j)_+}{q_m} \right\}. \tag{4.13}$$

We now show that $q_m > 0$ for some m. Let Q be an orthogonal projection in H of rank m with $QH \subset V$. Let $\{\phi_j\}_{j \in \mathbb{N}}$ be an orthonormal basis of H with $\phi_1, \ldots, \phi_m$ being a basis of QH. Then we have

$$\text{Tr } F'(u) \circ Q = \sum_{j=1}^m (F'(u)\phi_j, \phi_j) = \sum_{j=1}^m \|\phi_j\|_{S_0^1}^2 + \int_\Omega f'(u)\rho \, dxdy,$$

$$\geq \sum_{j=1}^m \|\phi_j\|_{S_0^1}^2 - \ell \int_\Omega \rho \, dxdy,$$

where

$$\rho(x, y) = \sum_{j=1}^m |\phi_j(x, y)|^2.$$

Let $\{\lambda_j\}$ be the sequence of eigenvalues of A in $L^2(\Omega)$ such that $0 < \lambda_1 \leq \lambda_2 \leq \cdots$. Then we see that

$$\sum_{j=1}^m \|\phi_j\|_{S_0^1}^2 \geq \lambda_1 + \cdots + \lambda_m.$$

Moreover, we have

$$\lambda_1 + \cdots + \lambda_m \geq c_1 m^{1+(2/N(s))} - c_2 m,$$

where $c_1 = c_1(N, s, |\Omega|, \alpha_{N-1})$ is a constant depending on $N, s, |\Omega|$ and α_{N-1}, and $c_2 = c_2(s)$ is a constant depending on s. Here α_{N-1} is the area of the unit sphere in $\mathbb{R}^N$ (for more details, see Theorem 1.1 in [3]). Hence we get

$$\sum_{j=1}^m \|\phi_j\|_{S_0^1}^2 \geq c_1 m^{1+(2/N(s))} - c_2 m.$$

Since each ϕ_j is mutually orthonormal, we have

$$\text{Tr } F'(u) \circ Q \geq c_1 m^{1+(2/N(s))} - m(c_2 + \ell),$$

where $\int_\Omega \rho \, dxdy = m$. By Young's inequality, we obtain

$$m(c_2 + \ell) \leq \frac{1}{2} c_1 m^{1+(2/N(s))} + c_3(c_2 + \ell)^{1+(N(s)/2)},$$

where c_3 depends on $N, s, |\Omega|$ and α_{N-1}. Consequently, we get

$$\operatorname{Tr} F'(u) \circ Q_{\geq} \frac{1}{2} c_1 m^{1+(2/N(s))} - c_3(c_2 + \ell)^{1+(N(s)/2)}.$$

Hence by (4.12), we have

$$q_m \geq \kappa_1 m^{1+(2/N(s))} - \kappa_2,$$

where $\kappa_1 = \frac{1}{2} c_1$ and $\kappa_2 = c_3(c_2 + l)^{1+(N(s)/2)}$. Let $m \in \mathbb{N}$ be such that

$$q_m > 0 \text{ and } m - 1 < \left(\frac{2\kappa_2}{\kappa_1}\right)^{N(s)/(2+N(s))} \leq m.$$

Then we obtain

$$\frac{(-q_j)_+}{q_m} \leq 1, \quad \forall j = 1, \ldots, m - 1.$$

Hence we conclude that $d_F \mathcal{A} \leq 2m$ from (4.13). $\qquad\square$

We now find a lower bound of the fractal dimension of the global attractor $\mathcal{A}$ induced by the equation (1.1). For simplicity, we assume $g \equiv 0$ and $f(0) = 0$ in the equation (1.1). Then we can see the following properties:

- the mapping $(t, u_0) \mapsto S(t)u_0$ from $\mathbb{R}_+ \times H$ into H is continuous;

- $\mathbf{0} \in H$ is a fixed point of $S(t)$.

Moreover, we can check that $S(t)$ is Fréchet differentiable at $\mathbf{0}$ for all $t \in \mathbb{R}_+$ with differential $S'(t)(\mathbf{0}) = L(t, \mathbf{0})$. Let us write $S'(t) = S'(t)(\mathbf{0})$ for simplicity. Then we can show that the spectrum of $S'(t)$ consists of its eigenvalues and 0. To show this, let us consider the following first variation equation which is obtained from (4.5) by putting $u = \mathbf{0}$:

$$\begin{cases} U_t - G_s U + f'(0)U = 0, \\ U(0) = \xi. \end{cases}$$

Then we have

$$-G_s w_k = \lambda_k w_k, \quad w_k \in S_0^1(\Omega),$$

and may assume that

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_k \to \infty \text{ as } k \to \infty.$$

Without loss of generality, we can suppose that $f'(0) + \lambda_k \neq 0$ for all $k \in \mathbb{N}$. Put $\mu_k = f'(0) + \lambda_k$. Then we obtain

$$-G_s w_k + f'(0)w_k = \mu_k w_k.$$

Consequently, we see that the eigenvalues of $S'(t)$ are the numbers $e^{-\mu_k t}$, and the corresponding eigenfunctions are w_k for $k \in \mathbb{N}$. If $\lambda \notin \{e^{-\mu_k t}, 0 : k \in \mathbb{N}\}$, then $S'(t)\zeta - \lambda\zeta \neq \mathbf{0}$ for $\zeta \in H \setminus \{\mathbf{0}\}$. This means that the spectrum of $S'(t)$ consists of eigenvalues and 0.

Let n be the number of λ'_ks with $\lambda_k < -f'(0)$, and let H_+ be the subspace of H spanned by the eigenvectors corresponding to the eigenvalues $e^{-\mu_k t}$ of $S'(t)$ with $\mu_k < 0$. Then we know that $\dim H_+ = n$. This shows that $z = \mathbf{0}$ is a hyperbolic fixed point of $S'(t)$ since the spectrum of $S'(t)$ does not intersect with the unit circle. Moreover, we see that $S'(t)$ satisfies a Lipschitz condition:

$$\|S'(t)u_1 - S'(t)u_2\| \leq \|u_1 - u_2\| \text{ for } u_1, u_2 \in H.$$

For $r > 0$, we let

$$W_+^r(z) = \{u_0 \in B_r(z) : \exists u_n \in H \text{ s.t. } S(n)u_n = u_0 \text{ and } u_n \to z \text{ as } n \to \infty\},$$

where $B_r(z)$ is the ball of radius r centered at z under a suitable norm on H. Then we have that $W_+^r(z) \subset \mathcal{A}$ for sufficiently small $r > 0$, and $W_+^r(z)$ is a C^1 manifold of dimension equal to $\dim H_+ = n$. Consequently, we have proved the following theorem.

Theorem 4.2. *The fractal dimension $d_F\mathcal{A}$ of the global attractor $\mathcal{A}$ of the equation (1.1) is greater than or equal to n, where n =$\dim H_+$.*

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